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Chapter 1

Introduction

For convenience we turn off the spell checker by typing

```
Off@General::spell1D
```

à 1.1 Preliminary Remarks

à 1.2 Symbolic Computation

The *Mathematica* programming language allows not only for interactive manipulations of a wide range of objects, but it also encourages the automation of complicated computations. *Mathematica* actually works by using systems of rewrite rules, and the user can freely create and use his or her own systems of rules. This distinguishes it from traditional programming languages, which normally have no such features. We have applied rule-based and functional programming techniques in *Mathematica* throughout this book. Hence, we briefly cover the basic ingredients of rule-based programming in Sections 1.2.1–1.2.5 and some functional programming features in Sections 1.2.6–1.2.8.

For further information on *Mathematica*, consult the standard *Mathematica* reference: *The Mathematica Book*, Stephen Wolfram, Third Edition, Wolfram Media and Cambridge University Press, 1996. Full descriptions of the functions in the standard *Mathematica* packages are available in *Mathematica 3.0 Standard Add-On Packages manual*, Wolfram Media and Cambridge University Press, 1996. Alternatively, simply click on the Help menu to view the Help Browser.

In the Text cells of this book, we use **bold-magenta** to indicate **user-defined functions** or **names** and **bold-black** to indicate *Mathematica's* **built-in functions**.

§ 1.2.1 Global Rules

Global rules are applied whenever the appropriate left-hand side is encountered or matched. The advantage of using global rules is that one does not need to apply explicitly a rule to get things done. However, it is almost impossible to prevent such global rules from being applied to a certain expression. There are two types of user-defined global rewrite rules: those using **=** and those using **:=**. The distinction between them lies in **when** the right-hand side is evaluated. Moreover, each of these rules has two forms, depending on **where** the rule is stored, resulting in four rules indicated by **=**, **^=**, **:=**, and **^:=**.

§ 1.2.1.1 = rules

We can think of `=`, an assignment statement in analogy with traditional procedural programming languages, as a (global) rewrite rule. The characteristic property of using the rule `=` is that the right-hand side is evaluated immediately when the assignment is made, and all subsequent matched left-hand sides are replaced by the evaluated right-hand side. For example, let us consider the rule

```
a = 2;
```

The output is suppressed by ending the input with a semicolon. From now on, whenever `a` is encountered in any expression, *Mathematica* will replace it by its value 2.

```
expr1 = a2 + 3 - a + Tan@aD
```

```
5 + Tan@2D
```

Let us try to evaluate `expr1` for another value,

```
a = 3;
```

```
expr1
```

```
5 + Tan@2D
```

The new value of `a` is not recognized in `expr1`.

Let us try to clear the definition of `a` by using `Clear` or

```
a = .
```

Then, let us evaluate `expr1`,

```
expr1
```

```
5 + Tan@2D
```

Again, `expr1` is not changed accordingly.

To be able to evaluate `expr1` for different values of `a`, we need to either define `expr1` before the assignment of `a` or use `:=`, instead of `=`. The precise meaning of `:=` is taken up in the next section. Here, we illustrate the former choice.

```
Clear@aD; expr1 = a2 + 3 - a + Tan@aD
```

```
3 - a + a2 + Tan@aD
```

```
a = 2; expr1
```

```
5 + Tan@2D
```

```
a = 3; expr1
```

```
9 + Tan@3D
```

In traditional programming languages, the left-hand side of an assignment statement is required to be a simple identifier (i.e., a symbol). In computer algebra, the left-hand side can be arbitrarily complicated. For example,

```
look@here + 9D = Expand@H1 + xL^2 + yD
```

```
1 + 2 x + x^2 + y
```

Note that the output of an `=` expression is the evaluated form of the right-hand side. *Mathematica* regards the left-hand side as a [pattern](#). Whenever *Mathematica* finds *something* that matches this pattern, it replaces the *something* by the evaluated right-hand side.

```
look@there + look@here + 5 + 2^2DD
```

```
look@1 + there + 2 x + x^2 + yD
```

In this evaluation, the pattern `look@here + 5 + 2^2D` simplifies to `look@here + 9D`, which is replaced by `1 + 2 x + x^2 + y`. The resulting expression does not match any pattern involving `look` and so it is left in unevaluated form. This rule is stored with `look`.

```
? look
```

```
Global`look
```

```
look@9 + hereD = 1 + 2 * x + x^2 + y
```

Again, we see the evaluated form on the right-hand side.

There are some problems associated with left-hand sides that are not symbols. For example, suppose we try to make the following rule:

```
x + y = z
```

```
Set::write : Tag Plus in x+y is Protected.
```

```
z
```

We get an error message stating that **Plus** is protected, which means new rules cannot be added for **Plus**. Every time *Mathematica* encounters **Plus**, it searches through the rules for **Plus** to see if anything applies. If we add a new rule for **Plus**, then that rule would have to be examined at every subsequent addition. When a rule of the form $x + y = z$ is given, *Mathematica* interprets it as a rule of the form `Plus[x, y] = z`. Rules have to be stored somewhere and the default rule is that for the head of the left-hand side. One way of adding a new rule for **Plus** is to unprotect **Plus**, make the rule, and then reprotect it.

```
Unprotect@PlusD;
```

```
x + y = z;
```

```
Protect@PlusD;
```

Now whenever *Mathematica* sees $x + y$, it rewrites it as z .

```
x + s + y + t
```

```
s + t + z
```

Alternatively, we can use **UpValues** to associate the rule with the unprotected argument of the left-hand side. (Definitions that attach a value to the head of the left-hand side are called **DownValues** of the head.) For example,

```
s + t ^= u
```

```
u
```

Note the caret ^ before the = sign. This rule is associated with the symbol *s* or *t*.

```
? s
```

```
Global` s
```

```
s •: s + t = u
```

```
t + s + t
```

```
t + u
```

A given symbol can have both up and down values. Let's give *s* a down value in addition to the up value it already has.

```
s@x_D := Exp@I a xD;
```

Then looking at *s* shows both kinds of values:

```
? s
```

```
Global` s
```

```
s •: s + t = u
```

```
s@x_D := Exp@I * a * xD
```

Finally, we can access the up and down values individually.

```
8UpValues@sD, DownValues@sD<
```

```
88HoldPattern@s + tD | u<, 8HoldPattern@s@x_DD | Exp@I a xD<<
```

Now let us try naively define the sine function using an = rule according to

```
m@xD = Sin@xD;
```

It works properly for the symbol *x* but not for anything else.

```
8m@xD, m@yD, m@1D<
```

```
8Sin@xD, m@yD, m@1D<
```


This is where the special symbol `_` comes in. The form `x_` means a [pattern](#) named `x`. We can show the internal representation of any expression, such as `x_`, by either using

```
FullForm@x_D
Pattern@x, Blank@DD
```

or using

```
x_ •• FullForm
Pattern@x, Blank@DD
```

An underscore `_` in a pattern matches [anything](#), so it is a dummy variable or a "wild card". If it appears on the left-hand side of "=" rule with a name, like `x`, then the left-hand side is rewritten as the right-hand side with `x` replaced by the desired variable. Let us use this to redefine [m](#); that is,

```
m@x_D = Sin@xD;
```

Now [m](#) works for any argument:

```
8m@xD, m@yD, m@1D, m@s - tD<
8Sin@xD, Sin@yD, Sin@1D, Sin@s - tD<
```

Thus, one can use `=` rules to define functions.

§ 1.2.1.2 := rules

Rules using `:=` do not evaluate the right-hand side immediately but instead leave it unevaluated until the function is actually used. They can be used with simple left-hand sides or with left-hand sides containing patterns. As an example, let us define and compare two rules using `=` or `:=`.

```
Clear@xD;
lhs1 = Expand@Hx + 1L ^2D;
lhs2 := Expand@Hx + 1L ^2D;
```

If these are evaluated, they give the same result.

```
8lhs1, lhs2<
81 + 2 x + x^2, 1 + 2 x + x^2<
```

If we now give a value to `x`, then [lhs1](#) and [lhs2](#) will use this value in different ways.

```
x = p + q;
8lhs1, lhs2<
81 + 2 Hp + qL + Hp + qL^2, 1 + 2 p + p^2 + 2 q + 2 p q + q^2<
```

If the left-hand side of a $:=$ rule contains a pattern, then on a subsequent occurrence of the left-hand side with actual arguments, the formal (dummy) arguments (or names of patterns) on the right-hand side are replaced by the actual arguments and then the right-hand side is evaluated. Thus, each time the left-hand side of such a rule matches something, it is replaced by a new evaluation of the right-hand side. To see the difference, we again set up the two rules

```
f@k_D = Expand@k^2D
```

```
k^2
```

```
g@k_D := Expand@k^2D
```

We note that the right-hand side of the rule for **f** is evaluated immediately when it is defined, whereas the right-hand side of **g** is kept unevaluated. Now, try out these two definitions on the same value.

```
8f@p + qD, g@p + qD<
```

```
8Hp + qL^2, p^2 + 2 p q + q^2<
```

Because the right-hand side of the rule for **f** was evaluated immediately when it was defined, there is nothing to expand, and k is replaced with $p+q$ without expanding the result. On the other hand, the right-hand side of the rule for **g** was not expanded when defined and it retains the whole expression **Expand**[k^2]. When the two functions are subsequently used, $f@p+qD$ is just replaced by $Hp+qL^2$, while $g@p+qD$ is replaced by **Expand**[$Hp+qL^2$], which evaluates to $p^2 + 2pq + q^2$. The internal representation of such a definition has the following form:

```
FullForm@g@k_D := pD
```

```
Null
```

To see the actual head of an expression before it is evaluated, we wrap the expression with **Hold** and then request its **FullForm**

```
FullForm@Hold@g@k_D := pDD
```

```
Hold@SetDelayed@g@Pattern@k, Blank@DDD, pDD
```

Thus, the symbol $:=$ is the infix form of **SetDelayed**. This makes dramatically clear the distinction between the evaluation [when the rule is given](#) and the evaluation [when the rule is used](#).

When the head of the left-hand side is protected, we can use **UpValues** to associate the rule with the unprotected argument of the left-hand side. For example, when we try to differentiate a user-defined definite integral with constant bounds, we obtain

```
Clear@int, f, xD
```

```
D@int@f@xD, 8x, 0, 1<D, xD
```

```
intH0,81,0,0<L@f@xD, 8x, 0, 1<D + fc@xD intH1,80,0,0<L@f@xD, 8x, 0, 1<D
```

which is not what we intended. We can either unprotect **D** and add a rule or use $\wedge:=$ to assign the definition with **int**; that is,

```
D@int@x_, a_D, y_D ^:= int@D@x, yD, aD
```

Now we check to see if *Mathematica* gives us the answer we want.

```
D@int@f@xD, 8x, 0, 1<D, xD
```

```
int@f@xD, 8x, 0, 1<D
```

As a second example, we define $\text{Cos}[n] = H - 1L^n$ and $\text{Sin}[n] = 0$ for integer n :

```
Unprotect@Cos, SinD;
```

```
Cos@n_ pD := H-1L^n •; IntegerQ@nD
```

```
Sin@n_ pD := 0 •; IntegerQ@nD
```

```
Protect@Cos, SinD;
```

If we declare m to be an integer (here `UpValues` is used),

```
IntegerQ@mD ^= True;
```

then

```
8Cos@m pD, Sin@m pD<
```

```
8H-1L^m, 0<
```

§ 1.2.2 Local Rules

Local rewrite rules ($lhs \rightarrow rhs$ or $lhs :> rhs$) are useful for making substitutions without making the definitions permanent. These rules are applied to an expression using the operation `•` (means **ReplaceAll**) or `••` (means **ReplaceRepeated** in *Mathematica*).

§ 1.2.2.1 $\rightarrow$ rules

We can use `Solve` to obtain the solutions of some algebraic equations. For example,

```
eq1 = u^2 - 3 u + 2 == 0;
```

```
sol1 = Solve@eq1, uD
```

```
88u ® 1<, 8u ® 2<<
```

We can check the result by substituting `sol1` into `eq1` and obtain

```
eq1 •. sol1
```

```
8True, True<
```

The output of `Solve` is a list of lists of local rules. Thus, $u \rightarrow 1$ is a local rule, which is the analog of the global rule $u = 1$. The rule is applied to an expression by using `/.`, so the expression `eq1/sol1` means "use the rewrite rules $u \rightarrow 1$ and $u \rightarrow 2$ in

eq1". The result of this is $80 == 0, 0 == 0$, which is then evaluated as **{True, True}**. The usual form of the right-hand side of **/.** is a list of local rules for some of the symbols that appear on the left-hand side.

```
u v w •. 8u -> 3, w -> 5<
```

```
15 v
```

If the right-hand side is a list of lists of local rules, then the result is a list of modified expressions, one for each substitution in the list.

```
u v w •. 88u -> 3, w -> 5<, 8v -> 2, w -> 3<<
```

```
815 v, 6 u<
```

Local replacement rules also work on expressions which have a **Hold** wrapped around them. For example, without a **Hold**, **¶_x Cos@xD** immediately evaluates to **-Sin@xD**.

```
¶x Cos@xD
```

```
-Sin@xD
```

However, the rewrite rule goes under the **Hold** to make the replacement.

```
Hold@¶x Cos@xDD •. Cos -> Sin
```

```
Hold@¶x Sin@xDD
```

Local rules with **"->"** share with **"=="** rules the property that their right-hand sides are evaluated at the time they are defined.

§ 1.2.2.2 **>** rules

The local analog of a **":="** rule is a **":>"** rule; i.e., a local rule that evaluates its right-hand side only when it is used. For example,

```
expr1 = Exp@I Hw2 - w1 L T0 D;
```

```
Clear@aD
```

```
expr1 •. Exp@a_D -> Exp@Expand@aDD
```

```
¶I T0 H-w1+w2 L
```

```
8expr1 •. Exp@a_D > Exp@Expand@aDD, expr1 •• ExpandAll<
```

```
8E-I T0 w1+I T0 w2, E-I T0 w1+I T0 w2<
```

where the immediate rule ($lhs \rightarrow rhs$) fails to do the expansion. The left-hand sides of the local rules involve patterns rather than just symbols. The difference is that the immediate rule for `Exp@a_D` replaces it by the evaluation of `Exp@Expand@aDD` which equals `Exp@aD`. Hence, when this is used with a equal to $I^H W_2 - W_1 L T_0$, we get the result $E^{I T_0 H - W_1 + W_2 L}$. On the other hand, the delayed rule ($lhs \rightarrow rhs$) for `Exp@a_D` replaces it by the unevaluated `Exp@Expand@aDD` which, when used with a equal to $I^H W_2 - W_1 L T_0$, gives $E^{-I T_0 W_1 + I T_0 W_2}$. Alternatively, for this simple example, we can use `ExpandAll` to obtain the same result.

We can check how these expressions are represented internally.

```
FullForm@Hold@p •. q -> rDD
Hold@ReplaceAll@p, Rule@q, rDDD

FullForm@Hold@p •. q :> rDD
Hold@ReplaceAll@p, RuleDelayed@q, rDDD
```

Thus, `/.` is the infix form of `ReplaceAll`, the arrow `->` is the infix form of `Rule`, and the arrow `:>` is the infix form of `RuleDelayed`, corresponding to `Set` and `SetDelayed` for `=` and `:=`.

§ 1.2.2.3 •. and ••.

There is another form of `•.` given by `••.` which applies a local rule repeatedly until the expression no longer changes. Internally, `••.` is represented by

```
p ••. q -> r •• FullForm •• HoldForm
ReplaceRepeated@p, Rule@q, rDD
```

An example of the difference between `•.` and `••.` follows. This example uses a list of rules rather than just a single rule. When a list of rules is applied to a single expression, then the rule for each symbol is tried from the left until a match is found. In the following example, the right-hand side of the `•.` expression consists of a list of two rules for the same symbol, `fac`. This list is searched from the left until a pattern is found that matches the left-hand side of the `•.` expression. In the first case using `•.`, as soon as a match is found, the evaluation is finished. In the second case using `••.`, the rules are tried repeatedly from the left on the output of the previous evaluation until no matches are found.

```
fac@5D •. 8fac@1D -> 1, fac@i_D -> i fac@i - 1D<
5 fac@4D

fac@5D ••. 8fac@1D -> 1, fac@i_D -> i fac@i - 1D<
120
```

In the first case, the left-hand side of the rule `fac@1D -> 1` does not match anything in `fac@5D`, but `fac@i_D -> i fac@i - 1D` does with `i` equal to 5, so the output is `5 fac@4D`. In the second case, the left-hand side of the rule `fac@i_D -> i fac@i - 1D` continues to match a part of the existing expression until `120 fac@1D` is obtained. Then the left-hand side of the rule `fac@1D -> 1` matches, leading to `120 * 1`, which simplifies to 120 where neither rule matches, so the output is 120.

If such rules are given globally, then the order in which they are given does not matter since *Mathematica* will put the more specific rule, $\text{fac}[1] = 1$, first. However, in a list of local rules, applied with **••**, we are completely responsible for the [ordering](#). Thus, the following gives the wrong answer:

```
fac@5D ••. 8fac@i_D -> i fac@i - 1D, fac@1D -> 1<
0
```

If several local rules are given for different symbols, then these rules are applied simultaneously. For example,

```
8u, v, w< •. 8u -> v, v -> w, w -> s<
8v, w, s<
```

If the substitutions are carried out sequentially, then the results are quite different.

```
8u, v, w< •. 8u -> v< •. 8v -> w< •. 8w -> s<
8s, s, s<
```

In particular, this means that variables can be interchanged without introducing an intermediate temporary variable.

```
8u, v< •. 8u -> v, v -> u<
8v, u<
```

§ 1.2.3 Pattern Matching

Mathematica is a language based on pattern matching. Patterns refer to the structure of expressions. The symbol **_** by itself, without any symbol on the left, can be used to describe a pattern. (Recall that the **FullForm** of **_** is **Blank[]**). For instance, the expression **_ ^ _** matches anything of the form x^y , where x and y are any (dummy) expressions.

```
Clear@a, xD
g1@_, _ ^ _D := u;
8g1@a, a ^ aD, g1@a, b ^ cD<
8u, u<
```

A pattern of the form **x_** is matched by any expression, which is then used for evaluating the right-hand side.

```
g2@x_ ^ y_, z_D := u@x, y, v@zDD;
8g2@a ^ a, aD, g2@a ^ b, cD<
8u@a, a, v@aDD, u@a, b, v@cDD<
```

```

g3@x_^y_, x_D := u@y, v@xDD;
8g3@a^a, aD, g3@a^b, cD, g3@a^b, aD<

8u@a, v@xDD, g3@a^b, cD, u@b, v@xDD<

```

A pattern of the form x_h is matched by any expression whose head is h .

```

g4@x_^y_Integer, z_D := u@x, y, v@zDD;
8g4@a^a, aD, g4@a^3, arg2D<

8g4@a^a, aD, u@a, 3, v@arg2DD<

```

If we give a rule for an expression involving two separate arguments with underscores, then we are constructing a function of two variables. Such a function only works if it is given exactly two arguments.

```

g5@x_, y_D := x + y;
8g5@aD, g5@a, bD, g5@a, b, cD<

8g5@aD, a + b, g5@a, b, cD<

```

Besides rule-schemes using a single underscore_, there are rule-schemes using double and triple underscores. A double underscore, __, is matched by any number of arguments, excluding zero arguments, separated by commas, while a triple underscore, ___, is matched by any number of arguments, including zero arguments. The form $x_$ means a sequence of one or more arguments, named x , and x_Head means a sequence of one or more arguments, named x , with the head of each argument is $Head$. Similarly, the form $x_$ means zero or more arguments, named x , and x_Head means zero or more arguments, named x , with the head of each argument is $Head$.

```

g6@x__D := Length@8x<D;
8g6@D, g6@aD, g6@a, bD, g6@a, b, cD<

8g6@D, 1, 2, 3<

g7@x___D := Length@8x<D;
8g7@D, g7@aD, g7@a, bD, g7@a, b, cD<

80, 1, 2, 3<

```

Default values and double and triple underscores are important techniques, which give optional number of arguments to functions; that is, a variable number of arguments can be given to such functions. Alternatively, there is another rule in which a specific argument can be optional.

```

g8@x_^y_, z_D := u@x, y, zD;
8g8@a^b, cD, g8@a, cD<

8u@a, b, cD, g8@a, cD<

```

Because $a = a^1$, one might think that `g8@a, cD` should match the pattern `g8@a^1, cD`, which would mean that it should be rewritten as `u@a, 1, cD`, but of course *Mathematica* cannot guess that this is what we intend. However, there is a provision to take care of such default values. To assign a default value v to a pattern, one can write `_ : v`. So, the effect we wanted to achieve is given by the form

```
g9@x_^Hy_: 1L, z_D := u@x, y, zD;
8g9@a^b, cD, g9@a, cD<

8u@a, b, cD, u@a, 1, cD<
```

In this case, the default value 1 for the exponent is the natural and obvious choice, and *Mathematica* knows this. It has standard built-in default values for a number of such positions. The notation `_.` tells *Mathematica* to use the built-in default value. Note the almost invisible period after the underscore. Thus, the effect we wanted at the beginning is given by a tiny modification of the original form.

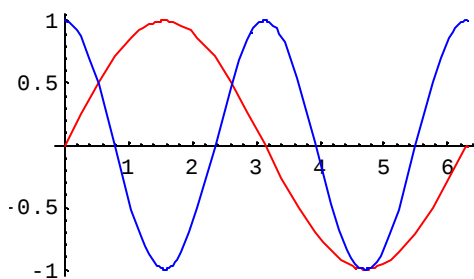
```
g10@x_^y_., z_D := u@x, y, zD;
8g10@a^b, cD, g10@a, cD<

8u@a, b, cD, u@a, 1, cD<
```

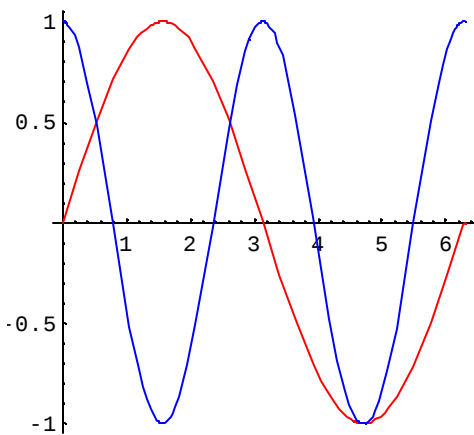
There is still another way that optional arguments occur in *Mathematica*. Some functions, such as **Plot**, can take named optional arguments such as `AspectRatio -> 1`. By incorporating such functions into our own definitions, we can use these named optional arguments too. Consider an example of a plotting function.

```
plotwithSin@func_, var_, opts___D := Plot@8Sin@varD, func@varD<,
      8var, 0, 2 Pi<, opts, PlotStyle -> 8RGBColor@1, 0, 0D, RGBColor@0, 0, 1D<D;

plotwithSin@Cos@2 #D &, xD;
```




```
plotwithSin@Cos@2 #D &, x, AspectRatio -> 1D;
```



In an expression of the form `g[x_^n_Integer, z_]`, the patterns are named with the symbols `x`, `n`, and `z`, but there is no name for the whole compound pattern `x_^n_Integer`. There is a way to give names to such compound patterns so that they can be referred to directly on the right-hand side. The syntax consists of a name followed by a colon followed by the compound pattern.

```
g11@expr : x_^y_Real, z_D := z expr;
8g11@Ha + b - 2L^0.5, 2D, g11@Ha + b - 2L^3, 2D<

82 H- 2 + a + bL^0.5, g11@H- 2 + a + bL^3, 2D<
```

§ 1.2.4 Using Patterns in Rules

[Patterns](#) play an important role in both global and local rules.

§ 1.2.4.1 Patterns in Global Rules

For example, it is very easy to give rewrite rules for differentiating polynomials of one variable.

```
diff@x_^n_., x_D := n x^Hn - 1L;
diff@a_ + b_, x_D := diff@a, xD + diff@b, xD;
```

Notice the default value for `n` in the first rule. Try it out on some typical functions.

```
8diff@y, yD, diff@z^2.1, zD, diff@u^2 + v^3, uD<

81, 2.1 z^1.1, 2 u + diff@v^3, uD<
```

But notice that `diff` does not know what to do with a constant times `x`, or just a constant for that matter, and we have no obvious way as yet to teach it what to do.

```
8diff@3, xD, diff@3 x, xD<
```

```
8diff@3, xD, diff@3 x, xD<
```

We could try the following: first, give a rule for products.

```
diff@a_ b_, xD := a diff@b, xD + b diff@a, xD
```

Using this for $a x$ gives

```
diff@a x, xD
```

```
a + x diff@a, xD
```

The program does not know that a is supposed to be a constant, so we have to tell it that explicitly, with a last rule.

```
diff@a, xD = 0;
```

Then it gives the "correct" answer.

```
diff@a x, xD
```

```
a
```

However, this is not very satisfactory. We would like some general way to say that a is not a function of x . Section 1.2.5 below will continue this discussion.

§ 1.2.4.2 Patterns in Local Rules

We will encounter this type of application quite often in the following chapters. In the following example, we show how to change the appearance of a matrix. Using **Array** to make a matrix with indexed entries, we then use a local rule to display the indices as subscripts.

```
Array@x, 83, 5<D •. x@i__D -> xi •• MatrixForm
```

```
{ {X1,1 X1,2 X1,3 X1,4 X1,5
  {X2,1 X2,2 X2,3 X2,4 X2,5
  {X3,1 X3,2 X3,3 X3,4 X3,5 {
```

where the two consecutive underscores next to i on the left-hand side of the rule represent one or more arguments.

We can also use pattern matching to replace the coefficients of various terms any way we want. For example,

```
Clear@fD
```

```
eq1 = u^2 @tD + u@tD f^e @x_0D +  $\frac{1}{2}$  u@tD^2 f^2 @x_0D +  $\frac{1}{6}$  u@tD^3 f^3L @x_0D == 0;
```

```
eq2 = eq1 •. _ . u@tD^b_ -> kb u@tD^b
```

```
k1 u@tD + k2 u@tD^2 + k3 u@tD^3 + u^2 @tD == 0
```

Sometimes we may obtain, in a bifurcation analysis, a data set, which includes a set of bifurcation points x_i and the corresponding control parameters $cpSearch$ and $cpFix$, such as `88x1, x2, x3, x4, cpSearch<, cpFix<`. If we want to switch these two parameters and then use the same data format as our initial input to a program, we have at least two choices. First, we take apart the data list and then assemble them:

```
data1 = 88x1, x2, x3, x4, cp1<, cp2<;

8Append@Drop@data1@@1DD, -1D, data1@@2DDD, data1@@1, -1DD<

88x1, x2, x3, x4, cp2<, cp1<
```

Second, we simply apply a local rule using patterns to the data list as a whole:

```
data1 •. 88a___, b_<, c_< -> 88a, c<, b<

88x1, x2, x3, x4, cp2<, cp1<
```

§ 1.2.5 Restricting Pattern Matching with Predicates

So far, all of the rules we have considered have been "context free" rewrite rules. Whenever the pattern is matched, the rewriting is carried out. There can be a restriction on the head of the matching expression included in the pattern. However, there are also **conditional rewrite rules**, which are only applied when some condition is satisfied. First, we have to discuss **predicates** (functions that return the value **True** or **False**) in *Mathematica* because all of the conditions will always be expressed in terms of them. All built-in predicates that are defined for all expressions end with **Q**. It is easy to display all of them.

?*Q

ArgumentCountQ	LinkConnectedQ	OrderedQ
AtomQ	LinkReadyQ	PartitionsQ
DigitQ	ListQ	PolynomialQ
EllipticNomeQ	LowerCaseQ	PrimeQ
EvenQ	MachineNumberQ	SameQ
ExactNumberQ	MatchLocalNameQ	StringMatchQ
FreeQ	MatchQ	StringQ
HypergeometricPFQ	MatrixQ	SyntaxQ
InexactNumberQ	MemberQ	TrueQ
IntegerQ	NameQ	UnsameQ
IntervalMemberQ	NumberQ	UpperCaseQ
InverseEllipticNomeQ	NumericQ	ValueQ
LegendreQ	OddQ	VectorQ
LetterQ	OptionQ	

Specifically requesting information on our function gives the usage message.

?OddQ

OddQ@exprD gives True if expr is an odd integer, and False otherwise.

Predicates are used to control pattern matching. In general, they are applied using **•**; , which is the infix form of **Condition**.

```
FullForm@Hold@x •; yDD
```

```
Hold@Condition@x, yDD
```

However, the position of the predicate in an expression allows it to be used in different ways. If the predicate is placed at the end of a global rule definition, it will restrict the application of the rule. For instance, we can use predicates to extend our definition of differentiation in Section 1.2.4.1 to deal with arbitrary polynomials in a very natural way by adding a single conditional rule.

```
Clear@diffD
```

```
diff@x_^n_, x_D := n x^Hn - 1L;
diff@a_ + b_, x_D := diff@a, xD + diff@b, xD;
diff@a_ b_, x_D := a diff@b, xD + b diff@a, xD;
diff@a_, x_D := 0 •; FreeQ@a, xD
```

Now constants and products are handled properly

```
diff@a@xD + b x^2 + c, xD
```

```
2 b x + diff@a@xD, xD
```

Using **rule•;Predicate** restricts the rule to those situations in which the predicate evaluates to **True**; that is, to those expressions belonging to the type given by the predicate. An unrestricted rule is the same as a conditional rule where the predicate always equals **True**.

If a predicate is immediately applied after a pattern, it will restrict the pattern matching rather than the rule application. For instance,

```
fac1@1D = 1; fac1@n_D := n fac1@n - 1D
```

This will work perfectly well if positive integers are given as arguments.

```
fac1@5D
```

```
120
```

However, if another argument is given, then it will fail badly.

```
8fac1@-3D, fac1@endlessD<;
```

```
$RecursionLimit::reclim : Recursion depth of 256 exceeded.
```

```
$RecursionLimit::reclim : Recursion depth of 256 exceeded.
```

```
$RecursionLimit::reclim : Recursion depth of 256 exceeded.
```

```
General::stop : Further output of $RecursionLimit::reclim will be suppressed during this calculation.
```

A very large output is suppressed. What happens, of course, in these cases is that the value 1 is never encountered as an argument, so the function keeps calling itself recursively until the built-in recursion limit is reached. This bad behavior can be corrected by using a conditional rule.

```
fac2@1D = 1; fac2@n_D := n fac2@n - 1D •; n > 1;

8fac2@5D, fac2@-2D, fac2@somethingD<

8120, fac2@-2D, fac2@somethingD<
```

One can also use the form `_?Predicate`, which restricts the pattern to something for which the predicate evaluates to **True**.

```
fac3@1D = 1; fac3@n_?PositiveD := n fac3@n - 1D;
fac4@1D = 1; fac4@n_ •; Positive@nDD := n fac4@n - 1D;

8fac3@5D, fac3@-2D, fac4@5D, fac4@-2D<

8120, fac3@-2D, 120, fac4@-2D<
```

In `n_?Positive`, **Positive** is a pure function, while in the form using `•`, the condition is the value of the predicate for the name of the pattern. In either case, **Positive** or **Positive**[n] is a positive test in the sense that the pattern is matched and the rule applied only if the test succeeds. Now try **fac4** on a real number and see what happens.

```
fac4@3.6D

8.9856 fac4@-0.4D
```

The rule is applied until the negative value -0.4 is reached, where the condition fails so **fac4**[-0.4] is returned in unevaluated form. Of course, the real problem is that we only intend the function to apply to integers. But this additional restriction can easily be added.

```
fac5@1D = 1; fac5@n_Integer?PositiveD := n fac5@n - 1D;

8fac5@5D, fac5@3.6D, fac5@-2D<

8120, fac5@3.6D, fac5@-2D<
```

The predicate that appears after `•`; or `?` can also be a user defined expression.

```
test@x_Integer?HH# > 2L &LD := x + 1

8test@1D, test@2D, test@3D, test@4D, test@5D<

8test@1D, test@2D, 4, 5, 6<
```

The form `? Predicate` can only be used after single slots, but the form `•; Predicate` can be used after any pattern, simple or compound. For instance,

```
f1@a_, b_D •; EvenQ@a + bD := a ^ b;
f1@a_, b_D •; OddQ@a + bD := a ^H- bL;
```

```
8f1@1, 3D, f1@2, 3D, f1@3, 3D<
```

```
91,  $\frac{1}{8}$ , 27=
```

Predicates also play an important role in manipulating lists.

```
Select@Range@-5, 5D, PositiveD
```

```
81, 2, 3, 4, 5<
```

```
Cases@Range@-5, 5D, _?PositiveD
```

```
81, 2, 3, 4, 5<
```

```
Cases@Range@-5, 5D, a_.; a > 0D
```

```
81, 2, 3, 4, 5<
```

There is also another form of **Cases** in which an operation is applied to the entries that are selected.

```
Cases@Range@-5, 5D, Ha_.; a > 0L :> Sqrt@aDD
```

```
91,  $\sqrt{2}$ ,  $\sqrt{3}$ , 2,  $\sqrt{5}$ =
```

The following example was the 1992 *Mathematica* programming competition question. The problem is to write a function called **maxima** that starts with a list of numbers and constructs the sublist of the numbers bigger than all previous ones from the given list. For instance, **maxima**[{3, 6, 4, 2, 8, 7, 9}] should return {3, 6, 8, 9}. The winning entry used a pattern with a condition in a local rule.

```
maxima@list_ListD := list /. 8a____, x_, y_, b____<.; y <= x -> 8a, x, b<
```

```
maxima@83, 6, 4, 2, 8, 7, 9<D
```

```
83, 6, 8, 9<
```

§ 1.2.6 Pure Functions

Given an expression *expr* involving a variable *x*, we can think of *expr* as describing a function with *x* being the argument. If we need to refer to this function by name, we can either define a rule for **f** in the form $f@x_D := expr$ or use the object **Function**[*x*, *expr*]. Either **f** or **Function**[*x*, *expr*] is a name for this function, and we can use the two interchangeably. To apply them to an argument, we write **f**[*arg*] in the familiar way or in the so-called pure function form **Function**[*x*, *expr*][*arg*].

```
f@x_D := 1 + x^2
```

```
g = Function@x, 1 + x^2D;
```

```
8f@2D, g@2D<
```

```
85, 5<
```

The name of the formal argument in a pure function does not matter. **Function**[x, x^2] is the same function as **Function**[y, y^2]. This fact is easy to see if you apply such a function to an argument, the result is the same in both cases.

```
8Function@x, x^2@aD, Function@y, y^2@aD, Sin@aD<
8a^2, a^2, Sin@aD<
```

Note that **Sin** and all built-in functions, with no arguments, are pure functions. Because the names of the variables in a pure function do not matter, *Mathematica* denotes these variables with symbols. The symbols #1, #2, ... are used for the first, second, ... variable. The internal form of #i is **Slot**[i]. If we use these forms, then we do not give the arguments of **Function**[] in order to declare the names of the variables. Thus, instead of **Function**[x, x^2] we simply write **Function**[#1^2]. A convenient abbreviation for #1 is #, so we can simplify our example a bit more and write **Function**[#^2]. Finally, there is a postfix operator **&** for **Function**. That is, **body&** is the same as **Function**[body]. Using this operator, we arrive at the shortest form #^2 **&** that our example can take. We have used this form quite frequently in this book. Let us see the internal form of this short form of a pure function.

```
FullForm@#^2 &D
Function@Power@Slot@1D, 2DD
```

The operator **&** has a very low priority, just above assignment. Therefore, $x = \text{body} \&$ is understood as $x = \text{Function}[\text{body}]$, but $x \rightarrow \text{body} \&$ is interpreted as **Function**[$x \rightarrow \text{body}$]. If we want the right-hand side of the rule to be the pure function, we use $x \rightarrow (\text{body} \&)$. Beware of $x \rightarrow (\text{body} \&)$. Another case where the low priority of pure functions requires the use of parentheses is in predicates for patterns: $x_? \text{body} \&$ is not the same as $x_?(\text{body} \&)$. The latter is usually correct.

```
1 + # + #^2 & @3D
13
```

The whole expression to the left of **&** is part of the body of the pure function. No parentheses are necessary. In other situations, parentheses are necessary.

```
h@aD . h -> H#^2 &L
a^2
```

Parentheses around the pure function are needed here; otherwise, the whole rule would be considered a part of the body of the function.

Pure functions written in the form with # and & are very concise and useful, especially, in functional operations, such as **Map**, **Apply**, **Thread**, and iteration functions, such as **FoldList**. For example,

```
eq1 = ¶t,t u@x, tD . a + u@x, tD == F@xD . a
u@x, tD +  $\frac{u^{H0,2L} @ x, tD}{a} == \frac{F@xD}{a}$ 
Clear@lhs2D
```

```
eq2 = lhs2 == rhs2;
eq3 = lhs3 == rhs3;
```

We can multiply both sides of **eq1** by a and use **Thread** and obtain

```
eq1 •• Thread@a #, EqualD & •• ExpandAll
a u@x, tD + uH0,2L@x, tD == F@xD
```

Moreover, we can add **eq2** to **eq3** by

```
eq2 + eq3 •• Thread@#, EqualD &
lhs2 + lhs3 == rhs2 + rhs3
```

§ 1.2.7 Map and Apply

In *Mathematica*, two important commands that take functions as arguments are **Map**[] and **Apply**[]. The various versions of **Map** act on the arguments of an expression, while **Apply** acts only on its head.

§ 1.2.7.1 Mapping Functions onto Expressions

The operation **Map**[f , $list$] or in infix form $f \bullet \text{list}$ applies the function f to each element of the list $list$. The second argument of **Map**[] need not be a list, however. Any expression of the form $h[e_1, e_2, \dots, e_n]$ will do. The result of the mapping is the expression $h[f@e_1, f@e_2, \dots, f@e_n]$.

```
Clear@fD
8Map@f, a + b + cD, #^2 & •• 81, 2, 3<<
8f@aD + f@bD + f@cD, 81, 4, 9<<
```

Map[f , $expr$, $levels$ spec] has an optional third argument that specifies the levels at which to map. The default level is {1}; that is, at the elements of $expr$. In a *matrix*, the entries are at level 2 (there are two levels of lists). If we want to map a function at these entries, we can use **Map**[f , $matrix$, {2}].

```
expr1 = 88a1,1, a1,2<, 8a2,1, a2,2<<;
Map@entry, expr1, 82<D •• MatrixForm
j entry@a1,1D entry@a1,2D
k entry@a2,1D entry@a2,2D {
```

This is a very powerful facility and is one of our main tools in manipulating expressions. The purpose of the operations based on **Map** is to make it possible to treat lists as wholes. For instance, a really poor way to square the entries in a list is as follows:


```
list1 = 8a, b, c<;
Table[list1[[i]]^2, {i, Length[list1]}<D
8a^2, b^2, c^2<
```

The term `list1[[i]]` tears apart the original list by extracting its parts one at a time, 2 squares each part, and then **Table** reassembles the parts into a new list. **Map**, however, allows us to treat mathematical structures as wholes so that we do not tear them apart and rebuild them again.

Map[] always maps the function at all elements of the given levels. The command **MapAt**[*f*, *expr*, *poslist*] allows you to apply a function at selective positions.

```
MapAt@G, a b + c d + e f, 82, 1<D
a b + e f + d G@cD
expr2 = a + b • a + c Exp@a + 1D
a +  $\frac{b}{a}$  + c E1+a
```

If we want to map a function *f* at all occurrences of *a*, we first find all the positions of *a* in our expression.

```
Position@expr2, aD
881<, 82, 1, 1<, 83, 2, 2, 2<<
```

This list of positions is in the right form for **MapAt** and can be used directly.

```
MapAt@f, expr2, %D
c E1+f@aD +  $\frac{b}{f@aD}$  + f@aD
```

This particular example could have been done more easily with a replacement rule.

```
expr2 •. a -> f@aD
c E1+f@aD +  $\frac{b}{f@aD}$  + f@aD
```

In some applications, the function to be performed on an element e_i may depend on its position *i*. The operation **MapIndexed**[*f*, { $e_1, e_2, \dots, e_n$ }] behaves essentially like **Map**[], but it passes the position of each element as a second argument to *f*. The resulting expression is

```
MapIndexed@1 + #^2 &, 8a, b, c, d<D
81 + a^2, 1 + b^2, 1 + c^2, 1 + d^2<
MapIndexed@1 + #^2 &, 88a, b<, 8c, d<<, 82<D
881 + a^2, 1 + b^2<, 81 + c^2, 1 + d^2<<
```


§ 1.2.8 Nest and Fold

Nest and its related operations **NestList** and **FixedPoint** apply a function to its argument many times. **Nest**[*function*, *x*, *n*] applies *function* to *x* and repeats the application *n* times

```
NestAH1 + #L2 &, x, 3E
```

```
J1 + H1 + H1 + xL2L2N2
```

```
NestListAH1 + #L2 &, x, 3E
```

```
9x, H1 + xL2, H1 + H1 + xL2L2, J1 + H1 + H1 + xL2L2N2 =
```

Here is an example producing a simple continued fraction.

```
Nest@1 • H1 + #L &, a, 3D
```

```

      1
 1 + -----
      1 + -----
            1 + -----
                  1 + -----
                        1 + -----

```

An operation that is closely related to **Nest** is **FixedPoint**, which nests its operation until there is no change. For instance, everyone is familiar with what happens if the **Cos** key on a pocket calculator is pushed repeatedly. In principle, **FixedPoint** is what happens if it is pushed forever.

```
8Nest@Cos, 0.3, 5D, Nest@Cos, 0.3, 10D, FixedPoint@Cos, 0.3D<
```

```
80.784436, 0.732698, 0.739085<
```

Actually, **FixedPoint** stops after machine accuracy is achieved. We can use the option **SameTest** to get some control of **FixedPoint**.

```
FixedPoint@Cos, 0.3, SameTest -> HAbs@#1 - #2D < 10-5 &LD
```

```
0.739089
```

The second pair of functions, **Fold** and **FoldList**, do something similar to **Nest** and **NestList**, but for functions of two variables.

```
Fold@f, u, 8x, y, z<D
```

```
f@f@f@u, xD, yD, zD
```

```
FoldList@f, u, 8x, y, z<D
```

```
8u, f@u, xD, f@f@u, xD, yD, f@f@f@u, xD, yD, zD<
```

Chapter 2

The Duffing Equation

```
Off@General::spell1D
```

To symbolize some notations, we need to load the package

```
Needs@"Utilities`Notation`"D
```

2.1 The Duffing Equation

The free oscillations of many conservative systems having a single degree of freedom are governed by an equation of the form

```
eq21a = x''@tD + f[x@tD] == 0;
```

```
H* An equation is set and then assigned to eq21a *L
```

where f is a nonlinear function of $x(t)$. Here, $x''(t)$ is the acceleration of the system, whereas $f(x(t))$ is the restoring force. Let $x = x_0$ be an equilibrium position of the system. Then x_0 is a constant and hence $f(x_0) = 0$. In order for *Mathematica* to interpret x_0 as a constant, we need to symbolize it. Otherwise, if we replace x with an expression $g(t)$, *Mathematica* will replace x_0 with $g(t)$ sub 0 and not treat it as a constant. For example, replacing x with $\sin(t)$ in the expression $x^2 + x_0$ yields

```
x^2 + x_0 /. x -> Sin@tD
```

```
Sin@tD^2 + HSin@tDL_0
```

Clearly, x_0 is treated as a variable x with subscript 0, which is wrong. Therefore, we need to symbolize x_0 as (enter and select x_0 followed by clicking **Symbolize** from NotationPalette)

```
Symbolize@x_0D;
```

```
H* Note that symbol form is also required in certain built-in functions *L
```

Then, *Mathematica* treats the symbolized expression x_0 as a constant. Consequently, replacing x with $\sin(t)$ in the expression $x^2 + x_0$ yields

```
x^2 + x_0 /. x -> Sin@tD
```

```
x_0 + Sin@tD^2
```

which is correct.

Next, we assume that f is an analytic function of x at $x = x_0$ and expand it in a Taylor series around x_0 as

```
fexp = Series[f[x@tDD, 8x@tD, x0, 3<D
```

$$f @ x_0 D + f' @ x_0 D H x @ t D - x_0 L + \frac{1}{2} f'' @ x_0 D H x @ t D^2 - x_0 L^2 + \frac{1}{6} f^{H3L} @ x_0 D H x @ t D^3 - x_0 L^3 + O[x @ t D - x_0 D^4$$

where terms up to cubic are retained. Using **Normal** to truncate the higher-order terms from **fexp**, we have

```
fpoly = fexp . Normal
```

$$f @ x_0 D + H - x_0 + x @ t D L f' @ x_0 D + \frac{1}{2} H - x_0 + x @ t D L^2 f'' @ x_0 D + \frac{1}{6} H - x_0 + x @ t D L^3 f^{H3L} @ x_0 D$$

Substituting **fpoly** for $f[x@tD]$ and using the equilibrium condition $f[x_0] = 0$, we obtain

```
temp = eq21a . 8f[x@tDD -> fpoly< . f[x0D -> 0
```

$$H - x_0 + x @ t D L f' @ x_0 D + \frac{1}{2} H - x_0 + x @ t D L^2 f'' @ x_0 D + x^2 @ t D + \frac{1}{6} H - x_0 + x @ t D L^3 f^{H3L} @ x_0 D == 0$$

Now, we introduce the transformation $x[t] \rightarrow x_0 + u[t]$ in **temp** to obtain the final form of the equation that will be used in the subsequent analysis. To accomplish this, we need to express this transformation in a pure function form as $x \rightarrow Hx_0 + u$ &L; otherwise, *Mathematica* will not transform the derivatives of $x[t]$ in terms of the derivatives of $u[t]$ unless we explicitly define the rules for those derivatives; that is, it will not replace $x'[t]$ with $u'[t]$ and $x''[t]$ with $u''[t]$ by simply substituting $x[t] \rightarrow x_0 + u[t]$. Moreover, it will not replace $x[p]$ with $x_0 + u[p]$ as desired. To see the difference between these two transformations, we compare the following two statements:

```
8x@tD, x@pD, x'@tD, x''@tD< . x@tD -> x0 + u@tD
```

```
8x0 + u@tD, x@pD, x'@tD, x''@tD<
```

```
8x@tD, x@pD, x@a, bD, y@tD, x'@tD, x''@tD< . x -> Hx0 + u@#D &L
```

```
8x0 + u@tD, x0 + u@pD, x0 + u@aD, y@tD, u'@tD, u''@tD<
```

We note that the pure function form can be used for any expression with the head x , especially, for the derivatives. Consequently, we introduce this transformation in pure function form into **temp** and obtain

```
eq21b = temp . x -> Hx0 + u@#D &L
```

$$u @ t D f' @ x_0 D + \frac{1}{2} u @ t D^2 f'' @ x_0 D + u^2 @ t D + \frac{1}{6} u @ t D^3 f^{H3L} @ x_0 D == 0$$

To simplify the notation in **eq21b**, we denote the coefficient of $u[t]^b$ by k_b . The result is

```
eq21c = eq21b . ._. u@tD^b_. -> kb u@tD^b
```

$$k_1 u @ t D + k_2 u @ t D^2 + k_3 u @ t D^3 + u^2 @ t D == 0$$

We relate the k_b to the derivatives of the function $f[x]$ by using the built-in function **Cases** as

CasesAeq21b, a_ . u@tD^b_ -> 8k_b -> a<, 82<E •• Flatten

$$9k_1 \otimes f^{\epsilon} @ x_0 D, \quad k_2 \otimes \frac{1}{2} f^2 @ x_0 D, \quad k_3 \otimes \frac{1}{6} f^{H3L} @ x_0 D =$$

Most of this chapter is devoted to the special case of cubic nonlinearities. Thus, we set the coefficient of the quadratic term in **eq21c** equal to zero and obtain

eq21d = eq21c . k₂ -> 0

$$k_1 u @ t D + k_3 u @ t D^3 + u^2 @ t D == 0$$

which is called the **Duffing equation**. We note that k_1 and k_3 may be positive or negative.

It is a good practice to nondimensionalize the governing equations before treating them with perturbation methods. To this end, we nondimensionalize u and t using a characteristic length U and a characteristic time T of the motion and denote the nondimensional variables by an asterisk. In nondimensional form, **eq21d** becomes

eq21e = eq21d . u -> HU u* @ # . TD & L . t -> T t*

$$U k_1 u^* @ t^* D + U^3 k_3 u^* @ t^* D^3 + \frac{U H u^* L^2 @ t^* D}{T^2} == 0$$

Next, we multiply the left-hand side of **eq21e** (using **eq21e[[1]]**) by $T^2 \cdot U$ so that the coefficient of $u^{*2} @ t^* D$ is unity. The result is

eq21f = l eq21e @ 1 D D T^2 . U •• ExpandM == 0

$$T^2 k_1 u^* @ t^* D + T^2 U^2 k_3 u^* @ t^* D^3 + H u^* L^2 @ t^* D == 0$$

We choose T so that the linear natural frequency of the system is unity; that is, we let $k_1 T^2 = 1$, for ease of notation, we let $e = k_3 T^2 U^2$, drop the asterisk, and rewrite **eq21f** as

DuffingEq = eq21f . 9k₁ T^2 -> 1, k₃ T^2 U^2 -> e = . anything_* -> anything

$$u @ t D + e u @ t D^3 + u^2 @ t D == 0$$

We note that e is a nondimensional quantity, which is a measure of the strength of the nonlinearity.

2.2 Straightforward Expansion

To solve the **DuffingEq**, we augment it with the initial conditions

$$ic = 8u @ 0 D == u0, \quad u^{\epsilon} @ 0 D == v0<;$$

The solution u of the initial-value problem (IVP), consisting of the **DuffingEq** and associated initial conditions **ic**, is a function of the independent variable t and the parameter e . Next, we determine an approximation of the IVP for weak nonlinearity; that is, for small but finite e .

First, we set ϵ equal to zero in the **DuffingEq**, augment the result with the initial conditions by using the *Mathematica* command **Join**, and obtain the following linear IVP:

```
linearIVP = Join@8DuffingEq /.  $\epsilon \rightarrow 0$  <, icD
```

```
8u@tD + u2@tD == 0, u@0D == u0, u'c@0D == v0<
```

The solution of this linear IVP can be obtained by using the *Mathematica* command **DSolve**. The result is

```
linearSol = DSolve@linearIVP, u@tD, tD@01DD
```

```
8u@tD ® u0 Cos@tD + v0 Sin@tD<
```

When ϵ is small but different from zero, the solution of the nonlinear initial-value problem is no longer given by **linearSol**, but deviates from it. We try a correction in the form of a powerseries in ϵ ; that is, we expand $u(t)$ in a power series in ϵ as

```
uExpRule@maxOrder_D := u -> ISumAei ui@#D, 8i, 0, maxOrder<E &M
```

where $u_0(t)$ is the solution of the linear problem. For example, to first order, we have

```
u@tD •. uExpRule@1D
```

```
u0@tD +  $\epsilon$  u1@tD
```

and to third order, we have

```
u@tD •. uExpRule@3D
```

```
u0@tD +  $\epsilon$  u1@tD +  $\epsilon^2$  u2@tD +  $\epsilon^3$  u3@tD
```

and to fifth order, we have

```
u@tD •. uExpRule@5D
```

```
u0@tD +  $\epsilon$  u1@tD +  $\epsilon^2$  u2@tD +  $\epsilon^3$  u3@tD +  $\epsilon^4$  u4@tD +  $\epsilon^5$  u5@tD
```

Next, we restrict our discussion to first order; that is, we keep two terms in the power series. Thus, we apply the rule **uExpRule[1]** to the **DuffingEq**, expand the result, and obtain

```
eq22atemp = DuffingEq •. uExpRule@1D •• ExpandAll
```

```
u0@tD +  $\epsilon$  u0@tD3 +  $\epsilon$  u1@tD + 3  $\epsilon^2$  u0@tD2 u1@tD + 3  $\epsilon^3$  u0@tD u1@tD2 +  $\epsilon^4$  u1@tD3 + u0'@tD +  $\epsilon$  u1'@tD == 0
```

Since we kept only terms up to $O(\epsilon)$ in the expansion of $u(t)$, we need to neglect terms of order higher than ϵ in **eq22atemp** for consistency. To accomplish this, we use the rule $\epsilon^{n*}; n > k \rightarrow 0$ to discard terms with order higher than ϵ^k and obtain

```
eq22a = eq22atemp •.  $\epsilon^{n*}; n > 1 \rightarrow 0$ 
```

```
u0@tD +  $\epsilon$  u0@tD3 +  $\epsilon$  u1@tD + u0'@tD +  $\epsilon$  u1'@tD == 0
```

Setting the coefficients of like powers of ϵ in **eq22a** equal to zero, we obtain the following hierarchy of linear equations:

```
eqEps = CoefficientList@eq22a@@1DD, eD == 0 •• Thread
```

$$8u_0(t) + u_0^2(t) = 0, \quad u_0(t)D^3 + u_1(t)D + u_1^2(t) = 0$$

which needs to be solved in succession. That is, we first solve the first equation in `eqEps` to obtain $u_0(t)$ and then substitute the result into the second equation in `eqEps` to obtain a linear nonhomogeneous equation, which can be solved for $u_1(t)$.

The general solution of the zeroth-order problem (`eqEps[[1]]`) can be expressed in Cartesian form as

```
DSolve@eqEps@@1DD, u_0@tD, tD@@1DD
```

$$8u_0(t) \propto C[2] \cos(t) - C[1] \sin(t)$$

Alternatively, we can express the solution of the zeroth-order problem in the polar form $u_0(t) \propto a_0 \cos(t + b_0)$ by adding two appropriate initial conditions to `eqEps[[1]]` as

```
sol0 =
```

```
DSolve@8eqEps@@1DD, u_0@tD == a_0 Cos@b_0D, u_0^'@tD == -a_0 Sin@b_0D<, u_0@tD, tD@@1DD •• Simplify
```

$$8u_0(t) \propto \cos(t + b_0) a_0$$

where a_0 and b_0 are arbitrary constants. Because `eqEps[[1]]` represents a linear undamped oscillator, one can write down its general solution in either Cartesian or polar form without using the *Mathematica* command `DSolve`.

Substituting for $u_0(t)$ in the first-order equation (`eqEps[[2]]`) and determining the general solution of the resulting equation for $u_1(t)$, we obtain

```
sol1temp = DSolve@eqEps@@2DD •. sol0, u_1@tD, tD@@1DD •• Simplify
```

$$9u_1(t) \propto$$

$$\frac{1}{32} H[3, 2] \cos(t) - C[1] \sin(t) + H[3, 2] \cos(3t) + b_0 D - 12 t \sin(t) + b_0 D a_0^3 =$$

We note that the homogeneous part of `sol1temp` consists of the terms involving the $C[i]$ (the usual case) and $\cos(t + b_0)$ (obtained from `DSolve` in this case). Hence, we can replace them by $a_1 \cos(t + b_1)$ and rewrite `sol1temp` as

```
sol1 = u_1@tD @ Hsol1temp@@1, 2DD •. 8C[_D -> 0, Cos@t + _D -> 0<L + a_1 Cos@t + b_1D
```

$$u_1(t) \propto \frac{1}{32} H \cos(3t) + b_0 D - 12 t \sin(t) + b_0 D a_0^3 + \cos(t) + b_1 D a_1$$

Combining the solutions of the zeroth- and first-order problems, we obtain the general solution of the *DuffingEq*, to first order, as

```
sol = u@tD == Hu@tD •. uExpRule@1D •. sol0 •. sol1L
```

$$u(t) = \cos(t + b_0) a_0 + e^{\int \frac{1}{32} H \cos(3t) + b_0 D - 12 t \sin(t) + b_0 D a_0^3 + \cos(t) + b_1 D a_1 dt}$$

where a_0, a_1, b_0 , and b_1 are arbitrary constants.

We started with a second-order equation that can support two initial conditions but it appears that we ended up with four arbitrary constants. It turns out that only two of the constants a_0 , a_1 , b_0 , and b_1 are arbitrary and that the two initial conditions in **ic** are sufficient to determine all of them. To see this, we first change the **Equal** form in **ic** to the **Rule** form as

```
icRule = ic . Equal -> Rule
```

```
8u@0D ® u0, u'@0D ® v0<
```

Then, we impose the initial conditions **icRule** by evaluating **sol** and its derivative at $t = 0$ according to

```
eq22b = 8sol, D@sol, tD< . t -> 0 . icRule
```

$$\begin{aligned} 9u0 == \cos(b_0 D a_0 + e \int_0^1 \frac{1}{32} \cos(3 b_0 D a_0^3 + \cos(b_1 D a_1) \frac{1}{2}, \\ v0 == -\sin(b_0 D a_0 + e \int_0^1 \frac{1}{32} H - 12 \sin(b_0 D - 3 \sin(3 b_0 D L a_0^3 - \sin(b_1 D a_1) \frac{1}{2} = \end{aligned}$$

We equate the coefficient of each power of e in **eq22b** and obtain four algebraic equations for the constants a_0 , a_1 , b_0 , and b_1 . The result is

```
eq22c = Table@Map@Coefficient@#, e, iD &, eq22b, 82<D, 8i, 0, 1<D
```

```
98u0 == Cos@b0D a0, v0 == -Sin@b0D a0<, \\
```

$$90 == \frac{1}{32} \cos(3 b_0 D a_0^3 + \cos(b_1 D a_1), 0 == \frac{1}{32} H - 12 \sin(b_0 D - 3 \sin(3 b_0 D L a_0^3 - \sin(b_1 D a_1) ==$$

Thus, once a_0 and b_0 are known from **eq22c**[[1]], a_1 and b_1 can be calculated from **eq22c**[[2]].

Alternatively, instead of including the homogeneous solution at each order and consider the arbitrary constants to be independent of e as done above, one may disregard the homogeneous solution at all orders except the first and consider the arbitrary constants to depend on e in imposing the initial conditions. With the latter approach, the solution to first order can be rewritten as

```
sol . 8a0 -> a, b0 -> b, a1 -> 0< . ExpandAll
```

$$u@tD == a \cos@t + bD + \frac{1}{32} a^3 e \cos@3 t + 3 bD - \frac{3}{8} a^3 t e \sin@t + bD$$

We note that the correction term $\frac{1}{32} a^3 e \cos@3 t + 3 bD - \frac{3}{8} a^3 t e \sin@t + bD$ is small compared to the principal term $a \cos@t + bD$, as it is supposed to be, only when $e t$ is small compared with unity. When $e t$ is $O(1)$, the term that is supposed to be a small correction becomes the same order as the principal term. Moreover, when $e t > O(1)$, the "small-correction" term becomes larger than the principal term. Hence, the above obtained straightforward expansion is valid only for times such that $e t < O(1)$; that is, $t < O(e^{-1})$. Consequently, we say such expansions are nonuniform or breakdown for long times and we call them **pedestrian** or **naïve expansions**. The reason for the breakdown of the above expansion is the presence of the term $t \sin@t + bD$, a product of algebraic and circular terms. Such terms are called mixed-secular terms. In subsequent sections, we implement four methods that avoid secular terms, and hence yield uniform expansions.


```
order1Eq = eqEps@@2DD •. sol0 •• Simplify
```

$$a^3 \cos(b + tD^3 - 2a \cos(b + tD) w_1 + u_1(tD) + u_1^2(tD) == 0$$

whose particular solution is

```
u1pSol = DSolve@order1Eq, u1@tD, tD@@1DD •. C@_D -> 0 •• Simplify
```

$$9u_1(tD) \approx$$

$$\frac{1}{32} a^2 H - 6 \cos(b + tD) + \cos(3Hb + tLD) - 12 t \sin(b + tDL) + 16 H \cos(b + tD) + 2 t \sin(b + tDL) w_1 L =$$

It is clear that **u1pSol** contains a mixed-secular term, which makes the expansion nonuniform. In contrast with the straightforward expansion, where the secular term cannot be annihilated unless u is trivial, in this case, we can choose the parameter w_1 to eliminate the secular term according to

```
omg1Rule = Solve@Coefficient@u1pSol@@1, 2DD, Sin@b + tDD == 0, w1D@@1DD
```

$$:w_1 \approx \frac{3a^2}{8} >$$

Substituting for w_1 into **u1pSol**, we have

```
sol1 = u1pSol •. omg1Rule •• Simplify
```

$$9u_1(tD) \approx \frac{1}{32} a^3 \cos(3Hb + tLD) =$$

We note that, to determine the condition for the elimination of the secular term from u_1 , we do not need to determine the particular solution first as done above. Instead, we only need to inspect **order1Eq** and choose w_1 so that the coefficient of $\cos(b + tD)$, which produces secular terms in u_1 , is equal to zero. To this end, we calculate the nonhomogeneous part of **order1Eq** and simplify the result using trigonometric identities as

```
expr23a = order1Eq@@1DD •. u1 -> H0 &L •• TrigReduce
```

$$\frac{1}{4} H^3 a^3 \cos(b + tD) + a^3 \cos(3b + 3tD) - 8a \cos(b + tD) w_1 L$$

Next, we choose w_1 to annihilate the term in **expr23a** that produces secular terms, namely $\cos(b + tD)$, and obtain

```
Solve@Coefficient@expr23a, Cos@b + tDD == 0, w1D@@1DD
```

$$:w_1 \approx \frac{3a^2}{8} >$$

Substituting **sol0** and **sol1** into **uExpRule[1]** yields

```
u@tD = Hu@tD •. uExpRule@1D •. sol0 •. sol1 •• SimplifyL
```

$$a \cos(b + tD) + \frac{1}{32} a^3 e \cos(3Hb + tLD)$$

Replacing t with wt , substituting for w , using the fact that $w_0 = 1$, and expanding the arguments of the trigonometric functions, we obtain

$$\begin{aligned} \text{solLP} = \\ u@tD == Hu@tD \cdot t \rightarrow wt \cdot \text{omgRule@1D} \cdot \text{omg1Rule} \cdot w_0 \rightarrow 1 \cdot f_{@arg_D} := f@arg \cdot \text{ExpandDL} \\ u@tD == a \cos At + b + \frac{3}{8} a^2 t e E + \frac{1}{32} a^3 e \cos A^3 t + 3 b + \frac{9}{8} a^2 t e E \end{aligned}$$

Clearly, the expansion **solLP** is free of secular terms and the correction term $\frac{1}{32} a^3 e \cos A^3 t + 3 b + \frac{9}{8} a^2 t e E$ is small compared with the principal term $a \cos At + b + \frac{3}{8} a^2 t e E$ for all t and hence it is uniformly valid.

2.4 The Method of Multiple Scales

We note from **solLP** that the functional dependence of u on t and e is not disjoint. In fact, u depends on the combination et as well as on the individual t and e . Carrying out the expansion to higher order, we find that u depends on the combinations $et, e^2 t, e^3 t, \dots$ as well as on the individual t and e . Hence, $u = u(et, e^2 t, e^3 t, \dots; e)$. For small e , $T_n = e^n t$, for $n = 1, 2, 3, \dots$, represent different time scales. For example, if $e = \frac{1}{60}$, variations on the time scales T_0 , T_1 , and T_2 can be observed, respectively, on the second, minute, and hour arms of a watch. Thus, T_0 represents a fast time scale, T_1 represents a slower time scale, T_2 represents an even slower time scale, and so on. Since the dependence of u on t and e occurs on different time scales, we imagine that we have a watch and attempt to observe the behavior of u using the different scales of the watch. Therefore, instead of determining u as a function of t , we determine u as a function of $T_0, T_1, T_2, \dots$. To this end, we change the independent variable t in the governing equation from t to $T_0, T_1, T_2, \dots$, which are symbolized by

$$\begin{aligned} \text{Symbolize@T}_0\text{D}; \text{Symbolize@T}_1\text{D}; \text{Symbolize@T}_2\text{D}; \\ \text{timeScales} = \{T_0, T_1, T_2\}; \end{aligned}$$

We define the derivative operators as

$$\begin{aligned} \text{dt@1D@expr_D} &:= \text{SumAe}^i \text{D@expr, timeScales@@i+1DDD, 8i, 0, maxOrder<E}; \\ \text{dt@2D@expr_D} &:= \text{Hdt@1D@dt@1D@exprDD} \cdot \text{ExpandL} \cdot e^{i \cdot i > \text{maxOrder}} \rightarrow 0; \end{aligned}$$

and treat the time scales T_0 , T_1 , and T_2 as independent variables.

Next, we use the method of multiple scales to determine an approximate solution for the **DuffingEq**. In Section 2.4.1, we treat the **DuffingEq** in its second-order form; in Section 2.4.2, we transform it first into a system of two real-valued equations; and in Section 2.4.3, we transform it first into a single complex-valued equation.

2.4.1 Second-Order Real-Valued System

$$\text{DuffingEq} = u''@tD + u@tD + e u@tD^3 == 0;$$

To determine a first-order uniform expansion of the solution of the Duffing equation, we use the two time scales T_0 and T_1 and hence let

$$\text{maxOrder} = 1;$$

Expressing the time derivative in **DuffingEq** in terms of the two time scales T_0 and T_1 , we have

$$\begin{aligned} \text{eq241a} &= \text{DuffingEq} \cdot \cdot \text{8u@tD} \rightarrow \text{u@T}_0, \text{T}_1\text{D}, \text{Derivative@n_D@u@tD} \rightarrow \text{dt@nD@u@T}_0, \text{T}_1\text{DD} < \\ \text{u@T}_0, \text{T}_1\text{D} + \epsilon \text{u@T}_0, \text{T}_1\text{D}^3 + 2 \epsilon \text{u}^{\text{H1}, \text{1L}} @ \text{T}_0, \text{T}_1\text{D} + \text{u}^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} &= 0 \end{aligned}$$

Consequently, we have replaced the original ordinary-differential equation by a partial-differential equation, and it appears that, as a result, we have complicated the problem. This is partially true, but experience with this method has shown that the disadvantages of introducing this complication are far outweighed by the advantages. Not only does this method provide a uniform expansion, but it also provides all the various nonlinear resonance phenomena, as we shall see in subsequent chapters.

We seek a uniform second-order expansion of the solution of **eq241a** in the form

$$\text{solRule} = \text{u} \rightarrow \text{ISumAe}^i \text{u}_i @ \#1, \#2\text{D}, \text{8i}, \text{0}, \text{maxOrder} < \text{E} \& \text{M};$$

Substituting this expansion into **eq241a**, expanding the result, and discarding terms of order higher than ϵ , we obtain

$$\begin{aligned} \text{eq241b} &= \text{Heq241a} \cdot \cdot \text{solRule} \cdot \cdot \text{ExpandAllL} \cdot \cdot \text{e}^{\text{n_}} ; \text{n} > \text{maxOrder} \rightarrow 0 \\ \text{u}_0 @ \text{T}_0, \text{T}_1\text{D} + \epsilon \text{u}_0 @ \text{T}_0, \text{T}_1\text{D}^3 + \epsilon \text{u}_1 @ \text{T}_0, \text{T}_1\text{D} + 2 \epsilon \text{u}_0^{\text{H1}, \text{1L}} @ \text{T}_0, \text{T}_1\text{D} + \text{u}_0^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} + \epsilon \text{u}_1^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} &= 0 \end{aligned}$$

Equating coefficients of like powers of ϵ in **eq241b** yields

$$\begin{aligned} \text{eqEps} &= \text{CoefficientList@eq241b@@1DD}, \text{eD} == 0 \cdot \cdot \text{Thread} \\ 9 \text{u}_0 @ \text{T}_0, \text{T}_1\text{D} + \text{u}_0^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} &= 0, \text{u}_0 @ \text{T}_0, \text{T}_1\text{D}^3 + \text{u}_1 @ \text{T}_0, \text{T}_1\text{D} + 2 \text{u}_0^{\text{H1}, \text{1L}} @ \text{T}_0, \text{T}_1\text{D} + \text{u}_1^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} &= 0 = \end{aligned}$$

The general solution of **eqEps**[[1]] can be written in a pure function form as

$$\text{sol0} = \text{u}_0 \rightarrow \text{Ha} @ \#2\text{D} \text{Cos} @ \#1 + \text{b} @ \#2\text{DD} \& \text{L};$$

where $\#1$ stands for the scale T_0 and $\#2$ stands for the scale T_1 . The functional dependence of a and b on T_1 is not known at this level of approximation; it is determined at subsequent levels of approximation by eliminating the secular terms.

Substituting **sol0** into **eqEps**[[2]] and moving the nonhomogeneous parts to the right-hand side of the resulting equation yields

$$\begin{aligned} \text{eq241c} &= \text{HeqEps} @ \#1, \text{1DD} \cdot \cdot \text{u}_0 \rightarrow \text{u}_1\text{L} - \# \& \cdot \checkmark \text{Reverse@eqEps} @ \#2\text{DD} \cdot \cdot \text{sol0} \cdot \cdot \text{SimplifyD} \\ \text{u}_1 @ \text{T}_0, \text{T}_1\text{D} + \text{u}_1^{\text{H2}, \text{0L}} @ \text{T}_0, \text{T}_1\text{D} &= \\ - \text{a} @ \text{T}_1\text{D}^3 \text{Cos} @ \text{T}_0 + \text{b} @ \text{T}_1\text{DD}^3 + 2 \text{Sin} @ \text{T}_0 + \text{b} @ \text{T}_1\text{DD} \text{a}^{\epsilon} @ \text{T}_1\text{D} + 2 \text{a} @ \text{T}_1\text{D} \text{Cos} @ \text{T}_0 + \text{b} @ \text{T}_1\text{DD} \text{b}^{\epsilon} @ \text{T}_1\text{D} & \end{aligned}$$

Next, we expand the right-hand side of **eq241c** in a Fourier series using trigonometric identities and obtain

eq241c@@2DD = eq241c@@2DD •• TrigFactor •• Expand

$$-\frac{3}{4} a T_1 D^3 \cos T_0 + b T_1 D D - \frac{1}{4} a T_1 D^3 \cos 3 T_0 + 3 b T_1 D D + 2 \sin T_0 + b T_1 D D a' T_1 D + 2 a T_1 D \cos T_0 + b T_1 D D b' T_1 D$$

To produce a uniform expansion, we eliminate the terms that produce secular terms from **eq241c**[[2]]; that is, we choose $a_1' T_1 D$ and $b_1' T_1 D$ to annihilate each of the coefficients of $\sin T_0 + b T_1 D D$ and $\cos T_0 + b T_1 D D$. The result is

eq241d = Coefficient@eq241c@@2DD, #D == 0 & •ž 8sin@T₀ + b@T₁DD, Cos@T₀ + b@T₁DD<

$$92 a' T_1 D == 0, -\frac{3}{4} a T_1 D^3 + 2 a T_1 D b' T_1 D == 0 =$$

Solving **eq241d** for $a_1' T_1 D$ and $b_1' T_1 D$ when $a T_1 D \neq 0$ yields

SCond = Solve@eq241d, 8a'@T₁D, b'@T₁D<D@@1DD

$$9 a' T_1 D \approx 0, b' T_1 D \approx \frac{3}{8} a T_1 D^2 =$$

The solution of **SCond**[[1]] is $a = a_0 = \text{constant}$. Then, the solution of **SCond**[[2]] is

betaRule = DSolve@eq241d@@2DD •. a@T₁D -> a₀, b@T₁D, T₁D@@1DD •. C@1D -> b₀

$$9 b T_1 D \approx \frac{3}{8} T_1 a_0^2 + b_0 =$$

With the above solvability conditions, **eq241c** becomes

eq241e = eq241c •. SCond

$$u_1 @ T_0, T_1 D + u_1^{H2, 0L} @ T_0, T_1 D == -\frac{1}{4} a T_1 D^3 \cos 3 T_0 + 3 b T_1 D D$$

The particular solution of **eq241e** can be obtained as

sol1 = DSolve@eq241e, u₁@T₀, T₁D, 8T₀, T₁<D@@1DD •. C@_D -> H0 &L •• Simplify

$$9 u_1 @ T_0, T_1 D \approx \frac{1}{32} a T_1 D^3 \cos 3 T_0 + b T_1 D D =$$

eq241f = u@T₀, T₁D == Hu@T₀, T₁D •. solRule •. sol0 •. sol1 •. a@T₁D -> a₀ •. betaRuleL

$$u @ T_0, T_1 D == \cos A T_0 + \frac{3}{8} T_1 a_0^2 + b_0 E a_0 + \frac{1}{32} e \cos A 3 \int_k T_0 + \frac{3}{8} T_1 a_0^2 + b_0 \frac{1}{2} E a_0^3$$

where a_0 and b_0 are constants to within the order of the error indicated. In terms of the original variable t , **eq241f** can be expressed as

u@tD == eq241f@@2DD •. 8T₀ -> t, T₁ -> e t<

$$u @ t D == \cos A t + \frac{3}{8} t e a_0^2 + b_0 E a_0 + \frac{1}{32} e \cos A 3 \int_k t + \frac{3}{8} t e a_0^2 + b_0 \frac{1}{2} E a_0^3$$

in agreement with the expansion obtained by using the Lindstedt-Poincaré technique.

In the higher-order approximations, we include the scales $T_0, T_1, \dots, T_N$ but we do not include the term $O(\epsilon^N)$ in an N th-order expansion.

Before closing this section, we present an alternate representation of the solutions of the perturbation equations. Instead of the real-valued form **sol0**, we represent the solution of **eqEps**[[1]] in a complex-valued form; that is,

$$\text{sol0} = u_0 \rightarrow i A \exp(i T_1 D) + \bar{A} \exp(-i T_1 D) + M;$$

where $\bar{A}$ is the complex conjugate of A and

$$\text{ruleA} = 9A \rightarrow \sum_{k=2}^{\infty} \frac{1}{k!} a^{(k)} \exp(i b^{(k)} D) + \frac{1}{k!} \bar{a}^{(k)} \exp(-i b^{(k)} D);$$

Substituting **sol0** into **eqEps**[[2]] and moving the nonhomogeneous terms to the right-hand side of the resulting equation, we have

$$\text{eq241g} = u_1 \exp(i T_0 D) + u_1^{(2,0L)} \exp(i T_0 D) - \epsilon \text{HeqEps} \cdot \text{sol0L} \cdot \text{ExpandAll} \cdot \text{Reverse}$$

$$u_1 \exp(i T_0 D) + u_1^{(2,0L)} \exp(i T_0 D) == -\epsilon^3 \exp(i T_0 D) A \exp(i T_1 D)^3 - 3 \epsilon \exp(i T_0 D) A \exp(i T_1 D)^2 \dot{A} \exp(i T_1 D) - 3 \epsilon \exp(i T_0 D) A \exp(i T_1 D) \dot{A} \exp(i T_1 D)^2 - \epsilon^3 \exp(i T_0 D) \dot{A} \exp(i T_1 D)^3 - 2 \epsilon \exp(i T_0 D) A \exp(i T_1 D) + 2 \epsilon \exp(i T_0 D) \dot{A} \exp(i T_1 D)$$

We note that the terms proportional to $\exp(i T_0 D)$ and $\exp(-i T_0 D)$ produce secular terms in the particular solution of u_1 . Thus, to produce a uniform expansion, we set the coefficient of each of these functions equal to zero; that is,

$$\text{eq241h} = \text{Coefficient}[\text{eq241g}, \exp(i T_0 D), \exp(-i T_0 D)] == 0 \& \cdot \epsilon \cdot \text{HeqEps} \cdot \text{ExpandAll} \cdot \text{Reverse}$$

$$: -3 A \exp(i T_1 D)^2 \dot{A} \exp(i T_1 D) - 2 \epsilon A \exp(i T_1 D) == 0, -3 A \exp(i T_1 D) \dot{A} \exp(i T_1 D)^2 + 2 \epsilon \dot{A} \exp(i T_1 D) == 0$$

These two equations are not independent because they are complex conjugates of each other. Hence, if one of them is satisfied, the other is automatically satisfied.

To analyze the solutions of **eq241h**[[1]], we multiply it by $\exp(-i T_0 D)$ and replace A with its polar form **ruleA**. The result is

$$\text{eq241i} = \text{Expand}[\text{eq241h} \cdot \exp(-i T_0 D), \exp(i T_1 D)] \cdot \text{ruleA} == 0$$

$$- \frac{3}{8} a \exp(i T_1 D)^3 - \epsilon a \exp(i T_1 D) + a \exp(i T_1 D) b \exp(i T_1 D) == 0$$

Next, we separate the real and imaginary parts of **eq241i**. To accomplish this, we define the following rule:

$$\text{realRule} = \text{Re}[s_D] \rightarrow s, \text{Im}[s_D] \rightarrow 0;$$

Using the **realRule**, we find that the imaginary part of **eq241i** is

$$\text{ampEq} = \text{Im}[\text{eq241i}] \cdot \text{realRule} == 0 \cdot \text{realRule}$$

$$- a \exp(i T_1 D) == 0$$

which governs the modulation of the amplitude. Similarly, the real part of **eq241i** is

$$\text{phaseEq} = \text{Re}[\text{eq241i}] == 0 \cdot \text{realRule}$$

$$-\frac{3}{8} a T_1 D^3 + a T_1 D b' T_1 D == 0$$

which governs the modulation of the phase. These modulation equations are in agreement with those obtained above by expressing the solution in real form.

Comparing the complex-valued and the real-valued representations, we find it more convenient to use the complex-valued form. Therefore, the complex-valued form is used in the remainder of this book.

2.4.2 First-Order Real-Valued System

We let $v(t) = u^\epsilon(t)$ and transform the **DuffingEq** into the following set of two first-order equations:

$$\text{eq242a} = 9u^\epsilon T D == v T D, \quad v^\epsilon T D + u T D == -e u T D^3;$$

We seek a first-order uniform expansion of the solution of **eq242a** in terms of the two time scales T_0 and T_1 in the form

$$\text{solRule} = 9u \rightarrow \text{ISumAe}^j u_j \#1, \#2D, 8j, 0, 1 < E \&M, v \rightarrow \text{ISumAe}^j v_j \#1, \#2D, 8j, 0, 1 < E \&M;$$

Substituting **solRule** into **eq242a**, transforming the derivative with respect to t in terms of the derivatives with respect to T_0 and T_1 , and discarding terms of order higher than e , we obtain

$$\text{eq242b} = \text{Heq242a} \cdot 8u^\epsilon T D \rightarrow dt @1D @u @T_0, T_1 D D, u T D \rightarrow u @T_0, T_1 D < \cdot \text{solRule} \cdot \cdot \text{ExpandAll} \cdot \cdot e^{n \cdot}; n > 1 \rightarrow 0$$

$$9e u_0^{H0, 1L} @T_0, T_1 D + u_0^{H1, 0L} @T_0, T_1 D + e u_1^{H1, 0L} @T_0, T_1 D == v_0 @T_0, T_1 D + e v_1 @T_0, T_1 D,$$

$$u_0 @T_0, T_1 D + e u_1 @T_0, T_1 D + e v_0^{H0, 1L} @T_0, T_1 D + v_0^{H1, 0L} @T_0, T_1 D + e v_1^{H1, 0L} @T_0, T_1 D == -e u_0 @T_0, T_1 D^3 =$$

Equating coefficients of like powers of e , we have

$$\text{eqEps} = \text{Thread}[\text{CoefficientList}[\text{Subtract} \# \& \#2, e D == 0 D \& \cdot \# \text{eq242b} \cdot \cdot \text{Transpose}$$

$$99 - v_0 @T_0, T_1 D + u_0^{H1, 0L} @T_0, T_1 D == 0, u_0 @T_0, T_1 D + v_0^{H1, 0L} @T_0, T_1 D == 0,$$

$$9 - v_1 @T_0, T_1 D + u_0^{H0, 1L} @T_0, T_1 D + u_1^{H1, 0L} @T_0, T_1 D == 0,$$

$$u_0 @T_0, T_1 D^3 + u_1 @T_0, T_1 D + v_0^{H0, 1L} @T_0, T_1 D + v_1^{H1, 0L} @T_0, T_1 D == 0 ==$$

Zeroth-Order Problem: Linear System

The zeroth-order problem is given by the linear system

$$\text{linearSys} = \# @ @1DD \& \cdot \# \text{eqEps} @ @1DD$$

$$9 - v_0 @T_0, T_1 D + u_0^{H1, 0L} @T_0, T_1 D, u_0 @T_0, T_1 D + v_0^{H1, 0L} @T_0, T_1 D =$$

To determine the solution of this linear system, we seek a solution proportional to $\exp(i T_0)$ as

```
coefList = E-i T0 linearSys . 9u0 -> IP Ei # &M, v0 -> IQ Ei # &M= •• Expand
8I P - Q, P + I Q<
```

Next, we determine the coefficient matrix as

```
coefMat = Outer@Coefficient, coefList, 8P, Q<D
88I, -1<, 81, I<<
```

To determine the eigenvalues and eigenvectors of **coefMat**, we define the following conjugate rule and Hermitian matrix:

```
conjugateRule = 9A ->  $\dot{A}$ ,  $\dot{A}$  -> A, Complex@0, n_D -> Complex@0, -nD=;
hermitian@mat_? MatrixQD := mat . conjugateRule •• Transpose
```

Then the left and right eigenvectors of **coefMat** are given by

```
leftVec = 81, c1< . Solve@Hhermitian@coefMatD.81, c1<L@@1DD == 0, c1D@@1DD
81, I<
rightVec = 81, c1< . Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD
81, I<
```

To express the solution of the zeroth-order problem, we introduce the following basic function:

```
basicH = A@T1D Ei T0;
```

In terms of this function and the right eigenvector, the solution of the zeroth-order problem can be expressed in terms of

```
sol0Form = rightVec basicH
8Ei T0 A@T1D, I Ei T0 A@T1D<
```

and its complex conjugate as

```
sol0 =
8u0 -> Function@8T0, T1<, sol0Form@@1DD + Hsol0Form@@1DD . conjugateRuleL •• Evaluated,
v0 -> Function@8T0, T1<, sol0Form@@2DD + Hsol0Form@@2DD . conjugateRuleL •• Evaluated<
8u0 ® Function@8T0, T1<, Ei T0 A@T1D + E-i T0  $\dot{A}$ @T1DD,
v0 ® Function@8T0, T1<, I Ei T0 A@T1D - I E-i T0  $\dot{A}$ @T1DD<
```

We have expressed the solution of the zeroth-order problem in function form so that its partial derivatives can be readily evaluated.

§ First-Order Problem

Substituting the zeroth-order solution `sol0` into the first-order problem `eqEps[[2]]` and moving the nonhomogeneous terms to the right-hand sides of the resulting equations, we obtain

```
order1Eq = HlinearSys . u_0 -> u_1L ==
  HlinearSys . u_0 -> u_1L - HSubtract žž # & •ž eqEps@@2DD . sol0 •• ExpandL •• Thread

: - v_1@T_0, T_1D + u_1^{H1, OL}@T_0, T_1D == -E^{I T_0} A^@T_1D - E^{-I T_0} \dot{A}^@T_1D,
u_1@T_0, T_1D + v_1^{H1, OL}@T_0, T_1D == -E^{3 I T_0} A@T_1D^3 - 3 E^{I T_0} A@T_1D^2 \dot{A}^@T_1D -
3 E^{-I T_0} A@T_1D \dot{A}^@T_1D^2 - E^{-3 I T_0} \dot{A}^@T_1D^3 - I E^{I T_0} A^@T_1D + I E^{-I T_0} \dot{A}^@T_1D>
```

Eliminating the terms that produce secular terms in `order1Eq` demands that their right-hand sides are orthogonal to every solution of the adjoint homogeneous problem. To determine these conditions, we first calculate the coefficient vector of $\text{Exp}[I T_0]$ as

```
STerms = Coefficient@#@2DD, Exp@I T_0DD & •ž order1Eq
8 - A^@T_1D, - 3 A@T_1D^2 \dot{A}^@T_1D - I A^@T_1D<
```

Then, we demand that this vector is orthogonal to the left eigenvector and obtain the solvability condition

```
SCond = Expand@Conjugate@leftVecD.STermsD == 0
3 I A@T_1D^2 \dot{A}^@T_1D - 2 A^@T_1D == 0
```

which is in agreement with that obtained by treating the second-order form of the `DuffingEq`. Repeating the procedure for the coefficient vector of $\text{Exp}[-I T_0]$, we obtain the complex conjugate of `SCond`.

§ 2.4.3 First-Order Complex-Valued System

In this section, we determine a first-order uniform expansion of the solution of the `DuffingEq` by transforming it first into a single first-order complex-valued equation. To this end, we introduce the transformation

```
transfRule = 9u@tD -> z@tD + \dot{z}^@tD, u^@tD -> I I z@tD - \dot{z}^@tD==;
```

whose inverse is

```
zetaRule = SolveAtransfRule . Rule -> Equal, 9z@tD, \dot{z}^@tD==E@@1DD
9z@tD ® \frac{1}{2} Hu@tD - I u^@tDL, \dot{z}^@tD ® \frac{1}{2} Hu@tD + I u^@tDL=
```

It follows from the `DuffingEq` that the acceleration is related to the displacement according to

acceleration = Solve@DuffingEq, u²@tDD@@1DD

$$8u^2@tD \otimes -u@tD - e u@tD^3 <$$

Differentiating $z@t$, **zetaRule**[[1]], with respect to t and substituting for $u@t$, $u^6@t$, and $u^2@t$ using the **acceleration** and **transfRule**, we obtain

eq243a = D@zetaRule@@1DD, tD . acceleration . transfRule . Rule -> Equal . ExpandAll

$$z^6@tD == I z@tD + \frac{1}{2} I e z@tD^3 + \frac{3}{2} I e z@tD^2 \dot{z}@tD + \frac{3}{2} I e z@tD \dot{z}@tD^2 + \frac{1}{2} I e \dot{z}@tD^3$$

Next, we use the method of multiple scales to determine a first-order uniform expansion of the solution of **eq243a** in the form

$$\text{solRule} = 9z \rightarrow I \text{Sum} A e^j z_j@ \#1, \#2D, 8j, 0, 1 < E \&M, \dot{z} \rightarrow I \text{Sum} A e^j \dot{z}_j@ \#1, \#2D, 8j, 0, 1 < E \&M;$$

Substituting this expansion into **eq243a**, expanding the result, and discarding terms of order higher than e , we have

eq243b =

$$I \text{eq243a} . 9z@tD \rightarrow z@T_0, T_1D, \dot{z}@tD \rightarrow \dot{z}@T_0, T_1D, z^6@tD \rightarrow dt@1D@z@T_0, T_1DD == . \text{solRule} . \\ \text{ExpandAllM} . e^{n_1; n_2} \rightarrow 0$$

$$e z_0^{H0, 1L}@T_0, T_1D + z_0^{H1, 0L}@T_0, T_1D + e z_1^{H1, 0L}@T_0, T_1D == I z_0@T_0, T_1D + \frac{1}{2} I e z_0@T_0, T_1D^3 + \\ I e z_1@T_0, T_1D + \frac{3}{2} I e z_0@T_0, T_1D^2 \dot{z}_0@T_0, T_1D + \frac{3}{2} I e z_0@T_0, T_1D \dot{z}_0@T_0, T_1D^2 + \frac{1}{2} I e \dot{z}_0@T_0, T_1D^3$$

Equating coefficients of like power of e in **eq243b** yields

eqEps = CoefficientList@Subtract @@ eq243b, eD == 0 . Thread

$$9 - I z_0@T_0, T_1D + z_0^{H1, 0L}@T_0, T_1D == 0, \\ - \frac{1}{2} I z_0@T_0, T_1D^3 - I z_1@T_0, T_1D - \frac{3}{2} I z_0@T_0, T_1D^2 \dot{z}_0@T_0, T_1D - \frac{3}{2} I z_0@T_0, T_1D \dot{z}_0@T_0, T_1D^2 - \\ \frac{1}{2} I \dot{z}_0@T_0, T_1D^3 + z_0^{H0, 1L}@T_0, T_1D + z_1^{H1, 0L}@T_0, T_1D == 0 =$$

The solution of the zeroth-order problem, **eqEps**[[1]], can be expressed as

sol0Form = DSolve@eqEps@@1DD, z_0@T_0, T_1D, 8T_0, T_1 < D@@1DD . C@1D -> A

$$8z_0@T_0, T_1D \otimes E^{I T_0} A@T_1D <$$

which we express in function form as

sol0 = 8z_0 -> Function@8T_0, T_1 <, sol0Form@@1, 2DD . EvaluateD <

$$8z_0 \otimes \text{Function}@8T_0, T_1 <, E^{I T_0} A@T_1DD <$$

To evaluate the complex conjugates of A and Z , we define the conjugate rule

conjugateRule = 9A -> $\dot{A}$, $\dot{A}$ -> A, z -> $\dot{z}$, $\dot{z}$ -> z, Complex@0, n_D -> Complex@0, -nD=;

Substituting **sol0** into the first-order problem, **eqEps[[2]]**, using **conjugateRule**, and moving the nonhomogeneous terms to the right-hand side of the resulting equation, we obtain

order1Eq = HeqEps@@1, 1DD •. z0 -> z1L ==
HeqEps@@1, 1DD •. z0 -> z1L - HeqEps@@2, 1DD •. sol0 •. Hsol0 •. conjugateRuleLL

$$-I z_1 @ T_0, T_1 D + z_1^{H1, 0L} @ T_0, T_1 D ==$$

$$\frac{1}{2} I E^{3 I T_0} A @ T_1 D^3 + \frac{3}{2} I E^{I T_0} A @ T_1 D^2 \dot{A} @ T_1 D + \frac{3}{2} I E^{-I T_0} A @ T_1 D \dot{A} @ T_1 D^2 + \frac{1}{2} I E^{-3 I T_0} \dot{A} @ T_1 D^3 - E^{I T_0} A^c @ T_1 D$$

Eliminating the terms that lead to secular terms from **order1Eq** yields the solvability condition

SCond = CoefficientAorder1Eq@@2DD, E^{IT0}E == 0

$$\frac{3}{2} I A @ T_1 D^2 \dot{A} @ T_1 D - A^c @ T_1 D == 0$$

in agreement with that obtained by treating the second-order form of the **DuffingEq** as well as that obtained by transforming it first into two first-order real-valued equations.

à 2.5 Variation of Parameters

In the next section, we use the method of averaging to determine a first-order uniform expansion of the solution of the Duffing equation. To this end, we use the method of variation of parameters to transform it into a system of two first-order equations. The Duffing equation is repeated here; that is,

DuffingEq = u²@tD + u@tD + e u@tD³ == 0;

When $e = 0$, the solution of this equation can be written as

usolEq = u@tD == a Cos@t + bD;

where a and b are constants, which are sometimes referred to as **parameters**. It follows from **usolEq** that

cond01 = D@usolEq, tD

$$u^c @ tD == -a \sin @ t + bD$$

When $e \neq 0$, we assume that the solution of **DuffingEq** is still given by **usolEq** but with time-varying a and b ; that is,

tdepRule = 8a -> a@tD, b -> b@tD<;

and hence

usolEq = usolEq •. tdepRule

$$u @ tD == a @ tD \cos @ t + b @ tD$$

In other words, we consider **usolEq** as a transformation from $u(t)$ to $a(t)$ and $b(t)$. This is why this approach is called the **method of variation of parameters**. Using this view, we note that we have two equations, namely **DuffingEq** and **usolEq**, for the three unknowns $u(t)$, $a(t)$, and $b(t)$. Hence, we have the freedom of imposing a third condition (third equation). This condition is arbitrary except that it must be independent of **DuffingEq** and **usolEq**. This arbitrariness can be used to advantage, namely to produce a simple and convenient transformation. Out of all possible conditions, we choose to impose the condition **cond01**, thereby assuming that $u(t)$ as well as $u'(t)$ have the same form as in the linear case. This condition leads to a convenient transformation; it leads to a set of first-order rather than second-order equations for $a(t)$ and $b(t)$.

Differentiating **usolEq** with respect to t and recalling that a and b are functions of t , we have

$$\text{cond02} = \text{D@usolEq}, t$$

$$u'(t) == \cos t + b(t) a'(t) - a(t) \sin t + b(t) H1 + b'(t) L$$

Comparing **cond02** with **cond01**, we conclude that

$$\text{cond1} = \text{Expand@cond02} - \text{Hcond01} \cdot t \text{depRule} == 0$$

$$\cos t + b(t) a'(t) - a(t) \sin t + b(t) H1 + b'(t) L == 0$$

Differentiating **cond01** with respect to t , we obtain

$$\text{cond03} = \text{D@cond01} \cdot t \text{depRule}, t$$

$$u''(t) == -\sin t + b(t) a''(t) - a(t) \cos t + b(t) H1 + b'(t) L$$

Substituting for $u(t)$ and $u'(t)$ from **usolEq** and **cond03** into **DuffingEq**, we have

$$\text{cond2} = \text{DuffingEq} \cdot \text{H8usolEq}, \text{cond03} < \cdot \text{Equal} \rightarrow \text{Rule} \cdot \text{ExpandAll}$$

$$e a(t)^3 \cos t + b(t) D^3 - \sin t + b(t) a'(t) - a(t) \cos t + b(t) H1 + b'(t) L == 0$$

Solving **cond1** and **cond2** for $a'(t)$ and $b'(t)$, we obtain the desired two first-order equations

$$\text{transformedEq} = \text{HSolve@8cond1}, \text{cond2} <, 8a'(t), b'(t) < D @ 1 D \cdot \text{Simplify} \cdot \text{Rule} \rightarrow \text{Equal}$$

$$8a'(t) == e a(t)^3 \cos t + b(t) D^3 \sin t + b(t) D, b'(t) == e a(t)^2 \cos t + b(t) D^4 <$$

if $a \neq 0$. Thus, the original second-order **DuffingEq** for $u(t)$ has been replaced by two first-order equations for $a(t)$ and $b(t)$. We emphasize that no approximations have been made in deriving **transformedEq**. Comparing **transformedEq** with **DuffingEq**, we find that the transformed equations are more nonlinear than the original equation. Then, the question arises what is the value of this transformation? The answer depends on the value of e . If e is small, the major parts of a and b vary more slowly with t than u . This fact can be used to advantage analytically and numerically. The analytical advantage is utilized in the method of averaging, as discussed in the next section. Numerically, it is advantageous to solve the transformed equations instead of the original equation because a large step size can be used in the integration. This is the reason why astronomers numerically solve the variational equations rather than the original equations. Usually, astronomers and celestial mechanics refer to this approach as the **special method of perturbations**.

2.6 The Method of Averaging

In this section, we determine a first-order approximation to the transformed equations obtained in the preceding section. To this end, we rewrite `transformedEq` as

$$\text{ampEq} = \text{TrigReduce} \cdot \sum \text{transformedEq} @ 1 \text{DD}$$

$$a^{\epsilon} @ t D == \frac{1}{8} H2 e a @ t D^3 \sin 2 t + 2 b @ t D D + e a @ t D^3 \sin 4 t + 4 b @ t D D L$$

$$\text{phaseEq} = \text{TrigReduce} \cdot \sum \text{transformedEq} @ 2 \text{DD}$$

$$b^{\epsilon} @ t D == \frac{1}{8} H3 e a @ t D^2 + 4 e a @ t D^2 \cos 2 t + 2 b @ t D D + e a @ t D^2 \cos 4 t + 4 b @ t D D L$$

The major parts of a and b are slowly varying functions of time if ϵ is small. Hence, they change very little during the time interval p (the period of the circular functions) and, to the first approximation, they can be considered constant in the interval $[0, p]$. Hence, we replace $a @ t D$ and $b @ t D$ on the right-hand sides of `ampEq` and `phaseEq` with time-independent a and b and obtain

$$\text{ampEq} @ 2 \text{DD} = \text{ampEq} @ 2 \text{DD} \cdot \text{HReverse} \cdot \sum \text{tdepRuleL}$$

$$\frac{1}{8} H2 a^3 e \sin 2 t + 2 b D + a^3 e \sin 4 t + 4 b D L$$

$$\text{phaseEq} @ 2 \text{DD} = \text{phaseEq} @ 2 \text{DD} \cdot \text{HReverse} \cdot \sum \text{tdepRuleL}$$

$$\frac{1}{8} H3 a^2 e + 4 a^2 e \cos 2 t + 2 b D + a^2 e \cos 4 t + 4 b D L$$

We average the right-hand sides of `ampEq` and `phaseEq` over the interval $[0, p]$ and obtain

$$\text{list26a} = \frac{1}{p} \text{Integrate} @ \#, 8t, 0, p < D \& \cdot \sum 8 \text{ampEq} @ 2 \text{DD}, \text{phaseEq} @ 2 \text{DD} < \cdot \cdot \text{Expand}$$

Integrate::gener : Unable to check convergence

Integrate::gener : Unable to check convergence

$$: 0, \frac{3 a^2 e}{8} >$$

It follows from the above result that

$$\text{averagingEq} = 8 a^{\epsilon} @ t D, b^{\epsilon} @ t D < == \text{list26a} \cdot \cdot \text{Thread}$$

$$: a^{\epsilon} @ t D == 0, b^{\epsilon} @ t D == \frac{3 a^2 e}{8} >$$

This averaging method is usually referred to as the [Krylov-Bogoliubov](#) or [van der Pol technique](#).

Solving the averaged equations yields

```
rule26a = DSolve[averagingEq, a[t], b[t], {t} -> {a0, C[2] -> b0}
```

$$9a(t) \approx a_0, \quad b(t) \approx \frac{1}{8} \left(3a^2 t + 8b_0 \right)$$

Substituting for $a(t)$ and $b(t)$ into **usolEq**, we obtain, to the first approximation, that

```
usolEq /. tdepRule /. rule26a /. Cos[arg_D] -> Cos[Expand[arg]]
```

$$u(t) == \cos at + \frac{3}{8} a^2 t + b_0 \cos a_0$$

in agreement with the solutions obtained by using the Lindstedt-Poincaré technique and the method of multiple scales.

Before closing this section, we note that one can arrive at the final results in **ampEq** and **phaseEq** without going through the averaging process. The right-hand sides of **ampEq** and **phaseEq** are the sum of two groups of terms — a group that is a linear combination of fast varying terms and a group that is a linear combination of slowly varying terms. Then, to the first approximation, $a(t)$ in **ampEq** is equal to the slowly varying group on its right-hand side, which is zero. And, to the first approximation, $b(t)$ in **phaseEq** is equal to the slowly varying group on its right-hand side, which is $\frac{3}{8} ea^2$.

Chapter 3

Systems with Quadratic and Cubic Nonlinearities

Off@General::spell1D

à 3.1 Nondimensional Equation of Motion

We consider the free oscillations of a particle of mass m under the action of gravity and restrained by a nonlinear spring. The equation of motion is

$$\begin{aligned} \text{eq31a} &= m \, D^2 x^* @ t^* D, \, 2 < D + f @ x^* @ t^* D D == m \, g \\ f @ x^* @ t^* D D + m \, H x^* L^2 @ t^* D &== g \, m \end{aligned}$$

where g is the gravitational acceleration and $f @ x^* D$ is the restraining force of the spring. We assume that $f @ x^* D$ is an odd cubic function of x^* ; that is,

$$\begin{aligned} \text{eq31b} &= \text{eq31a} \bullet f @ a_D \rightarrow k_1 a + k_3 a^3 \\ k_1 x^* @ t^* D + k_3 x^* @ t^* D^3 + m \, H x^* L^2 @ t^* D &== g \, m \end{aligned}$$

The equilibrium positions $x^* = x_s^* = \text{constant}$ can be obtained by dropping the acceleration term. The result is

$$\begin{aligned} \text{eq31c} &= \text{eq31b} \bullet x^* \rightarrow H x_s^* \&L \\ k_1 H x_s L^* + k_3 H H x_s L^* L^3 &== g \, m \end{aligned}$$

In this chapter, we investigate small oscillations about one of the equilibrium positions. To this end, we shift this equilibrium position to the origin by using the transformation

$$xRule = 8x^* \rightarrow Hx_s^* + u^* @ \#D \&L<;$$

Substituting this transformation into eq31b and using the equilibrium condition eq31c, we obtain

$$\begin{aligned} \text{eq31d} &= \text{Expand@Subtract} \, \check{\check{}} \, \text{eq31b} \bullet xRuleD == 0 \bullet Heq31c \bullet Equal \rightarrow RuleL \\ k_1 u^* @ t^* D + 3 k_3 H H x_s L^* L^2 u^* @ t^* D + 3 k_3 H x_s L^* u^* @ t^* D^2 + k_3 u^* @ t^* D^3 + m \, H u^* L^2 @ t^* D &== 0 \end{aligned}$$

As before, we introduce the following dimensionless quantities:

$$\text{dimenRule} = 8u^* \rightarrow Hx_s^* u @ \# \bullet TD \&L, \, t^* \rightarrow T \, t<;$$

into **eq31d** and obtain

$$\text{eq31e} = \text{Ieq31d} @ \text{1DD } T^2 \cdot H m x_s^* L \cdot \cdot \text{dimenRule} \cdot \cdot \text{ExpandM} == 0$$

$$\frac{T^2 k_1 u @ t D}{m} + \frac{3 T^2 k_3 H H x_s L^* L^2 u @ t D}{m} + \frac{3 T^2 k_3 H H x_s L^* L^2 u @ t D^2}{m} + \frac{T^2 k_3 H H x_s L^* L^2 u @ t D^3}{m} + u^2 @ t D == 0$$

We rewrite **eq31e** as

$$\text{eq31f} = \text{HCollect} @ \text{eq31e} @ \text{1DD}, u @ t D \cdot \cdot 8 _ u @ t D^{\wedge} n _ \cdot \rightarrow a_n u @ t D^{\wedge} n < L == 0$$

$$a_1 u @ t D + a_2 u @ t D^2 + a_3 u @ t D^3 + u^2 @ t D == 0$$

where the a_j are given by

$$\text{alphas} = \text{Ia} \# \rightarrow \text{CoefficientAeq31e} @ \text{1DD}, u @ t D^{\#} \text{EM} \& \cdot \checkmark 81, 2, 3 <$$

$$: a_1 @ \frac{T^2 k_1}{m} + \frac{3 T^2 k_3 H H x_s L^* L^2}{m}, a_2 @ \frac{3 T^2 k_3 H H x_s L^* L^2}{m}, a_3 @ \frac{T^2 k_3 H H x_s L^* L^2}{m} >$$

We choose T so that $a_1 = 1$ and hence the natural frequency is unity; that is,

$$\text{TRule} = \text{Solve} @ a_1 == 1 \cdot \cdot \text{alphas}, T D @ \text{2DD}$$

$$: T @ \frac{\frac{1}{m}}{k_1 + 3 k_3 H H x_s L^* L^2} >$$

Consequently, a_2 and a_3 can be rewritten as

$$\text{alphas} = \text{alphas} \cdot \cdot \text{TRule} \cdot \cdot \text{Rest}$$

$$: a_2 @ \frac{3 k_3 H H x_s L^* L^2}{k_1 + 3 k_3 H H x_s L^* L^2}, a_3 @ \frac{k_3 H H x_s L^* L^2}{k_1 + 3 k_3 H H x_s L^* L^2} >$$

With a_1 being unity, **eq31f** becomes

$$\text{eq31g} = \text{eq31f} \cdot \cdot a_1 \rightarrow 1$$

$$u @ t D + a_2 u @ t D^2 + a_3 u @ t D^3 + u^2 @ t D == 0$$

In contrast with the **DuffingEq**, **eq31g** contains a quadratic as well as a cubic term.

In the next section, we determine a second-order straightforward expansion to the solutions of **eq31g** for small but finite amplitudes. In Section 3.3, we determine a uniform second-order expansion by using the Lindstedt-Poincaré technique. In Section 3.4, we determine a uniform second-order expansion by using the method of multiple scales. In Section 3.5, we show that the first approximation obtained with the method of averaging yields an incomplete solution. In Section 3.6, we introduce the generalized method of averaging and obtain a uniform second-order expansion. Finally, in Sections 3.7 and 3.8, we introduce the Krylov-Bogoliubov-Mitropolsky technique and the method of normal forms, respectively.

à 3.2 Straightforward Expansion

To carry out a straightforward expansion for small but finite amplitudes for **eq31g**, we need to introduce a small parameter ϵ as a bookkeeping parameter. In terms of this parameter, we seek a third-order expansion in the form

$$\mathbf{uExpRule@maxOrder_D := u \rightarrow I \text{Sum} A \epsilon^i u_i @ \#D, \{i, 1, \text{maxOrder}\} \&M}$$

where maxOrder is the order of the expansion sought. In what follows, we let $\text{maxOrder} = 3$ for an expansion of order three. Clearly, ϵ is a measure of the amplitude of oscillation and can be set equal to unity in the final solution if the amplitude is taken to be small, as described below. Substituting this expansion into **eq31g**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we have

$$\begin{aligned} \mathbf{eq32a} = & \mathbf{Heq31g} \cdot \mathbf{uExpRule@3D} \cdot \mathbf{ExpandAll} \cdot \epsilon^{\text{n_}; \text{n} > 3} \rightarrow 0 \\ & \epsilon u_1 @ tD + \epsilon^2 a_2 u_1 @ tD^2 + \epsilon^3 a_3 u_1 @ tD^3 + \epsilon^2 u_2 @ tD + \\ & 2 \epsilon^3 a_2 u_1 @ tD u_2 @ tD + \epsilon^3 u_3 @ tD + \epsilon u_1^2 @ tD + \epsilon^2 u_2^2 @ tD + \epsilon^3 u_3^2 @ tD == 0 \end{aligned}$$

Equating coefficients of like powers of ϵ in **eq32a** yields

$$\begin{aligned} \mathbf{eqEps} = & \mathbf{TableACoefficientAeq32a@@1DD, \{ \epsilon^i == 0, \{i, 3\} \}} \\ & 8u_1 @ tD + u_1^2 @ tD == 0, \quad a_2 u_1 @ tD^2 + u_2 @ tD + u_2^2 @ tD == 0, \quad a_3 u_1 @ tD^3 + 2 a_2 u_1 @ tD u_2 @ tD + u_3 @ tD + u_3^2 @ tD == 0 \end{aligned}$$

which can be solved sequentially for u_1 , u_2 , and u_3 .

The general solution of the first-order equation, **eqEps[[1]]**, can be expressed as

$$\begin{aligned} \mathbf{sol1} = & \mathbf{DSolve@8eqEps@@1DD, u_1 @ 0D == a \text{Cos}@bD, u_1' @ 0D == -a \text{Sin}@bD, u_1 @ tD, tD@@1DD} \cdot \mathbf{Simplify} \\ & 8u_1 @ tD \& a \text{Cos}@t + bD \end{aligned}$$

where a and b are constants. Substituting **sol1**, $u_1 @ tD$, into the second-order equation, **eqEps[[2]]**, and solving the resulting equation for $u_2 @ tD$, we obtain

$$\begin{aligned} \mathbf{sol2} = & \mathbf{DSolve@eqEps@@2DD} \cdot \mathbf{sol1, u_2 @ tD, tD@@1DD} \cdot \mathbf{C@_D} \rightarrow 0 \cdot \mathbf{Simplify} \\ & 9u_2 @ tD \& \frac{1}{6} a^2 H_{-3} + \text{Cos}@2 H t + b L D L a_2 = \end{aligned}$$

As before, we do not include the solution of the homogeneous problem for $u_2 @ tD$.

Substituting **sol1** and **sol2** into the third-order equation, **eqEps[[3]]**, and solving the resulting equation $u_3 @ tD$, we have

$$\begin{aligned} \mathbf{sol3} = & \mathbf{DSolve@eqEps@@3DD} \cdot \mathbf{sol1} \cdot \mathbf{sol2, u_3 @ tD, tD@@1DD} \cdot \mathbf{C@_D} \rightarrow 0 \cdot \mathbf{Simplify} \\ & 9u_3 @ tD \& \frac{1}{96} a^3 H_{-10} \text{Cos}@t + bD + \text{Cos}@3 H t + b L D + 20 t \text{Sin}@t + b D L a_2^2 + \\ & 3 H - 6 \text{Cos}@t + bD + \text{Cos}@3 H t + b L D - 12 t \text{Sin}@t + b D L a_3 L = \end{aligned}$$

Simply letting $C_0 = 0$ did not remove all of the solutions of the homogeneous equation in this case. Therefore, we clear it one more time and obtain

```
sol3 = sol3 /. Cos[t + _D -> 0 // ExpandAll
```

```
9u3[tD
```

$$\frac{1}{48} a^3 \cos(3t + 3bD a_2^2 + \frac{5}{12} a^3 t \sin(t + bD a_2^2 + \frac{1}{32} a^3 \cos(3t + 3bD a_3 - \frac{3}{8} a^3 t \sin(t + bD a_3 =$$

Combining the first-, second-, and third-order solutions, we obtain, to the third approximation, that

```
sol = u[tD == Hu[tD /. uExpRule[3D /. Flatten[8sol1, sol2, sol3]<DL
```

$$u[tD == a e \cos(t + bD + \frac{1}{6} a^2 e^2 H^{-3} + \cos(2Ht + bDL a_2 +$$

$$e^3 \int \frac{1}{48} a^3 \cos(3t + 3bD a_2^2 + \frac{5}{12} a^3 t \sin(t + bD a_2^2 + \frac{1}{32} a^3 \cos(3t + 3bD a_3 - \frac{3}{8} a^3 t \sin(t + bD a_3 \frac{y}{t}$$

We note that the dependence of u on e and a is in the combination ea . Thus, one can set $e = 1$ and consider a as the perturbation parameter.

The straightforward expansion [pink](#) breaks down for $t^3 O(e^{-1} a^{-1})$ because the third-order term is the same order or larger than the second-order term owing to the presence of the mixed-secular term. Next, we use the Lindstedt-Poincaré technique to determine a third-order uniform solution.

à 3.3 The Lindstedt-Poincaré Technique

To apply the Lindstedt-Poincaré technique to [eq31g](#), we introduce the transformation $t = \omega t$ and rewrite it as

```
eq33a = eq31g /. u -> Hu[w #D &L /. t -> t . w
```

$$u[tD + a_2 u[tD^2 + a_3 u[tD^3 + w^2 u'^2[tD == 0$$

To determine a uniform expansion of order `maxOrder` of the solution of [eq33a](#), we expand both w and $u[tD$ in powers of e as

```
omgRule[maxOrder_D := w -> 1 + Sum[Ae^i w_i, 8i, maxOrder<E
```

```
uExpRule[maxOrder_D := u -> I Sum[Ae^i u_i[#D, 8i, maxOrder + 1<E &M
```

As discussed in the preceding chapter, the first term in the expansion of w is taken to be the linear natural frequency of the system, which is unity in this case. We note that the order of the expansion for u is larger than that of w and the uniform expansion required. As discussed below, we eliminate the terms that lead to secular terms from the equation at order `maxOrder+1` but do not include in the solution of the resulting equation in the expansion.

For a uniform second order expansion, we let

```
maxOrder = 2;
```

Substituting the expansions of u and w into **eq33a**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq33b = Heq33a . 8uExpRule@maxOrderD, omgRule@maxOrderD< . ExpandAllL . en_; n>maxOrder+1 -> 0
e u1@tD + e2 a2 u1@tD2 + e3 a3 u1@tD3 + e2 u2@tD + 2 e3 a2 u1@tD u2@tD + e3 u3@tD + e u12@tD +
2 e2 w1 u12@tD + e3 w12 u12@tD + 2 e3 w2 u12@tD + e2 u22@tD + 2 e3 w1 u22@tD + e3 u32@tD == 0

eqEps = TableACoefficientAeq33b@@1DD, eiE == 0, 8i, 3<E
8u1@tD + u12@tD == 0, a2 u1@tD2 + u2@tD + 2 w1 u12@tD + u22@tD == 0,
a3 u1@tD3 + 2 a2 u1@tD u2@tD + u3@tD + w12 u12@tD + 2 w2 u12@tD + 2 w1 u22@tD + u32@tD == 0<
```

The general solution of the first-order equation, **eqEps[[1]]**, can be expressed as

```
sol1 = DSolve@8eqEps@@1DD, u1@tD == a Cos@bD, u16@tD == -a Sin@bD<, u1, tD@@1DD
8u1 ® Ha Cos@bD Cos@#1D - a Sin@bD Sin@#1D &L<
```

where a and b are constants. Instead of using $u_1@tD$, we use u_1 in the second argument to **DSolve** so that **sol1** can be used in the terms involved with derivatives.

Substituting **sol1**, $u_1@tD$, into the second-order equation, **eqEps[[2]]**, moving the nonhomogeneous terms to the right-hand side of the resulting equation, we obtain

```
eq33c = u2@tD + u22@tD - # & . Reverse@eqEps@@2DD . sol1 . SimplifyD
u2@tD + u22@tD == -a2 Cos@b + tD2 a2 + 2 a Cos@b + tD w1
```

Expanding the right-hand side of **eq33c** in a **FourierSeries** using trigonometric identities yields

```
eq33c@@2DD = eq33c@@2DD . TrigReduce
1
--- H - a2 a2 - a2 Cos@2 b + 2 tD a2 + 4 a Cos@b + tD w1L
2
```

Eliminating the terms, $\text{Cos@b} + tD$ and $\text{Sin@b} + tD$, demands that $w_1 = 0$. Then, the particular solution of **eq33c** can be expressed as

```
sol2 = DSolve@eq33c . w1 -> 0, u2@tD, tD@@1DD . C@_D -> 0 . Simplify
9u2@tD ® 1
--- a2 H - 3 + Cos@2 b + tDL a2 =
6
```

or in a pure function format as

```
sol21 = u2 -> Function@t, sol2@@1, 2DD . EvaluateD
u2 ® FunctionAt, 1
--- a2 H - 3 + Cos@2 b + tDL a2E
6
```

Substituting **sol1** and **sol21** into the third-order equation, **eqEps[[3]]**, and using the fact that $w_1 = 0$, we obtain

Symbolize@T₀D; Symbolize@T₁D; Symbolize@T₂D;

and list as

timeScales = 8T₀, T₁, T₂<;

In terms of the time scales T_n , the time derivatives become

**dt@1D@expr_D := SumAeⁱ D@expr, timeScales@@i+1DDD, 8i, 0, maxOrder<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD • ExpandL • e^{i • ; i>maxOrder -> 0;}**

For a uniform second-order expansion, we let

maxOrder = 2;

To represent some of the expressions in a more concise way, we introduce the following display rule:

displayRule =
9Derivative@a__DAu_i_E@__D := SequenceFormATimes žž MapIndexedAD^{#1}_{#2@@1DD-1} &, 8a<E, u_iE,
Derivative@a__D@AD@__D := SequenceFormATimes žž MapIndexedAD^{#1}_{#2@@1DD} &, 8a<E, AE,
Derivative@a__D@ĀD@__D := SequenceFormATimes žž MapIndexedAD^{#1}_{#2@@1DD} &, 8a<E, ĀE,
u_i_@__D -> u_i, A@__D -> A, Ā@__D -> Ā=;

§ 3.4.1 Second-Order Real-Valued System

Using the derivative rule, we transform **eq31g** into the partial-differential equation

eq341a = eq31g •. 8u@tD -> u@T₀, T₁, T₂D, Derivative@n_D@uD@tD -> dt@nD@u@T₀, T₁, T₂DD<

$u@T_0, T_1, T_2D + a_2 u@T_0, T_1, T_2D^2 + a_3 u@T_0, T_1, T_2D^3 + e^2 u^{H0,2,0L}@T_0, T_1, T_2D +$
 $2 e^2 u^{H1,0,1L}@T_0, T_1, T_2D + 2 e u^{H1,1,0L}@T_0, T_1, T_2D + u^{H2,0,0L}@T_0, T_1, T_2D == 0$

Again, to determine a uniform expansion of order maxOrder, we expand u to order maxOrder+1, eliminate the secular terms from the maxOrder+1 equation, but do not include its solution in the final approximate solution. Hence, we seek a uniform expansion of the solution of **eq341a** in the form

solRule = u -> ISumAeⁱ u_i@#1, #2, #3D, 8i, maxOrder + 1<E &M;

Substituting this expansion into **eq341a**, expanding the result for small e , and discarding terms of order higher than e^3 , we have

eq341b = Heq341a •. solRule •. ExpandAllL •. e^{n • ; n>maxOrder+1 -> 0}

$e u_1@T_0, T_1, T_2D + e^2 a_2 u_1@T_0, T_1, T_2D^2 + e^3 a_3 u_1@T_0, T_1, T_2D^3 + e^2 u_2@T_0, T_1, T_2D +$
 $2 e^3 a_2 u_1@T_0, T_1, T_2D u_2@T_0, T_1, T_2D + e^3 u_3@T_0, T_1, T_2D + e^3 u_1^{H0,2,0L}@T_0, T_1, T_2D +$
 $2 e^3 u_1^{H1,0,1L}@T_0, T_1, T_2D + 2 e^2 u_1^{H1,1,0L}@T_0, T_1, T_2D + 2 e^3 u_2^{H1,1,0L}@T_0, T_1, T_2D +$
 $e u_1^{H2,0,0L}@T_0, T_1, T_2D + e^2 u_2^{H2,0,0L}@T_0, T_1, T_2D + e^3 u_3^{H2,0,0L}@T_0, T_1, T_2D == 0$

Equating coefficients of like powers of ϵ in [eq341b](#), we obtain

```
eqEps = CoefficientList@eq341b@@1DD, eD == 0 •• Thread •• Rest
```

$$\begin{aligned} 9u_1 @ T_0, T_1, T_2 D + u_1^{H2,0,0L} @ T_0, T_1, T_2 D &== 0, \\ a_2 u_1 @ T_0, T_1, T_2 D^2 + u_2 @ T_0, T_1, T_2 D + 2 u_1^{H1,1,0L} @ T_0, T_1, T_2 D + u_2^{H2,0,0L} @ T_0, T_1, T_2 D &== 0, \\ a_3 u_1 @ T_0, T_1, T_2 D^3 + 2 a_2 u_1 @ T_0, T_1, T_2 D u_2 @ T_0, T_1, T_2 D + u_3 @ T_0, T_1, T_2 D + u_1^{H0,2,0L} @ T_0, T_1, T_2 D + \\ 2 u_1^{H1,0,1L} @ T_0, T_1, T_2 D + 2 u_2^{H1,1,0L} @ T_0, T_1, T_2 D + u_3^{H2,0,0L} @ T_0, T_1, T_2 D &== 0 = \end{aligned}$$

To place the linear operator on one side and the rest of the terms on the other, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •• u1 -> u1L - # & •ž eqEps@@iDD •• Reverse
```

Using [eqOrder\[i\]](#) and [displayRule](#), we rewrite [eqEps](#) in a concise way as

```
Array@eqOrder, 3D •• displayRule •• TableForm
```

$$\begin{aligned} D_0^2 u_1 + u_1 &== 0 \\ D_0^2 u_2 + u_2 &== -2 HD_0 D_1 u_1 L - u_1^2 a_2 \\ D_0^2 u_3 + u_3 &== -2 HD_0 D_1 u_2 L - D_1^2 u_1 - 2 HD_0 D_2 u_1 L - 2 u_1 u_2 a_2 - u_1^3 a_3 \end{aligned}$$

where $D_n u_i = \nabla u_i @ T_0, T_1, T_2 D \cdot \nabla T_n$.

The general solution of the first-order equation, [eqOrder\[1\]](#), can be expressed as

$$\text{sol1} = 9u_1 \rightarrow \text{IA} @ \#2, \#3D \text{Exp}@I \#1D + \dot{A} @ \#2, \#3D \text{Exp}@-I \#1D \&M=;$$

where $\dot{A}$ is the complex conjugate of A defined by the conjugate rule

$$\text{conjugateRule} = 9A \rightarrow \dot{A}, \dot{A} \rightarrow A, \text{Complex}@0, n_D \rightarrow \text{Complex}@0, -nD=;$$

Then, the second-order equation, [eqOrder\[2\]](#), becomes

```
eq341c = eqOrder@2D •• sol1 •• ExpandAll;
eq341c •• displayRule
```

$$D_0^2 u_2 + u_2 == -2 I E^{I T_0} HD_1 A L + 2 I E^{-I T_0} HD_1 \dot{A} L - A^2 E^{2 I T_0} a_2 - 2 A \dot{A} a_2 - E^{-2 I T_0} \dot{A}^2 a_2$$

Eliminating the term $E^{I T_0}$ that produces secular terms in u_2 from the right-hand side of [eq341c](#) demands that

$$\text{SCond1} = \text{Coefficient}@eq341c@@2DD, \text{Exp}@I T_0 DD == 0$$

$$-2 I A^{H1,0L} @ T_1, T_2 D == 0$$

or

```

SCond1Rule = SolveASCond1, AH1,0L@T1, T2DE@@1DD
H* either copy and paste AH1,0L@T1, T2D from SCond1 or
  use Derivative@1,0D@A@T1, T2D as the second argument to Solve *L
8AH1,0L@T1, T2D @ 0<

```

Eliminating the term $E^{-I}T_0$ that produces secular terms in u_2 from the right-hand side of [eq341c](#) yields the complex conjugate of [SCond1Rule](#); that is,

```

ccSCond1Rule = SCond1Rule •. conjugateRule
:AH1,0L@T1, T2D @ 0>

```

It follows from [SCond1](#) that $A = A@T_2D$. Substituting the solvability conditions into [eq341c](#), we have

```

eq341d = eq341c •. SCond1Rule •. ccSCond1Rule
u2@T0, T1, T2D + u2H2,0,0L@T0, T1, T2D ==
- E2IT0 A@T1, T2D2 a2 - 2 A@T1, T2D a2 A•@T1, T2D - E-2IT0 a2 A•@T1, T2D2

```

The particular solution of [eq341d](#) can be obtained by using [DSolve](#); the result is

```

u2Sol =
DSolve@eq341d, u2@T0, T1, T2D, timeScalesD@@1DD •. C@_D -> H0 &L •. TrigToExp •. ExpandAll
9u2@T0, T1, T2D @  $\frac{1}{3} E^{2IT_0} A@T_1, T_2D^2 a_2 - 2 A@T_1, T_2D a_2 \dot{A}@T_1, T_2D + \frac{1}{3} E^{-2IT_0} a_2 \dot{A}@T_1, T_2D^2 =$ 

```

whose right-hand side can be used directly to express u_2 in a pure function form as

```

sol2 = 8u2 -> Function@8T0, T1, T2<, u2Sol@@1, 2DD •. EvaluateD<
9u2 @
Function@8T0, T1, T2<,  $\frac{1}{3} E^{2IT_0} A@T_1, T_2D^2 a_2 - 2 A@T_1, T_2D a_2 \dot{A}@T_1, T_2D + \frac{1}{3} E^{-2IT_0} a_2 \dot{A}@T_1, T_2D^2 E=$ 

```

Substituting the first- and second-order solutions, [sol1](#) and [sol2](#), into the third-order equation, [eqOrder\[3\]](#), we obtain

```

eq341e = eqOrder@3D •. sol1 •. sol2 •. ExpandAll;
eq341e •. displayRule
D02u3 + u3 == - EIT0 HD12AL - E-IT0 HD12AL - 2 I EIT0 HD2AL + 2 I E-IT0 HD2AL -
 $\frac{8}{3} I A E^{2IT_0} HD_1AL a_2 + \frac{8}{3} I E^{-2IT_0} \dot{A} HD_1AL a_2 - \frac{2}{3} A^3 E^{3IT_0} a_2^2 + \frac{10}{3} A^2 E^{IT_0} \dot{A} a_2^2 +$ 
 $\frac{10}{3} A E^{-IT_0} \dot{A}^2 a_2^2 - \frac{2}{3} E^{-3IT_0} \dot{A}^3 a_2^2 - A^3 E^{3IT_0} a_3 - 3 A^2 E^{IT_0} \dot{A} a_3 - 3 A E^{-IT_0} \dot{A}^2 a_3 - E^{-3IT_0} \dot{A}^3 a_3$ 

```

Eliminating the terms that produce secular terms in u_3 from the right-hand side of [eq341e](#) demands that

$$\text{SCond2} = \text{Coefficient}[\text{eq341e}, \text{Exp}[I T_0 D] == 0$$

$$\frac{10}{3} A(T_1, T_2 D^2) \dot{a}_2^2 \dot{A}(T_1, T_2 D) - 3 A(T_1, T_2 D^2) \dot{a}_3 \dot{A}(T_1, T_2 D) - 2 I A^{(H_0, 1L)}(T_1, T_2 D) - A^{(H_2, 0L)}(T_1, T_2 D) == 0$$

Using **SCond1Rule** and the fact that $A = A(T_2 D)$, we can rewrite **SCond2** as

$$\text{SCond} = \text{SCond2} \cdot \text{D}[\text{SCond1Rule}, T_1 D] \cdot 9A \rightarrow \text{HA}[\#2] \&L, \dot{A} \rightarrow \text{IA}[\#2] \&M=$$

$$\frac{10}{3} A(T_2 D^2) \dot{a}_2^2 \dot{A}(T_2 D) - 3 A(T_2 D^2) \dot{a}_3 \dot{A}(T_2 D) - 2 I A^{(c)}(T_2 D) == 0$$

Expressing A in the polar form

$$\text{ruleA} = 9A \rightarrow \int_{-\infty}^{\infty} \frac{1}{k} a[\#D] \text{Exp}[I b[\#DD] \&Y], \dot{A} \rightarrow \int_{-\infty}^{\infty} \frac{1}{k} a[\#D] \text{Exp}[-I b[\#DD] \&Y];$$

where a and b are real and using the rule

$$\text{realRule} = \text{Re}[s_D] \rightarrow s, \text{Im}[s_D] \rightarrow 0;$$

we write **SCond** as

$$\text{eq341f} = \text{SCond}[\text{eq341e}] \text{Exp}[-I b(T_2 D)] \cdot \text{ruleA} \cdot \text{Expand}$$

$$\frac{5}{12} a(T_2 D^3) \dot{a}_2^2 - \frac{3}{8} a(T_2 D^3) \dot{a}_3 - I a^{(c)}(T_2 D) + a(T_2 D) b^{(c)}(T_2 D)$$

Separating the real and imaginary parts in **eq341f**, we obtain the modulation equations

$$\text{ampEq} = \text{Im}[\text{eq341fD}] == 0 \cdot \text{realRule}$$

$$-a^{(c)}(T_2 D) == 0$$

$$\text{phaseEq} = \text{Re}[\text{eq341fD}] == 0 \cdot \text{realRule}$$

$$\frac{5}{12} a(T_2 D^3) \dot{a}_2^2 - \frac{3}{8} a(T_2 D^3) \dot{a}_3 + a(T_2 D) b^{(c)}(T_2 D) == 0$$

The solution of **ampEq** is $a = a_0 = \text{const}$. Then, if $a_0 \neq 0$, the solution of **phaseEq** is

$$\text{betaRule} = \text{DSolve}[\text{phaseEq} \cdot a(T_2 D) \rightarrow a_0, b(T_2 D, T_2 D) \rightarrow b_0] \cdot \text{ExpandAll}$$

$$9b(T_2 D) \cdot \frac{5}{12} T_2 a_0^2 \dot{a}_2^2 + \frac{3}{8} T_2 a_0^2 \dot{a}_3 + b_0 =$$

where b_0 is a constant. Substituting **ruleA** into **sol1** and **sol2** and recalling that $T_0 = t$ and $T_2 = e^2 t$, we obtain, to the second approximation, that

```

u@tD == I I u@T0, T1, T2D . solRule . sol1 . sol2 . e^3 -> 0 . 9A@_D -> A@T2D, A@_D -> A@T2D .
ruleA . ExpToTrigM . 8a@T2D -> a0 . betaRule .
9T0 -> t, T2 -> e^2 t . Cos@arg_D -> Cos@Collect@arg, tDDM

u@tD == e CosAt  $\int_k 1 - \frac{5}{12} e^2 a_0^2 a_2^2 + \frac{3}{8} e^2 a_0^2 a_3^2 \frac{y}{\{}$  + b0E a0 -
 $\frac{1}{2} e^2 a_0^2 a_2 + \frac{1}{6} e^2 \text{CosAt} \int_k 2 - \frac{5}{6} e^2 a_0^2 a_2^2 + \frac{3}{4} e^2 a_0^2 a_3^2 \frac{y}{\{}$  + 2 b0E a0^2 a2

```

which is in full agreement with that obtained by using the Lindstedt-Poincaré technique.

§ 3.4.2 First-Order Real-Valued System

In this section, we first transform [eq31g](#) into a system of two real-valued first-order equations using the transformation $v@tD = u@tD$ and obtain

```

eq342a = 8u@tD == v@tD, eq31g . u^2@tD -> v@tD<
8u@tD == v@tD, u@tD + a2 u@tD^2 + a3 u@tD^3 + v@tD == 0<

```

To determine a second-order uniform expansion of the solution of [eq342a](#) using the method of multiple scales, we first introduce

```

multiScales = 8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D,
u@tD -> dt@1D@u@T0, T1, T2DD, v@tD -> dt@1D@v@T0, T1, T2DD<;

```

to transform the derivative with respect to t in terms of the derivatives with respect to the three time scales T_0 , T_1 , and T_2 and then expand u and v in the form

```

solRule = 9u -> I Sum Ae^j u_j@#1, #2, #3D, 8j, 3<E &M, v -> I Sum Ae^j v_j@#1, #2, #3D, 8j, 3<E &M=;

```

Substituting [multiScales](#) and [solRule](#) into [eq342a](#), expanding the result for small e , and discarding terms of order higher than e^3 , we obtain

```

eq342b = Heq342a . multiScales . solRule . ExpandAllL . e^n_.; n>3 -> 0;

```

Equating coefficients of like powers of e in [eq342b](#) yields

```

eqEps = Thread@CoefficientList@Subtract zz #, eD == 0D & . zz eq342b . Transpose . Rest;

```

To place the linear operator on one side and the rest of the terms on the other, we define

```

eqOrder@i_D := H#@@1DD & . zz eqEps@@1DD . u_{-1} -> u_iL ==
H#@@1DD & . zz eqEps@@1DD . u_{-1} -> u_iL - H#@@1DD & . zz eqEps@@iDDL . Thread

```

Using [eqOrder\[i\]](#) and [displayRule](#), we rewrite [eqEps](#) in a concise way as

```

eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm

D0u1 - v1 == 0
D0v1 + u1 == 0

D0u2 - v2 == -HD1u1L
D0v2 + u2 == -HD1v1L - u12 a2

D0u3 - v3 == -HD1u2L - D2u1
D0v3 + u3 == -HD1v2L - D2v1 - 2 u1 u2 a2 - u13 a3

```

§ First-Order Equations: Linear System

To determine the solution of the first-order equations, **eqOrder[1]**, we list their left-hand sides and obtain

```

linearSys = #@@1DD & •ž eqOrder@1D

9 - v1@T0, T1, T2D + u1H1, 0, 0L@T0, T1, T2D, u1@T0, T1, T2D + v1H1, 0, 0L@T0, T1, T2D=

```

Next, we seek a solution of **linearSys** in the form $u_1 = P E^{I T_0}$ and $v_1 = Q E^{I T_0}$ and obtain

```

coefList = E-IT0 linearSys •. 9u1 -> IP EI# &M, v1 -> IQ EI# &M= •• Expand

8IP - Q, P + IQ<

```

The coefficient matrix of **coefList** is

```

coefMat = Outer@Coefficient, coefList, 8P, Q<D

88I, -1<, 81, I<<

```

and its adjoint is defined by

```

hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose

```

Hence, the right and left eigenvectors of **coefMat** are

```

rightVec = 81, c1< •. Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD

81, I<

leftVec = 81, c1< •. Solve@Hhermitian@coefMatD.81, c1<L@@1DD == 0, c1D@@1DD

81, I<

```

Then the solution of the first-order equations can be expressed as

$$\text{sol1Form} = \text{rightVec } A @ T_1, T_2 D E^{I T_0}$$

$$8 E^{I T_0} A @ T_1, T_2 D, I E^{I T_0} A @ T_1, T_2 D <$$

and hence u_1 and v_1 can be expressed in a pure function form as

$$\text{order1Sol} =$$

$$\begin{aligned} 8 u_1 &\rightarrow \text{Function}[8 T_0, T_1, T_2 <, \text{sol1Form} @ @ 1 D D + \text{Hsol1Form} @ @ 1 D D \cdot \text{conjugateRuleL} \cdot \cdot \text{Evaluated}, \\ v_1 &\rightarrow \text{Function}[8 T_0, T_1, T_2 <, \text{sol1Form} @ @ 2 D D + \text{Hsol1Form} @ @ 2 D D \cdot \text{conjugateRuleL} \cdot \cdot \text{Evaluated} < \end{aligned}$$

$$\begin{aligned} 8 u_1 &\otimes \text{Function}[8 T_0, T_1, T_2 <, E^{I T_0} A @ T_1, T_2 D + E^{-I T_0} \dot{A} @ T_1, T_2 D D, \\ v_1 &\otimes \text{Function}[8 T_0, T_1, T_2 <, I E^{I T_0} A @ T_1, T_2 D - I E^{-I T_0} \dot{A} @ T_1, T_2 D D < \end{aligned}$$

Second-Order Equations

Substituting the first-order solution, **order1Sol**, into the second-order equations, **eqOrder[2]**, we have

$$\begin{aligned} \text{order2Eq} &= \text{eqOrder} @ 2 D \cdot \cdot \text{order1Sol} \cdot \cdot \text{ExpandAll}; \\ \text{order2Eq} &\cdot \cdot \text{displayRule} \end{aligned}$$

$$\begin{aligned} : D_0 u_2 - v_2 &== - E^{I T_0} H D_1 A L - E^{-I T_0} H D_1 \dot{A} L, \\ D_0 v_2 + u_2 &== - I E^{I T_0} H D_1 A L + I E^{-I T_0} H D_1 \dot{A} L - A^2 E^{2 I T_0} a_2 - 2 A \dot{A} a_2 - E^{-2 I T_0} \dot{A}^2 a_2 > \end{aligned}$$

To eliminate the terms that produce secular terms in u_2 and v_2 , we determine the vector proportional to $E^{I T_0}$ in the right-hand sides of **order2Eq** and obtain

$$\text{STerms1} = \text{Coefficient} @ \# @ @ 2 D D, \text{Exp} @ I T_0 D D \& \cdot \cdot \text{order2Eq}$$

$$8 - A^{H1, 0L} @ T_1, T_2 D, - I A^{H1, 0L} @ T_1, T_2 D <$$

Then, the condition for the elimination of the terms that produce secular terms, solvability condition, demands that **STerms1** be orthogonal to the adjoint, left eigenvector. The result is

$$\text{SCond1} = \text{SolveAConjugate} @ \text{leftVecD} . \text{STerms1} == 0, A^{H1, 0L} @ T_1, T_2 D E @ @ 1 D D$$

$$8 A^{H1, 0L} @ T_1, T_2 D \otimes 0 <$$

whose complex conjugate is

$$\text{ccSCond1} = \text{SCond1} \cdot \cdot \text{conjugateRule}$$

$$: \dot{A}^{H1, 0L} @ T_1, T_2 D \otimes 0 >$$

With this solvability condition, **order2Eq** becomes

```

order2Eqm = order2Eq . SCond1 . ccSCond1;
order2Eqm . displayRule

:D0u2 - v2 == 0, D0v2 + u2 == -A^2 E^{2 I T0} a2 - 2 A \dot{A} a2 - E^{-2 I T0} \dot{A}^2 a2>

```

whose particular solution can be expressed as

```

DSolve@order2Eqm, 8u2@T0, T1, T2D, v2@T0, T1, T2D<, timeScalesD

DSolve::pde : Partial differential equation may not have
a general solution. Try loading Calculus`DSolveIntegrals` to find special solutions.

DSolveB:-v2@T0, T1, T2D + u2^{H1,0,0L}@T0, T1, T2D == 0, u2@T0, T1, T2D + v2^{H1,0,0L}@T0, T1, T2D ==
-E^{2 I T0} A@T1, T2D^2 a2 - 2 A@T1, T2D a2 \dot{A}@T1, T2D - E^{-2 I T0} a2 \dot{A}@T1, T2D^2>,
8u2@T0, T1, T2D, v2@T0, T1, T2D<, 8T0, T1, T2<F

```

Here, directly applying **DSolve** to solve for **order2Eqm** does not work. Since the differential operators on the left-hand sides of **order2Eqm** are essentially ordinary-differential operators in terms of T_0 , we can first transform **order2Eqm** into an ordinary-differential-equation form and then use **DSolve** to obtain the particular solution as

```

sol2pForm =
DSolve@order2Eqm . 8u2 -> Hu2@#1D &L, v2 -> Hv2@#1D &L<, 8u2@T0D, v2@T0D<, T0D@@1DD .
C@_D -> 0 . TrigToExp . ExpandAll

9u2@T0D @ \frac{1}{3} E^{2 I T0} A@T1, T2D^2 a2 - 2 A@T1, T2D a2 \dot{A}@T1, T2D + \frac{1}{3} E^{-2 I T0} a2 \dot{A}@T1, T2D^2,
v2@T0D @ \frac{2}{3} I E^{2 I T0} A@T1, T2D^2 a2 - \frac{2}{3} I E^{-2 I T0} a2 \dot{A}@T1, T2D^2 =

```

We can then express the solution of the second-order equations in a pure function form as

```

order2Sol = 8u2 -> Function@8T0, T1, T2<, sol2pForm@@1, 2DD . Evaluated,
v2 -> Function@8T0, T1, T2<, sol2pForm@@2, 2DD . EvaluatedD<

9u2 @
FunctionA8T0, T1, T2<, \frac{1}{3} E^{2 I T0} A@T1, T2D^2 a2 - 2 A@T1, T2D a2 \dot{A}@T1, T2D + \frac{1}{3} E^{-2 I T0} a2 \dot{A}@T1, T2D^2 E,
v2 @ FunctionA8T0, T1, T2<, \frac{2}{3} I E^{2 I T0} A@T1, T2D^2 a2 - \frac{2}{3} I E^{-2 I T0} a2 \dot{A}@T1, T2D^2 E=

```

Third-Order Equations

Substituting the first- and second-order solutions into the third-order equations, **eqOrder[3]**, yields

```
order3Eq = eqOrder@3D •. order1Sol •. order2Sol •• ExpandAll;
order3Eq •. displayRule
```

$$9D_0 u_3 - v_3 ==$$

$$-E^{I T_0} H D_2 A L - E^{-I T_0} H D_2 \dot{A} L - \frac{2}{3} A E^{2 I T_0} H D_1 A L a_2 + 2 \dot{A} H D_1 A L a_2 + 2 A H D_1 \dot{A} L a_2 - \frac{2}{3} E^{-2 I T_0} \dot{A} H D_1 \dot{A} L a_2,$$

$$D_0 v_3 + u_3 == -I E^{I T_0} H D_2 A L + I E^{-I T_0} H D_2 \dot{A} L - \frac{4}{3} I A E^{2 I T_0} H D_1 A L a_2 +$$

$$\frac{4}{3} I E^{-2 I T_0} \dot{A} H D_1 \dot{A} L a_2 - \frac{2}{3} A^3 E^{3 I T_0} a_2^2 + \frac{10}{3} A^2 E^{I T_0} \dot{A} a_2^2 + \frac{10}{3} A E^{-I T_0} \dot{A}^2 a_2^2 -$$

$$\frac{2}{3} E^{-3 I T_0} \dot{A}^3 a_2^2 - A^3 E^{3 I T_0} a_3 - 3 A^2 E^{I T_0} \dot{A} a_3 - 3 A E^{-I T_0} \dot{A}^2 a_3 - E^{-3 I T_0} \dot{A}^3 a_3 =$$

To eliminate the terms that lead to secular terms from **order3Eq**, we calculate the vector proportional to $E^{I T_0}$ in their right-hand sides and obtain

```
STerms2 = Coefficient@@@2DD, Exp@I T_0DD & •ž order3Eq
```

$$9 - A^{H_0, 1L} @T_1, T_2 D, \frac{10}{3} A @T_1, T_2 D^2 a_2^2 \dot{A} @T_1, T_2 D - 3 A @T_1, T_2 D^2 a_3 \dot{A} @T_1, T_2 D - I A^{H_0, 1L} @T_1, T_2 D =$$

Hence, the solvability condition at this order is

```
SCond2 = Conjugate@leftVecD.STerms2 == 0 •• ExpandAll
```

$$- \frac{10}{3} I A @T_1, T_2 D^2 a_2^2 \dot{A} @T_1, T_2 D + 3 I A @T_1, T_2 D^2 a_3 \dot{A} @T_1, T_2 D - 2 A^{H_0, 1L} @T_1, T_2 D == 0$$

This solvability condition is the same as that obtained by attacking the second-order form of the governing equation.

§ 3.4.3 First-Order Complex-Valued System

In this section, we first transform **eq31g** into a single first-order complex-valued equation using the transformation

$$\text{transfRule} = 9u@tD \rightarrow z@tD + \dot{z}@tD, u^@tD \rightarrow I I z@tD - \dot{z}@tD \text{DM};$$

whose inverse is given by

$$\text{zetaRule} = \text{SolveAtransfRule} •. \text{Rule} \rightarrow \text{Equal}, 9z@tD, \dot{z}@tD = E@@@1DD$$

$$9z@tD \otimes \frac{1}{2} H u@tD - I u^@tD L, \dot{z}@tD \otimes \frac{1}{2} H u@tD + I u^@tD L =$$

where $\bar{z}$ is the complex conjugate of z as defined by the complex conjugate rule

$$\text{conjugateRule} = 9A \rightarrow \dot{A}, \dot{A} \rightarrow A, z \rightarrow \bar{z}, \bar{z} \rightarrow z, \text{Complex}@0, n_D \rightarrow \text{Complex}@0, -nD;$$

It follows from **eq31g** that the acceleration is given by

```
acceleration = Solve@eq31g, u'^@tDD@@1DD
```

$$8u'^@tD \otimes -u@tD - a_2 u@tD^2 - a_3 u@tD^3 <$$

Differentiating **zetaRule[[1]]** once with respect to t and using **acceleration** and **transfRule**, we obtain the following first-order complex-valued equation:

```
eq343a = D@zetaRule@@1DD, tD •. acceleration •. transfRule •. Rule -> Equal •• ExpandAll
```

$$z'^@tD == I z@tD + \frac{1}{2} I a_2 z@tD^2 + \frac{1}{2} I a_3 z@tD^3 + I a_2 z@tD \dot{z}@tD + \frac{3}{2} I a_3 z@tD^2 \dot{z}@tD + \frac{1}{2} I a_2 \dot{z}@tD^2 + \frac{3}{2} I a_3 z@tD \dot{z}@tD^2 + \frac{1}{2} I a_3 \dot{z}@tD^3$$

To determine a second-order uniform expansion of the solution of **eq343a** using the method of multiple scales, we first introduce

```
multiScales = 9z@tD -> z@T0, T1, T2D, z'@tD -> z'@T0, T1, T2D, z''@tD -> dt@1D@z@T0, T1, T2DD=;
```

and then expand z and $\ddot{z}$ in the form

```
solRule = 9z -> I Sum Ae^j z_j@#1, #2, #3D, 8j, 3<E &M, z' -> I Sum Ae^j z_j'@#1, #2, #3D, 8j, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq343a**, expanding the result of small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq343b = Heq343a •. multiScales •. solRule •• ExpandAllL •. e^N_•; n>3 -> 0;
```

Equating coefficients of like powers of ϵ yields

```
eqEps = CoefficientList@Subtract @@ eq343b, eD == 0 •• Thread •• Rest;
```

To place the linear operator on one side and the rest of the terms on the other, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. z1 -> z1L - # & • z eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm
```

$$D_0 z_1 - I z_1 == 0$$

$$D_0 z_2 - I z_2 == -HD_1 z_1 L + \frac{1}{2} I a_2 z_1^2 + I a_2 z_1 \dot{z}_1 + \frac{1}{2} I a_2 \dot{z}_1^2$$

$$D_0 z_3 - I z_3 == -HD_1 z_2 L - D_2 z_1 + \frac{1}{2} I a_3 z_1^3 + I a_2 z_1 z_2 + \frac{3}{2} I a_3 z_1^2 \dot{z}_1 + I a_2 z_2 \dot{z}_1 + \frac{3}{2} I a_3 z_1 \dot{z}_1^2 + \frac{1}{2} I a_3 \dot{z}_1^3 + I a_2 \dot{z}_1^2$$

The general solution of the first-order equation, **eqOrder[1]**, can be expressed in a pure function form as

```
sol1 = DSolve@eqOrder@1D, z1, timeScalesD@@1DD •. C@1D -> A
```

$$8z_1 \otimes HE^{I \#1} A@#2, #3D \&L <$$

whose complex conjugate is

```
ccsol1 = sol1 •. conjugateRule
8z1 @ HEComplex@0, -1D #1 A@#2, #3D &L<
```

Substituting the first-order solution into the second-order equation, **eqOrder[2]**, we have

```
order2Eq = eqOrder@2D •. sol1 •. ccsol1;
order2Eq •. displayRule
```

$$D_0 z_2 - I z_2 == -E^{IT_0} HD_1 AL + \frac{1}{6} I A^2 E^{2IT_0} a_2 + I A \dot{A} a_2 + \frac{1}{6} I E^{-2IT_0} \dot{A}^2 a_2$$

Eliminating the terms that lead to secular terms in z_2 from **order2Eq** yields

```
SCond1 = CoefficientAorder2Eq@@2DD, E^{IT_0} E == 0
-A^{H1, 0L}@T1, T2D == 0
```

or

```
SCond1Rule = SolveASCond1, A^{H1, 0L}@T1, T2DE@@1DD
8A^{H1, 0L}@T1, T2D @ 0<
```

With this solvability condition, the particular solution of **order2Eq** can be expressed in a pure function form as

```
sol2 = DSolve@order2Eq •. SCond1Rule, z2, timeScalesD@@1DD •. C@_D -> H0 &L
```

$$9z_2 @ \int_k E^{I \#1} H0 \&L@#2, \#3D + \frac{1}{6} E^{-2I \#1} a_2 J3 E^{4I \#1} A@#2, \#3D^2 - 6 E^{2I \#1} A@#2, \#3D \dot{A}@#2, \#3D - \dot{A}@#2, \#3D^2 N \&Z = \{$$

whose complex conjugate is

```
ccsol2 = sol2 •. conjugateRule;
```

Substituting the first- and second-order solutions in the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D •. sol1 •. ccsol1 •. sol2 •. ccsol2 •. ExpandAll;
order3Eq •. displayRule
```

$$D_0 z_3 - I z_3 == -E^{IT_0} HD_2 AL - A E^{2IT_0} HD_1 AL a_2 + \dot{A} HD_1 AL a_2 + A HD_1 \dot{A} L a_2 + \frac{1}{3} E^{-2IT_0} \dot{A} HD_1 \dot{A} L a_2 + \frac{1}{3} I A^3 E^{3IT_0} a_2^2 - \frac{5}{3} I A^2 E^{IT_0} \dot{A} a_2^2 - \frac{5}{3} I A E^{-IT_0} \dot{A}^2 a_2^2 + \frac{1}{3} I E^{-3IT_0} \dot{A}^3 a_2^2 + \frac{1}{2} I A^3 E^{3IT_0} a_3 + \frac{3}{2} I A^2 E^{IT_0} \dot{A} a_3 + \frac{3}{2} I A E^{-IT_0} \dot{A}^2 a_3 + \frac{1}{2} I E^{-3IT_0} \dot{A}^3 a_3$$

Eliminating the terms that produce secular terms in z_3 from **order3Eq** demands the following condition


```
SCond2 = CoefficientAorder3Eq@@2DD, E^T0 E == 0;
SCond2 . displayRule
```

$$-HD_2AL - \frac{5}{3} I A^2 \dot{A} a_2^2 + \frac{3}{2} I A^2 \dot{A} a_3 == 0$$

which is in full agreement with that obtained by attacking directly [eq31g](#).

3.5 The Method of Averaging

As before, we use the method of variation of parameters to transform the dependent variable u into a and b according to

$$\text{eq35a} = u@tD == e a \cos@t + bD;$$

subject to the constraint

$$\text{eq35b} = D@eq35a, tD$$

$$u^e@tD == -a e \sin@t + bD$$

where a and b are functions of t

$$tdependentRule = 8a -> a@tD, b -> b@tD<;$$

and e is a small dimensionless parameter that is a measure of the amplitude of oscillation. Differentiating [eq35a](#) with respect to t yields

$$\text{eq35c} = D@eq35a . tdependentRule, tD$$

$$u^e@tD == e \cos@t + b@tDD a^e@tD - e a@tD \sin@t + b@tDD H1 + b^e@tDL$$

Comparing [eq35b](#) and [eq35c](#), we conclude that

$$\text{cond35a} = \text{Expand}@eq35c@@2DD - \text{Heq35b@@2DD} . tdependentRuleLD == 0$$

$$e \cos@t + b@tDD a^e@tD - e a@tD \sin@t + b@tDD b^e@tD == 0$$

Differentiating [eq35b](#) once with respect to t yields

$$\text{eq35d} = D@eq35b . tdependentRule, tD$$

$$u^2@tD == -e \sin@t + b@tDD a^e@tD - e a@tD \cos@t + b@tDD H1 + b^e@tDL$$

Substituting [eq35a](#) and [eq35d](#) into [eq31g](#), we have

$$\text{cond35b} = \text{eq31g} . H8eq35a . tdependentRule, eq35d< . Equal -> RuleL . ExpandAll$$

$$e^2 a@tD^2 \cos@t + b@tDD^2 a_2 + e^3 a@tD^3 \cos@t + b@tDD^3 a_3 - \\ e \sin@t + b@tDD a^e@tD - e a@tD \cos@t + b@tDD b^e@tD == 0$$

Solving **cond35a** and **cond35b** for $a^\epsilon(t)$ and $b^\epsilon(t)$, we have

transformedEq = Solve@8cond35a, cond35b<, 8a^ϵ(tD, b^ϵ(tD<D@@1DD •. Rule -> Equal •• Simplify

$$\begin{aligned} 8a^\epsilon(t) &= e a(t)^2 \cos t + b(t) D^2 \sin t + b(t) D^2 H a_2 + e a(t) \cos t + b(t) D^2 a_3 L, \\ b^\epsilon(t) &= e a(t) \cos t + b(t) D^3 H a_2 + e a(t) \cos t + b(t) D^2 a_3 L \end{aligned}$$

where a is assumed to be different from zero in arriving at **transformedEq[[2]]**. Using trigonometric identities, we expand the right-hand sides of **transformedEq** in Fourier series and obtain

ampEq1 = TrigReduce •ž transformedEq@@1DD

$$\begin{aligned} a^\epsilon(t) &= \frac{1}{8} H_2 e a(t)^2 \sin t + b(t) D^2 a_2 + 2 e a(t)^2 \sin 3 t + 3 b(t) D^2 a_2 + \\ &\quad 2 e^2 a(t)^3 \sin 2 t + 2 b(t) D^2 a_3 + e^2 a(t)^3 \sin 4 t + 4 b(t) D^2 a_3 L \end{aligned}$$

phaseEq1 = TrigReduce •ž transformedEq@@2DD

$$\begin{aligned} b^\epsilon(t) &= \frac{1}{8} H_6 e a(t) \cos t + b(t) D^2 a_2 + 2 e a(t) \cos 3 t + 3 b(t) D^2 a_2 + \\ &\quad 3 e^2 a(t)^2 a_3 + 4 e^2 a(t)^2 \cos 2 t + 2 b(t) D^2 a_3 + e^2 a(t)^2 \cos 4 t + 4 b(t) D^2 a_3 L \end{aligned}$$

Since e is small, $a^\epsilon(t)$ and $b^\epsilon(t)$ are slowly varying functions of t . Then, according to the method of averaging, we can neglect the fast-varying terms in **ampEq1** and **phaseEq1** and obtain the following averaged equations:

ampAvgEq = ampEq1 •. Sin@_. t + _D -> 0

$$a^\epsilon(t) = 0$$

phaseAvgEq = phaseEq1 •. Cos@_. t + _D -> 0

$$b^\epsilon(t) = \frac{3}{8} e^2 a(t)^2 a_3$$

Whereas **ampAvgEq** agrees with **ampEq**, **phaseAvgEq** does not agree with **phaseEq** obtained in the preceding section by using the method of multiple scales. There is a term $\frac{5}{12} e^2 a^2 a_2^2$ missing from **phaseAvgEq**. Following the details of the solution in the preceding section, one finds that this term is the result of the interaction of the first- and second-order approximations. This interaction was not taken into account in arriving at **ampAvgEq** and **phaseAvgEq**. To include the effect of this interaction, we need to carry out the solutions of **ampEq1** and **phaseEq1** to higher order. This is accomplished by using the generalized method of averaging, which is discussed next, or its variant the Krylov-Bogoliubov-Mitropolsky technique, which is discussed in Section 3.7.

à 3.6 The Generalized Method of Averaging

To apply this method, we introduce the variable $f = t + b$ and rewrite **ampEq1** and **phaseEq1** as

$$\text{ampEq2} = \text{ampEq1} \cdot \sin(n \cdot t + a_D) \rightarrow \sin \text{Expand} @ n \text{Hf} @ tD - b @ tDL + aDD$$

$$a^e @ tD == \frac{1}{8} H_2 e a @ tD^2 \sin @ f @ tDD a_2 + \\ 2 e a @ tD^2 \sin @ 3 f @ tDD a_2 + 2 e^2 a @ tD^3 \sin @ 2 f @ tDD a_3 + e^2 a @ tD^3 \sin @ 4 f @ tDD a_3 L$$

$$\text{phaseEq2} = f^e @ tD == 1 + \text{phaseEq1} @ 2DD \cdot \cos(n \cdot t + a_D) \rightarrow \cos \text{Expand} @ n \text{Hf} @ tD - b @ tDL + aDD$$

$$f^e @ tD == 1 + \frac{1}{8} H_6 e a @ tD \cos @ f @ tDD a_2 + 2 e a @ tD \cos @ 3 f @ tDD a_2 + \\ 3 e^2 a @ tD^2 a_3 + 4 e^2 a @ tD^2 \cos @ 2 f @ tDD a_3 + e^2 a @ tD^2 \cos @ 4 f @ tDD a_3 L$$

We seek approximate solutions to **ampEq2** and **phaseEq2** in the form

$$\text{asol} = a @ tD == a_0 @ tD + \text{Sum} A e^i a_i @ a_0 @ tD, f_0 @ tDD, 8i, 2 < E$$

$$a @ tD == a_0 @ tD + e a_1 @ a_0 @ tD, f_0 @ tDD + e^2 a_2 @ a_0 @ tD, f_0 @ tDD$$

$$\text{phisol} = f @ tD == f_0 @ tD + \text{Sum} A e^i f_i @ a_0 @ tD, f_0 @ tDD, 8i, 2 < E$$

$$f @ tD == f_0 @ tD + e f_1 @ a_0 @ tD, f_0 @ tDD + e^2 f_2 @ a_0 @ tD, f_0 @ tDD$$

where $a_0 @ tD$ and $f_0 @ tD$ are expanded in powerseries in e as

$$a_0 @ tD == \text{Sum} A e^i A_i @ a_0 @ tDD, 8i, 2 < E$$

$$a_0 @ tD == e A_1 @ a_0 @ tDD + e^2 A_2 @ a_0 @ tDD$$

$$f_0 @ tD == 1 + \text{Sum} A e^i F_i @ a_0 @ tDD, 8i, 2 < E$$

$$f_0 @ tD == 1 + e F_1 @ a_0 @ tDD + e^2 F_2 @ a_0 @ tDD$$

The functions $a_1, a_2, \dots, a_n$ and $f_1, f_2, \dots, f_n$ are fast varying functions of f_0 , while it follows from **a0Eq** and **phi0Eq** that a_0 , and hence the A_n and F_n are slowly varying functions of t .

To the second approximation, we differentiate $a @ tD$ and $f @ tD$ once with respect to t , use **a0Eq** and **phi0Eq**, expand the result for small e , discard terms of order higher than e^2 , and obtain

$$\text{aEq1} = \text{HD} @ \text{asol}, tD \cdot \text{H8a0Eq}, \text{phi0Eq} < \cdot. \text{Equal} \rightarrow \text{RuleL} \cdot \cdot \text{ExpandAllL} \cdot \cdot e^{n \cdot \cdot; n > 2} \rightarrow 0$$

$$a^e @ tD == e A_1 @ a_0 @ tDD + e^2 A_2 @ a_0 @ tDD + e a_1^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + \\ e^2 F_1 @ a_0 @ tDD a_1^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + e^2 a_2^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + e^2 A_1 @ a_0 @ tDD a_1^{H1, 0L} @ a_0 @ tD, f_0 @ tDD$$

$$\text{phiEq1} = \text{HD} @ \text{phisol}, tD \cdot \text{H8a0Eq}, \text{phi0Eq} < \cdot. \text{Equal} \rightarrow \text{RuleL} \cdot \cdot \text{ExpandAllL} \cdot \cdot e^{n \cdot \cdot; n > 2} \rightarrow 0$$

$$f^e @ tD == 1 + e F_1 @ a_0 @ tDD + e^2 F_2 @ a_0 @ tDD + e f_1^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + \\ e^2 F_1 @ a_0 @ tDD f_1^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + e^2 f_2^{H0, 1L} @ a_0 @ tD, f_0 @ tDD + e^2 A_1 @ a_0 @ tDD f_1^{H1, 0L} @ a_0 @ tD, f_0 @ tDD$$

Next, we substitute **asol** and **phisol** into **ampEq2** and **phaseEq2**, expand the right-hand sides for small e keeping terms up to $O(e^2)$, and rewrite their right-hand sides as

ampEqrhs = Series@ampEq2@@2DD •. H8asol, phisol< •. Equal -> RuleL, 8e, 0, 2<D •• Normal

$$\begin{aligned} & \frac{1}{8} e^2 H_2 \sin f_0 t D D a_2 a_0 t D^2 + 2 \sin^3 f_0 t D D a_2 a_0 t D^2 L + \\ & \frac{1}{8} e^2 H_2 \sin^2 f_0 t D D a_3 a_0 t D^3 + \sin^4 f_0 t D D a_3 a_0 t D^3 + \\ & 2 a_2 H_2 \sin f_0 t D D a_0 t D a_1 a_0 t D, f_0 t D D + \cos f_0 t D D a_0 t D^2 f_1 a_0 t D, f_0 t D D L + \\ & 2 a_2 H_2 \sin^3 f_0 t D D a_0 t D a_1 a_0 t D, f_0 t D D + 3 \cos^3 f_0 t D D a_0 t D^2 f_1 a_0 t D, f_0 t D D L L \end{aligned}$$

phaseEqrhs = Series@phaseEq2@@2DD •. H8asol, phisol< •. Equal -> RuleL, 8e, 0, 2<D •• Normal

$$\begin{aligned} & 1 + \frac{1}{8} e^2 H_6 \cos f_0 t D D a_2 a_0 t D + 2 \cos^3 f_0 t D D a_2 a_0 t D L + \\ & \frac{1}{8} e^2 H_3 a_3 a_0 t D^2 + 4 \cos^2 f_0 t D D a_3 a_0 t D^2 + \cos^4 f_0 t D D a_3 a_0 t D^2 + \\ & 6 a_2 H \cos f_0 t D D a_1 a_0 t D, f_0 t D D - \sin f_0 t D D a_0 t D f_1 a_0 t D, f_0 t D D L + \\ & 2 a_2 H \cos^3 f_0 t D D a_1 a_0 t D, f_0 t D D - 3 \sin^3 f_0 t D D a_0 t D f_1 a_0 t D, f_0 t D D L L \end{aligned}$$

Equating coefficients of like powers of e from **ampEq1**, **ampEqrhs**, **phiEq1**, and **phaseEqrhs**, we obtain

ampEq3 = CoefficientList@#, eD & •ž HaEq1@@2DD == ampEqrhsL •• Thread •• Rest

$$\begin{aligned} & 9A_1 a_0 t D D + a_1^{H_0, 1L} a_0 t D, f_0 t D D == \frac{1}{8} H_2 \sin f_0 t D D a_2 a_0 t D^2 + 2 \sin^3 f_0 t D D a_2 a_0 t D^2 L, \\ & A_2 a_0 t D D + F_1 a_0 t D D a_1^{H_0, 1L} a_0 t D, f_0 t D D + a_2^{H_0, 1L} a_0 t D, f_0 t D D + \\ & A_1 a_0 t D D a_1^{H_1, 0L} a_0 t D, f_0 t D D == \frac{1}{8} H_2 \sin^2 f_0 t D D a_3 a_0 t D^3 + \sin^4 f_0 t D D a_3 a_0 t D^3 + \\ & 2 a_2 H_2 \sin f_0 t D D a_0 t D a_1 a_0 t D, f_0 t D D + \cos f_0 t D D a_0 t D^2 f_1 a_0 t D, f_0 t D D L + \\ & 2 a_2 H_2 \sin^3 f_0 t D D a_0 t D a_1 a_0 t D, f_0 t D D + 3 \cos^3 f_0 t D D a_0 t D^2 f_1 a_0 t D, f_0 t D D L L = \end{aligned}$$

phaseEq3 = CoefficientList@#, eD & •ž HphiEq1@@2DD == phaseEqrhsL •• Thread •• Rest

$$\begin{aligned} & 9F_1 a_0 t D D + f_1^{H_0, 1L} a_0 t D, f_0 t D D == \frac{1}{8} H_6 \cos f_0 t D D a_2 a_0 t D + 2 \cos^3 f_0 t D D a_2 a_0 t D L, \\ & F_2 a_0 t D D + F_1 a_0 t D D f_1^{H_0, 1L} a_0 t D, f_0 t D D + \\ & f_2^{H_0, 1L} a_0 t D, f_0 t D D + A_1 a_0 t D D f_1^{H_1, 0L} a_0 t D, f_0 t D D == \\ & \frac{1}{8} H_3 a_3 a_0 t D^2 + 4 \cos^2 f_0 t D D a_3 a_0 t D^2 + \cos^4 f_0 t D D a_3 a_0 t D^2 + \\ & 6 a_2 H \cos f_0 t D D a_1 a_0 t D, f_0 t D D - \sin f_0 t D D a_0 t D f_1 a_0 t D, f_0 t D D L + \\ & 2 a_2 H \cos^3 f_0 t D D a_1 a_0 t D, f_0 t D D - 3 \sin^3 f_0 t D D a_0 t D f_1 a_0 t D, f_0 t D D L L = \end{aligned}$$

Next, we use the method of separation of variables to separate fast and slowly varying terms in the first-order equations, **ampEq3[[1]]** and **phaseEq3[[1]]**. The slowly varying parts yield

$$\text{cond36a} = 8A_1 a_0 t D D \rightarrow 0, F_1 a_0 t D D \rightarrow 0<;$$

whereas the fast varying parts yield

order1Eq = 8ampEq3@@1DD, phaseEq3@@1DD < . . cond36a

$$9a_1^{H0,1L} @a_0 @tD, f_0 @tDD == \frac{1}{8} H2 \sin @f_0 @tDD a_2 a_0 @tD^2 + 2 \sin @3 f_0 @tDD a_2 a_0 @tD^2 L,$$

$$f_1^{H0,1L} @a_0 @tD, f_0 @tDD == \frac{1}{8} H6 \cos @f_0 @tDD a_2 a_0 @tD + 2 \cos @3 f_0 @tDD a_2 a_0 @tDL =$$

To determine the solutions of **order1Eq** using **DSolve**, we transform them from partial-differential to ordinary-differential equations, replace $f_0 @tD$ with s , and obtain

order1EqTransf = order1Eq . 8a₁ -> Ha₁@#2D &L, f₁ -> Hf₁@#2D &L < . . f₀ @tD -> f . . f₀ -> s

$$9a_1^s @sD == \frac{1}{8} H2 \sin @sD a_0^2 a_2 + 2 \sin @3 sD a_0^2 a_2 L, f_1^s @sD == \frac{1}{8} H6 \cos @sD a_0 a_2 + 2 \cos @3 sD a_0 a_2 L =$$

Instead of replacing $f_0 @tD$, we could have symbolized it. Using **DSolve**, we find that a_1 and f_1 are given by

rule36@1D = DSolve@order1EqTransf@@1DD, a₁@sD, sD@@1DD . . C@_D -> 0 . .

8s -> f₀ @tD, a₀ -> a₀ @tD < . . a₁@arg_D -> a₁@a₀ @tD, argD

$$9a_1 @a_0 @tD, f_0 @tDD @ \frac{1}{12} H-3 \cos @f_0 @tDD a_2 a_0 @tD^2 - \cos @3 f_0 @tDD a_2 a_0 @tD^2 L =$$

rule36@2D = DSolve@order1EqTransf@@2DD, f₁@sD, sD@@1DD . . C@_D -> 0 . .

8s -> f₀ @tD, a₀ -> a₀ @tD < . . f₁@arg_D -> f₁@a₀ @tD, argD

$$9f_1 @a_0 @tD, f_0 @tDD @ \frac{1}{12} H9 \sin @f_0 @tDD a_2 a_0 @tD + \sin @3 f_0 @tDD a_2 a_0 @tDL =$$

which can be combined into

cond36b = Join@rule36@1D, rule36@2DD;

Substituting the slow- and fast- varying components of the first-order solution, **cond36a** and **cond36b**, into the second-order equations, we have

order2Eq = 8ampEq3@@2DD, phaseEq3@@2DD< •. cond36a •. cond36b

$$\begin{aligned}
 9A_2@a_0@tDD + a_2^{H0,1L}@a_0@tD, f_0@tDD == & \frac{1}{8} \int_k 2 \sin 2 f_0@tDD a_3 a_0@tD^3 + \sin 4 f_0@tDD a_3 a_0@tD^3 + \\
 & 2 a_2 \int_k \frac{1}{12} \cos f_0@tDD a_0@tD^2 H9 \sin f_0@tDD a_2 a_0@tD + \sin 3 f_0@tDD a_2 a_0@tDL + \\
 & \frac{1}{6} \sin f_0@tDD a_0@tD H-3 \cos f_0@tDD a_2 a_0@tD^2 - \cos 3 f_0@tDD a_2 a_0@tD^2 L \frac{1}{2} + \\
 & 2 a_2 \int_k \frac{1}{4} \cos 3 f_0@tDD a_0@tD^2 H9 \sin f_0@tDD a_2 a_0@tD + \sin 3 f_0@tDD a_2 a_0@tDL + \\
 & \frac{1}{6} \sin 3 f_0@tDD a_0@tD H-3 \cos f_0@tDD a_2 a_0@tD^2 - \cos 3 f_0@tDD a_2 a_0@tD^2 L \frac{1}{2} \frac{1}{2}, \\
 F_2@a_0@tDD + f_2^{H0,1L}@a_0@tD, f_0@tDD == & \frac{1}{8} \int_k 3 a_3 a_0@tD^2 + 4 \cos 2 f_0@tDD a_3 a_0@tD^2 + \cos 4 f_0@tDD \\
 & a_3 a_0@tD^2 + 6 a_2 \int_k -\frac{1}{12} \sin f_0@tDD a_0@tD H9 \sin f_0@tDD a_2 a_0@tD + \sin 3 f_0@tDD a_2 a_0@tDL + \\
 & \frac{1}{12} \cos f_0@tDD H-3 \cos f_0@tDD a_2 a_0@tD^2 - \cos 3 f_0@tDD a_2 a_0@tD^2 L \frac{1}{2} + \\
 & 2 a_2 \int_k -\frac{1}{4} \sin 3 f_0@tDD a_0@tD H9 \sin f_0@tDD a_2 a_0@tD + \sin 3 f_0@tDD a_2 a_0@tDL + \\
 & \frac{1}{12} \cos 3 f_0@tDD H-3 \cos f_0@tDD a_2 a_0@tD^2 - \cos 3 f_0@tDD a_2 a_0@tD^2 L \frac{1}{2} \frac{1}{2} =
 \end{aligned}$$

Next, we use trigonometric identities to expand the right-hand sides of the second-order equations in Fourier series and obtain

order2Eqrhs = H#@@2DD •• TrigReduce •• ExpandL & •ž order2Eq

$$\begin{aligned}
 9 - \frac{9}{32} \sin 2 f_0@tDD a_2^2 a_0@tD^3 + \frac{5}{24} \sin 4 f_0@tDD a_2^2 a_0@tD^3 + \\
 \frac{1}{96} \sin 6 f_0@tDD a_2^2 a_0@tD^3 + \frac{1}{4} \sin 2 f_0@tDD a_3 a_0@tD^3 + \frac{1}{8} \sin 4 f_0@tDD a_3 a_0@tD^3, \\
 - \frac{5}{12} a_2^2 a_0@tD^2 - \frac{3}{16} \cos 2 f_0@tDD a_2^2 a_0@tD^2 + \frac{1}{4} \cos 4 f_0@tDD a_2^2 a_0@tD^2 + \\
 \frac{1}{48} \cos 6 f_0@tDD a_2^2 a_0@tD^2 + \frac{3}{8} a_3 a_0@tD^2 + \frac{1}{2} \cos 2 f_0@tDD a_3 a_0@tD^2 + \frac{1}{8} \cos 4 f_0@tDD a_3 a_0@tD^2 =
 \end{aligned}$$

Since we are seeking an expansion valid to $O(\epsilon^2)$, we do not need to solve for a_2 and f_2 . All we need to do is to investigate the above expressions to determine the slowly varying parts and determine A_2 and F_2 . The result is

cond36c = 8A2@a0@tDD, F2@a0@tDD< -> Horder2Eqrhs •. 8Sin@_D -> 0, Cos@_D -> 0<L •• Thread

$$9A_2@a_0@tDD @ 0, F_2@a_0@tDD @ -\frac{5}{12} a_2^2 a_0@tD^2 + \frac{3}{8} a_3 a_0@tD^2 =$$

Substituting the fast-varying components of the first-order solution, **cond36b**, into the expansions for a and f and discarding terms of order higher than ϵ , we obtain, to the second approximation, that

asolF = asol •. cond36b •. ϵ^2 -> 0

$$a@tD == a_0@tD + \frac{1}{12} \epsilon H-3 \cos f_0@tDD a_2 a_0@tD^2 - \cos 3 f_0@tDD a_2 a_0@tD^2 L$$

$$\text{phisolF} = \text{phisol} \cdot \text{cond36b} \cdot e^2 \rightarrow 0$$

$$f_0(t) = \frac{1}{12} e^{\frac{1}{12} t} \left(9 \sin(f_0(t)) a_2 a_0(t) + \sin(3 f_0(t)) a_2 a_0(t) + f_0(t) \right)$$

Substituting the solw-varying components of the first- and second-order solutions, **cond36a** and **cond36c**, into **a0Eq** and **phi0Eq** yields

$$\text{a0Eq1} = \text{a0Eq} \cdot \text{cond36a} \cdot \text{cond36c}$$

$$a_0(t) = 0$$

$$\text{phi0Eq1} = \text{phi0Eq} \cdot \text{cond36a} \cdot \text{cond36c}$$

$$f_0(t) = 1 + e^2 \left(\frac{5}{12} a_2^2 a_0(t)^2 + \frac{3}{8} a_3 a_0(t)^2 \right)$$

It follows from the above equations that $a_0 = \text{constant}$ and

$$\text{phi0Rule} = \text{DSolve}[\text{phi0Eq1} \cdot a_0(t) \rightarrow a_0, f_0(t), t] \cdot C[1] \rightarrow b_0$$

$$9 f_0(t) = 1 + e^2 \left(\frac{5}{12} a_2^2 a_0^2 + \frac{3}{8} a_3 a_0^2 \right) + b_0$$

where b_0 is a constant. Substituting these expansions into **eq35a**, we obtain the following second-order expansion:

$$\text{eq36a} = u(t) = e a(t) \cos(f_0(t)) \cdot \text{H8asolF}, \text{phisolF} \cdot \text{Equal} \rightarrow \text{RuleL}$$

$$u(t) = e \cos\left(\frac{1}{12} e^{\frac{1}{12} t} \left(9 \sin(f_0(t)) a_2 a_0(t) + \sin(3 f_0(t)) a_2 a_0(t) + f_0(t) \right)\right) a_0(t) + \frac{1}{12} e^{\frac{1}{12} t} \left(9 \cos(f_0(t)) a_2 a_0(t)^2 - \cos(3 f_0(t)) a_2 a_0(t)^2 \right)$$

To compare the present solution with those obtained by using the method of multiple scales and the Lindstedt-Poincaré technique, we expand the circular functions in **eq36a** for small e about f_0 and obtain

$$u(t) = \text{HSeries}[\text{eq36a} @ 2 \text{DD}, \{e, 0, 2\} \cdot \text{Normal} \cdot \text{TrigReduce} \cdot \text{ExpandL}]$$

$$u(t) = e \cos(f_0(t)) a_0(t) - \frac{1}{2} e^2 a_2 a_0(t)^2 + \frac{1}{6} e^2 \cos(2 f_0(t)) a_2 a_0(t)^2$$

This expansion and **phi0Rule** are in full agreement with those obtained by using the method of multiple scales.

3.7 The Krylov-Bogoliubov-Mitropolsky Technique

In this section, we describe a variant of the generalized method of averaging, namely the Krylov-Bogoliubov-Mitropolsky technique. It is often referred to as the **asymptotic method**. When the nonlinear terms are neglected, the solution of **eq31g** is

$$\text{linearSol} = u = e a \cos(t) + b;$$

where a and b are constants and ϵ is a small dimensionless parameter that is a measure of the amplitude. When the nonlinear terms are included, we consider **linearSol** to be the first term in an approximate solution of **eq31g** but with slowly varying rather than constant a and b . Moreover, we introduce the fast scale $\tau = t + b$ and use a to represent the slow variations. Thus, we seek a second-order uniform expansion of the solution of **eq31g** in the form

$$\mathbf{uSol} = \mathbf{u} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} == \epsilon \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} + \text{Sum} \mathbf{A} \epsilon^i \mathbf{u}_i @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD}, \quad \mathbf{8i}, \quad 2, \quad 3 < \epsilon$$

$$\mathbf{u} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} == \epsilon \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} + \epsilon^2 \mathbf{u}_2 @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + \epsilon^3 \mathbf{u}_3 @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD}$$

Again, terms of order ϵ^3 have been included in the expansion, but these terms will not be included in the final result. Since a and b are slowly varying functions of t , we express them in power series of ϵ in terms of a as

$$\mathbf{apEq} = \mathbf{a} ' @ \mathbf{tD} == \text{Sum} \mathbf{A} \epsilon^i \mathbf{A}_i @ \mathbf{a} @ \mathbf{tDD}, \quad \mathbf{8i}, \quad 2 < \epsilon$$

$$\mathbf{a}^\epsilon @ \mathbf{tD} == \epsilon \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} + \epsilon^2 \mathbf{A}_2 @ \mathbf{a} @ \mathbf{tDD}$$

$$\mathbf{phipEq} = \mathbf{f} ' @ \mathbf{tD} == 1 + \text{Sum} \mathbf{A} \epsilon^i \mathbf{F}_i @ \mathbf{a} @ \mathbf{tDD}, \quad \mathbf{8i}, \quad 2 < \epsilon$$

$$\mathbf{f}^\epsilon @ \mathbf{tD} == 1 + \epsilon \mathbf{F}_1 @ \mathbf{a} @ \mathbf{tDD} + \epsilon^2 \mathbf{F}_2 @ \mathbf{a} @ \mathbf{tDD}$$

In what follows, we need $\mathbf{a}'' @ \mathbf{tD}$ and $\mathbf{f}'' @ \mathbf{tD}$. To this end, we differentiate **apEq** and **phipEq** with respect to t , use these equations to express $\mathbf{a}^\epsilon @ \mathbf{tD}$ and $\mathbf{f}^\epsilon @ \mathbf{tD}$ in terms of the $\mathbf{A}_i @ \mathbf{a} @ \mathbf{tDD}$ and $\mathbf{F}_i @ \mathbf{a} @ \mathbf{tDD}$, discard terms of order higher than ϵ^2 , and obtain

$$\mathbf{appEq} = \mathbf{HD} @ \mathbf{apEq}, \quad \mathbf{tD} \bullet \mathbf{HapEq} \bullet \mathbf{Equal} \rightarrow \mathbf{RuleL} \bullet \mathbf{ExpandAllL} \bullet \epsilon^{n_*}; n > 2 \rightarrow 0$$

$$\mathbf{a}'' @ \mathbf{tD} == \epsilon^2 \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{A}_1^\epsilon @ \mathbf{a} @ \mathbf{tDD}$$

$$\mathbf{phippEq} = \mathbf{HD} @ \mathbf{phipEq}, \quad \mathbf{tD} \bullet \mathbf{HapEq} \bullet \mathbf{Equal} \rightarrow \mathbf{RuleL} \bullet \mathbf{ExpandAllL} \bullet \epsilon^{n_*}; n > 2 \rightarrow 0$$

$$\mathbf{f}'' @ \mathbf{tD} == \epsilon^2 \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{F}_1^\epsilon @ \mathbf{a} @ \mathbf{tDD}$$

Thus, this method can be viewed as a multiple scales procedure with a and τ being the scales.

Substituting the assumed expansion for $\mathbf{u} @ \mathbf{tD}$, expressed in function form, and the expansions for the derivatives of $\mathbf{a} @ \mathbf{tD}$ and $\mathbf{f} @ \mathbf{tD}$ into **eq31g**, expanding the result for small ϵ , and keeping terms up to order ϵ^3 , we have

$$\mathbf{eq37a} = \mathbf{Heq31g} \bullet \mathbf{u} \rightarrow \mathbf{Function} @ \mathbf{t}, \quad \mathbf{uSol} @ \mathbf{2DD} \bullet \mathbf{EvaluateD} \bullet$$

$$\mathbf{H8apEq}, \quad \mathbf{phipEq}, \quad \mathbf{appEq}, \quad \mathbf{phippEq} < \bullet \mathbf{Equal} \rightarrow \mathbf{RuleL} \bullet \mathbf{ExpandAllL} \bullet \epsilon^{n_*}; n > 3 \rightarrow 0$$

$$\begin{aligned} & \epsilon^2 \mathbf{a} @ \mathbf{tD}^2 \cos @ \mathbf{f} @ \mathbf{tDD}^2 \mathbf{a}_2 + \epsilon^3 \mathbf{a} @ \mathbf{tD}^3 \cos @ \mathbf{f} @ \mathbf{tDD}^3 \mathbf{a}_3 - 2 \epsilon^2 \sin @ \mathbf{f} @ \mathbf{tDD} \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} - \\ & 2 \epsilon^3 \sin @ \mathbf{f} @ \mathbf{tDD} \mathbf{A}_2 @ \mathbf{a} @ \mathbf{tDD} + \epsilon^2 \mathbf{u}_2 @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + 2 \epsilon^3 \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} \mathbf{a}_2 \mathbf{u}_2 @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + \\ & \epsilon^3 \mathbf{u}_3 @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} - 2 \epsilon^2 \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} \mathbf{F}_1 @ \mathbf{a} @ \mathbf{tDD} - 2 \epsilon^3 \sin @ \mathbf{f} @ \mathbf{tDD} \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{F}_1 @ \mathbf{a} @ \mathbf{tDD} - \\ & \epsilon^3 \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} \mathbf{F}_1 @ \mathbf{a} @ \mathbf{tDD}^2 - 2 \epsilon^3 \mathbf{a} @ \mathbf{tD} \cos @ \mathbf{f} @ \mathbf{tDD} \mathbf{F}_2 @ \mathbf{a} @ \mathbf{tDD} + \epsilon^3 \cos @ \mathbf{f} @ \mathbf{tDD} \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{A}_1^\epsilon @ \mathbf{a} @ \mathbf{tDD} - \\ & \epsilon^3 \mathbf{a} @ \mathbf{tD} \sin @ \mathbf{f} @ \mathbf{tDD} \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{F}_1^\epsilon @ \mathbf{a} @ \mathbf{tDD} + \epsilon^2 \mathbf{u}_2^{\text{H0}, 2\text{L}} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + \\ & 2 \epsilon^3 \mathbf{F}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{u}_2^{\text{H0}, 2\text{L}} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + \epsilon^3 \mathbf{u}_3^{\text{H0}, 2\text{L}} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} + 2 \epsilon^3 \mathbf{A}_1 @ \mathbf{a} @ \mathbf{tDD} \mathbf{u}_2^{\text{H1}, 1\text{L}} @ \mathbf{a} @ \mathbf{tD}, \quad \mathbf{f} @ \mathbf{tDD} == 0 \end{aligned}$$

Equating coefficients of like powers of ϵ in **eq37a** yields


```
eqEps = TableACoefficientAeq37a@@1DD, eiE == 0, 8i, 2, 3<E •. f_@tD -> f
9a2 Cos@fD2 a2 - 2 Sin@fD A1@aD + u2@a, fD - 2 a Cos@fD F1@aD + u2H0,2L@a, fD == 0,
a3 Cos@fD3 a3 - 2 Sin@fD A2@aD + 2 a Cos@fD a2 u2@a, fD + u3@a, fD -
2 Sin@fD A1@aD F1@aD - a Cos@fD F1@aD2 - 2 a Cos@fD F2@aD + Cos@fD A1@aD A1ε@aD -
a Sin@fD A1@aD F1ε@aD + 2 F1@aD u2H0,2L@a, fD + u3H0,2L@a, fD + 2 A1@aD u2H1,1L@a, fD == 0=
```

Next, we use trigonometric identities to expand the nonhomogeneous terms in **eqEps[[1]]** in a Fourier series and obtain

```
order2Eq = TrigReduce •ž eqEps@@1DD
1
--- | a2 a2 + a2 Cos@2 fD a2 - 4 Sin@fD A1@aD + 2 u2@a, fD - 4 a Cos@fD F1@aD + 2 u2H0,2L@a, fD == 0
2
```

Eliminating the terms that produce secular terms in u_2 from **order2Eq** demands that

```
rule37a =
Solve@Coefficient@order2Eq@@1DD, 8Cos@fD, Sin@fD<D == 0 •• Thread, 8F1@aD, A1@aD<D@@1DD
8F1@aD ® 0, A1@aD ® 0<
```

Then, the solution of the resulting **order2Eq** can be written as

```
rule37b = DSolve@order2Eq •. rule37a, u2@a, fD, 8a, f<D@@1DD •. C@_D -> H0 &L •• Simplify
9u2@a, fD ® 1
--- a2 H - 3 + Cos@2 fDL a2=
6
```

which can be expressed in function form as

```
rule37c = 8u2 -> Function@8a, f<, rule37b@@1, 2DD •• EvaluateD<
9u2 ® FunctionA8a, f<, 1
--- a2 H - 3 + Cos@2 fDL a2E=
6
```

Substituting **rule37a** and **rule37c** into **eqEps[[2]]**, we obtain

```
eq37b = eqEps@@2DD •. rule37a •. rule37c
1
--- a3 Cos@fD H - 3 + Cos@2 fDL a22 + a3 Cos@fD3 a3 -
3
2 Sin@fD A2@aD + u3@a, fD - 2 a Cos@fD F2@aD + u3H0,2L@a, fD == 0
```

Eliminating the terms that produce secular terms in u_3 from **eq37b** demands that

```
eq37c = Coefficient@eq37b@@1DD •• TrigReduce, 8Cos@fD, Sin@fD<D == 0 •• Thread
: - 5
--- a3 a22 + 3 a3 a3
4 - 2 a F2@aD == 0, - 2 A2@aD == 0>
```

or

```
rule37d = Solve@eq37c, 8A2@aD, F2@aD<D@@1DD
```

$$9A_2@aD \otimes 0, F_2@aD \otimes -\frac{1}{24} a^2 H_{10} a_2^2 - 9 a_3 L =$$

Substituting **rule37a** and **rule37d** into **apEq** and **phipEq** yields

```
apEq •. Hrule37a~Join~rule37d •. a -> a@tDL
```

$$a^{\epsilon}@tD == 0$$

```
phipEq •. Hrule37a~Join~rule37d •. a -> a@tDL •• ExpandAll
```

$$f^{\epsilon}@tD == 1 - \frac{5}{12} e^2 a@tD^2 a_2^2 + \frac{3}{8} e^2 a@tD^2 a_3$$

which are in full agreement with those obtained by using the method of multiple scales and the generalized method of averaging.

à 3.8 The Method of Normal Forms

In this section, we use the method of normal forms to determine a second-order uniform expansion of the solution of **eq31g**. To this end, we start with the corresponding first-order complex-valued equation

$$\begin{aligned} \text{eq343a} = z^{\epsilon}@tD = & \frac{1}{2} \mathbf{I} a_2 z@tD^2 + \frac{1}{2} \mathbf{I} a_3 z@tD^3 + \mathbf{I} a_2 z@tD \dot{z}@tD + \\ & \frac{3}{2} \mathbf{I} a_3 z@tD^2 \dot{z}@tD + \frac{1}{2} \mathbf{I} a_2 \dot{z}@tD^2 + \frac{3}{2} \mathbf{I} a_3 z@tD \dot{z}@tD^2 + \frac{1}{2} \mathbf{I} a_3 \dot{z}@tD^3; \end{aligned}$$

According to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = 8h@tD, h̄@tD<;
```

```
zetaRule = 9
```

```
z -> FunctionAt, e h@tD + SumAe^j h_j z̄ z basicTerms, 8j, 2, 3<E •• EvaluateE,
```

```
z̄ -> FunctionAt, e h̄@tD + SumAe^j h̄_j z̄ z basicTerms, 8j, 2, 3<E •• EvaluateE=
```

```
8z @ Function@t, e h@tD + e^2 h_2@h@tD, h̄@tDD + e^3 h_3@h@tD, h̄@tDDD,
```

```
z̄ @ Function@t, e h̄@tD + e^2 h̄_2@h@tD, h̄@tDD + e^3 h̄_3@h@tD, h̄@tDDD<
```

that results in the simplest possible equation

```
etaRule = 9h^{\epsilon}@tD -> \mathbf{I} h@tD + SumAe^j g_j@tD, 8j, 2<E=
```

$$8h^{\epsilon}@tD \otimes \mathbf{I} h@tD + e g_1@tD + e^2 g_2@tD<$$

where the overbar denotes the complex conjugate

```
conjugateRule = 8h -> h̄, h̄ -> h, g -> ḡ, ḡ -> g, Complex@0, n_D -> Complex@0, -nD<;
```

Substituting the expansion for z , the **zetaRule**, into **eq343a**, using **etaRule**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we have

```
eq38a =
  Heq343a . zetaRule . etaRule . HetaRule . conjugateRuleL . ExpandAll . en-*; n>3 -> 0;
```

Equating coefficients of like powers of ϵ in **eq38a** yields

```
eqEps = TableACoefficientASubtract @@ eq38a, eiE == 0, 8i, 2, 3<E
9 -  $\frac{1}{2} I a_2 h @ t D^2 - I a_2 h @ t D \dot{h} @ t D - \frac{1}{2} I a_2 \dot{h} @ t D^2 + g_1 @ t D -$ 
 $I h_2 @ h @ t D, \dot{h} @ t D D - I \dot{h} @ t D h_2^{H0, 1L} @ h @ t D, \dot{h} @ t D D + I h @ t D h_2^{H1, 0L} @ h @ t D, \dot{h} @ t D D == 0,$ 
 $-\frac{1}{2} I a_3 h @ t D^3 - \frac{3}{2} I a_3 h @ t D^2 \dot{h} @ t D - \frac{3}{2} I a_3 h @ t D \dot{h} @ t D^2 - \frac{1}{2} I a_3 \dot{h} @ t D^3 + g_2 @ t D -$ 
 $I a_2 h @ t D h_2 @ h @ t D, \dot{h} @ t D D - I a_2 \dot{h} @ t D h_2 @ h @ t D, \dot{h} @ t D D - I h_3 @ h @ t D, \dot{h} @ t D D -$ 
 $I a_2 h @ t D h_2 @ h @ t D, \dot{h} @ t D D - I a_2 \dot{h} @ t D h_2 @ h @ t D, \dot{h} @ t D D + g_1 @ t D h_2^{H0, 1L} @ h @ t D, \dot{h} @ t D D -$ 
 $I \dot{h} @ t D h_3^{H0, 1L} @ h @ t D, \dot{h} @ t D D + g_1 @ t D h_2^{H1, 0L} @ h @ t D, \dot{h} @ t D D + I h @ t D h_3^{H1, 0L} @ h @ t D, \dot{h} @ t D D == 0 =$ 
```

Y Second-Order Solution

We choose the h_i to eliminate as many terms from **eqEps**, thereby reducing them into their simplest possible form. It turns out that we can eliminate all nonresonance terms. To determine the resonance terms in **eqEps[[1]]**, we note that all of the possible forms of its nonhomogeneous terms are

```
possibleQTerms = Outer@Times, basicTerms, basicTermsD . Flatten . Union
9 h @ t D^2, h @ t D \dot{h} @ t D, \dot{h} @ t D^2 =
```

It follows from the linear parts of the **etaRule** that

```
form = 9 h @ t D -> EI t, \dot{h} @ t D -> E-I t =;
```

Hence, the possible resonance terms are given by

```
ResonantQTerm = IE-I t possibleQTerms . form . E-t -> 0 M possibleQTerms . Union . Rest
8<
```

Consequently, there are no resonance terms in the second-order problem and the nonresonance terms can be defined as

```
NRQT = Complement@possibleQTerms, ResonantQTermD
9 h @ t D^2, h @ t D \dot{h} @ t D, \dot{h} @ t D^2 =
```

We associate with each of them a coefficient according to

coeffsQ = Table@G_j, 8j, Length@NRQTD<D

8G₁, G₂, G₃<

Therefore, h_2 and its complex conjugate have the form

hFormQ = 9h₂ -> HEvaluate@coeffsQ.NRQT •. Thread@basicTerms -> 8#1, #2<DD &L,
h₂ -> HEvaluate@coeffsQ.NRQT •. conjugateRule •. Thread@basicTerms -> 8#1, #2<DD &L=
 $8h_2 \otimes H\#1^2 G_1 + \#1 \#2 G_2 + \#2^2 G_3 \&L, \dot{h}_2 \otimes H\#2^2 G_1 + \#1 \#2 G_2 + \#1^2 G_3 \&L<$

Substituting for h_2 in **eqEps[[1]]**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting equations for the G_i , we obtain

coeffsQRule = Solve@Coefficient@eqEps@@@1, 1DD •. hFormQ, NRQTD == 0 •• Thread, coeffsQD@@1DD
 $:G_1 \otimes \frac{a_2}{2}, G_2 \otimes -a_2, G_3 \otimes -\frac{a_2}{6}>$

We choose g_1 to eliminate the resonance terms in **eqEps[[1,1]]** according to

gRuleQ = g₁@tD -> - Coefficient@eqEps@@@1, 1DD, ResonantQTermD.ResonantQTerm
 $g_1@tD \otimes 0$

In this case, there are on resonance terms and hence $g_1 = 0$.

Y Third-Order Equations

Substituting the second-order results into the third-order equation yields

order3expr =
eqEps@@2, 1DD •. hFormQ •. coeffsQRule •. gRuleQ •. HgRuleQ •. conjugateRuleL •• Expand
 $-\frac{1}{3} \int a_2^2 h@tD^3 - \frac{1}{2} \int a_3 h@tD^3 + \frac{5}{3} \int a_2^2 h@tD^2 \dot{h}@tD - \frac{3}{2} \int a_3 h@tD^2 \dot{h}@tD +$
 $\frac{5}{3} \int a_2^2 h@tD \dot{h}@tD^2 - \frac{3}{2} \int a_3 h@tD \dot{h}@tD^2 - \frac{1}{3} \int a_2^2 \dot{h}@tD^3 - \frac{1}{2} \int a_3 \dot{h}@tD^3 + g_2@tD -$
 $\int h_3@h@tD, \dot{h}@tDD - \int \dot{h}@tD h_3^{H0,1L}@h@tD, \dot{h}@tDD + \int h@tD h_3^{H1,0L}@h@tD, \dot{h}@tDD$

The nonhomogeneous terms in **order3expr** are proportional to

possibleCTerms = Outer@Times, possibleQTerms, basicTermsD •• Flatten •• Union
 $9h@tD^3, h@tD^2 \dot{h}@tD, h@tD \dot{h}@tD^2, \dot{h}@tD^3=$

Next, we determine the resonance terms according to

```

ResonantCTerm = IE-I t possibleCTerms •. form •. E-t -> OM possibleCTerms •• Union •• Rest
8h@tD2 ḣ@tD<

```

Then, the nonresonance terms are the complement of these resonance terms; that is,

```

NRCT = Complement@possibleCTerms, ResonantCTermD
9h@tD3, h@tD ḣ@tD2, ḣ@tD3=

```

Again, we associate coefficients with these terms as follows:

```

coeffsC = Table@Lj, 8j, Length@NRCTD<D
8L1, L2, L3<

```

Consequently, h_3 and its complex conjugate have the forms

```

hFormC = 9h3 -> HEvaluate@coeffsC.NRCT •. Thread@basicTerms -> 8#1, #2<DD &L,
ḣ3 -> HEvaluate@coeffsC.NRCT •. conjugateRule •. Thread@basicTerms -> 8#1, #2<DD &L=
8h3 ® H#13 L1 + #1 #22 L2 + #23 L3 &L, ḣ3 ® H#23 L1 + #12 #2 L2 + #13 L3 &L<

```

Substituting for h_3 into **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the L_i , we obtain

```

coeffsCRule = Solve@Coefficient@order3expr •. hFormC, NRCTD == 0 •• Thread, coeffsCD@@1DD
9L1 ®  $\frac{1}{12} H^2 a_2^2 + 3 a_3 L$ , L2 ®  $\frac{1}{12} H^{10} a_2^2 - 9 a_3 L$ , L3 ®  $\frac{1}{24} H - 2 a_2^2 - 3 a_3 L$ =

```

Choosing g_2 to eliminate the resonance terms from **order3expr** yields

```

gRuleC = g2@tD -> - Coefficient@order3expr, ResonantCTermD.ResonantCTerm
g2@tD ® -  $\int_k \frac{5 I a_2^2}{3} - \frac{3 I a_3 y}{2} h@tD^2 ḣ@tD$ 

```

Combining **etaRule**, **gRuleQ**, and **gRuleC** and letting $h = A@tD E^{I t}$, we obtain the modulation equation

```

moduEq = 2 I E-I t Hhε@tD - Hhε@tD •. etaRule •. gRuleQ •. gRuleCLL == 0 •.
9h -> IA@#D EI # &M, ḣ -> I A@#D E-I # &M= •• ExpandAll
-  $\frac{10}{3} e^2 A@tD^2 a_2^2 \dot{A}@tD + 3 e^2 A@tD^2 a_3 \dot{A}@tD + 2 I A^ε@tD == 0$ 

```

which is in full agreement with that obtained by using the other techniques.

Chapter 4

Forced Oscillations of the Duffing Equation

Off@General::spell1D

In contrast with the preceding chapters, which deal with free oscillations, this chapter and the next two deal with forced oscillations. We consider

$$\text{FDuffingEq} = u''@tD + w^2 u@tD + 2 m u^{\epsilon}@tD + d u@tD^2 + a u@tD^3 == F \text{Cos}@W tD;$$

where m is a positive constant. In this chapter, we determine second-order solutions to **FDuffingEq** beginning with the straightforward expansion in the next section. We investigate this straightforward expansion and determine under what conditions it breaks down. This leads to the so-called resonance values of W . In Section 4.2, we use the method of multiple scales to determine second-order uniform expansions of the solutions of **FDuffingEq** for all resonance cases, including the effect of light viscous damping. In Sections 4.3 and 4.4, we use the generalized method of averaging and the method of normal forms, respectively, to determine second-order uniform expansions for the case of subharmonic resonance of order one-half; that is, $W \gg 2w$.

à 4.1 Straightforward Expansion

We seek a straightforward expansion for the solutions of **FDuffingEq** in the form

$$\text{uSol} = u \rightarrow \text{ISumAe}^i u_i@tD, \text{ 8i, } 3 < E \&M;$$

Substituting **uSol** into **FDuffingEq**, letting $F = e f$ and $m = e m$, expanding the result for small e , and discarding terms of order higher than e^3 , we have

$$\begin{aligned} \text{eq41a} = & \text{HFDuffingEq} \cdot \text{uSol} \cdot \text{8F} \rightarrow e f, m \rightarrow e m \cdot \cdot \text{ExpandA11L} \cdot \cdot e^{n \cdot}; n > 3 \rightarrow 0 \\ & e w^2 u_1@tD + d e^2 u_1@tD^2 + a e^3 u_1@tD^3 + e^2 w^2 u_2@tD + 2 d e^3 u_1@tD u_2@tD + \\ & e^3 w^2 u_3@tD + 2 e^2 m u_1^{\epsilon}@tD + 2 e^3 m u_2^{\epsilon}@tD + e u_1^2@tD + e^2 u_2^2@tD + e^3 u_3^2@tD == f e \text{Cos}@t WD \end{aligned}$$

Equating coefficients of like powers of e on both sides yields

$$\text{eqEpsa} = \text{CoefficientList@Subtract} \check{z} \check{z} \text{ eq41a, } eD == 0 \cdot \cdot \text{Thread} \cdot \cdot \text{Rest}$$

$$\begin{aligned} 8 - f \text{Cos}@t WD + w^2 u_1@tD + u_1^2@tD &== 0, d u_1@tD^2 + w^2 u_2@tD + 2 m u_1^{\epsilon}@tD + u_2^2@tD == 0, \\ a u_1@tD^3 + 2 d u_1@tD u_2@tD + w^2 u_3@tD + 2 m u_2^{\epsilon}@tD + u_3^2@tD &== 0 < \end{aligned}$$

The particular solution of the first-order equation, linear and undamped problem, can be expressed as

```
sol1p = DSolve@eqEpsa@@1DD, u1@tD, tD@@1DD •. C@_D -> 0 •• TrigToExp •• Simplify
```

$$:u_1@tD \otimes - \frac{E^{-I t W} H_1 + E^{2 I t W} L f}{2 H - w^2 + W^2 L} >$$

We note that **sol1p** consists of a term whose denominator is very small when $W \gg w$. Such a term is called a **small-divisor term**. Moreover, $u@tD$ tends to infinity as $W \otimes w$, and the excitation is referred to as a **resonance excitation**. Because the small-divisor term appears in the first-order problem, we speak of a **primary** or **main resonance**. When the small-divisor terms appear in the higher-order problems, we speak of **secondary resonances**. In the case of primary resonance, the scaling $F \otimes e f$ is not valid. Physically, as soon as the motion becomes large, the damping and nonlinearity are activated to counter the effect of the resonance. Consequently, to obtain a uniform expansion in this case, we rescale F and m , as discussed in Section 4.2, so that the influence of the damping and nonlinearity balances the influence of the primary resonance.

When W is away from w , we add the homogeneous solution to **sol1p** and obtain the general solution of the first-order problem as

$$fRule = 9f \rightarrow 2 L I W^2 - w^2 M;$$

```
sol1 = u1 -> FunctionAt, A E^{I w t} + \dot{A} E^{-I w t} + sol1p@@1, 2DD •. fRule •• Expand •• EvaluateE
```

$$u_1 \otimes Function@t, A E^{I t w} - E^{-I t W} L - E^{I t W} L + E^{-I t w} \dot{A} D$$

where A is a complex-valued constant. Substituting **sol1** into the second-order equation, **eqEpsa[2]**, yields

```
order2Eqa = u2^2@tD + w^2 u2@tD - # & •ž eqEpsa@@2DD •. sol1 •• ExpandAll •• Reverse
```

$$\begin{aligned} & w^2 u_2@tD + u_2^2@tD = \\ & -A^2 E^{2 I t w} d + 2 A E^{I t w - I t W} d L + 2 A E^{I t w + I t W} d L - 2 d L^2 - E^{-2 I t W} d L^2 - E^{2 I t W} d L^2 - 2 I A E^{I t w} m w - \\ & 2 I E^{-I t W} L m W + 2 I E^{I t W} L m W - 2 A d \dot{A} + 2 E^{-I t w - I t W} d L \dot{A} + 2 E^{-I t w + I t W} d L \dot{A} + 2 I E^{-I t w} m w \dot{A} - E^{-2 I t w} d \dot{A}^2 \end{aligned}$$

Solving for the particular solution of **order2Eqa**, we obtain

```
sol2p = Hu2@tD •. DSolve@order2Eqa@@1DD == #, u2@tD, tD@@1DD •. C@_D -> 0 •• TrigToExp •• SimplifyL & •ž order2Eqa@@2DD
```

$$\begin{aligned} & \frac{A^2 E^{2 I t w} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{A E^{I t w} m H I + 2 t w L}{2 w} + \frac{2 A E^{I t H w - W L} d L}{H^2 w - W L W} - \\ & \frac{2 A E^{I t H w + W L} d L}{W H^2 w + W L} + \frac{2 I E^{I t w} L m W}{w^2 - W^2} + \frac{2 I E^{-I t w} L m W}{-w^2 + W^2} + \frac{E^{-2 I t W} d L^2}{-w^2 + 4 W^2} + \frac{E^{2 I t W} d L^2}{-w^2 + 4 W^2} - \\ & \frac{2 A d \dot{A}}{w^2} - \frac{E^{-I t w} m H - I + 2 t w L \dot{A}}{2 w} + \frac{2 E^{-I t H w - W L} d L \dot{A}}{H^2 w - W L W} - \frac{2 E^{-I t H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I t w} d \dot{A}^2}{3 w^2} \end{aligned}$$

Clearly, **sol2p** breaks down because it contains secular terms proportional to the damping coefficient m . Moreover, **sol2p** breaks down when $W \gg 2w$, $W \gg w \cdot 2$, or $W \gg 0$ because they produce small-divisor terms and hence secondary resonances. As discussed in Section 4.2, the case $W \gg 2w$ is called **subharmonic resonance of order one-half** and the case $W \gg w \cdot 2$ is called **superharmonic resonance of order two**.

When W is away from w , $2w$, $w/2$, or 0 , **sol2p** is valid only if we rescale the damping term so that it first appears in the third-order equation; that is, $m \rightarrow e^2 m$. Using this scaling, we rewrite **eq41a** as

$$\mathbf{eq41b} = \mathbf{eq41a} \cdot m \rightarrow e m \cdot e^4 \rightarrow 0$$

$$e w^2 u_1 @ t D + d e^2 u_1 @ t D^2 + a e^3 u_1 @ t D^3 + e^2 w^2 u_2 @ t D + 2 d e^3 u_1 @ t D u_2 @ t D + e^3 w^2 u_3 @ t D + 2 e^3 m u_1 @ t D + e u_1 @ t D + e^2 u_2 @ t D + e^3 u_3 @ t D == f e \cos @ t W D$$

$$\mathbf{eqEpsb} = \mathbf{CoefficientList@Subtract} \, \mathbf{\&\&} \, \mathbf{eq41b}, e D == 0 \cdot \cdot \mathbf{Thread} \cdot \cdot \mathbf{Rest}$$

$$8 - f \cos @ t W D + w^2 u_1 @ t D + u_1 @ t D^2 == 0, d u_1 @ t D^2 + w^2 u_2 @ t D + u_2 @ t D^2 == 0, a u_1 @ t D^3 + 2 d u_1 @ t D u_2 @ t D + w^2 u_3 @ t D + 2 m u_1 @ t D + u_3 @ t D == 0$$

Comparing **eqEpsb** to **eqEpsa**, we immediately obtain its solution from **sol2p** as

$$\mathbf{sol2pNew} = \mathbf{sol2p} \cdot m \rightarrow 0$$

$$\frac{A^2 E^{2 I t w} d}{3 w^2} - \frac{2 d L^2}{w^2} + \frac{2 A E^{I t H w - W L} d L}{H^2 w - W L W} - \frac{2 A E^{I t H w + W L} d L}{W H^2 w + W L} + \frac{E^{-2 I t w} d L^2}{-w^2 + 4 W^2} + \frac{E^{2 I t w} d L^2}{-w^2 + 4 W^2} - \frac{2 A d \dot{A}}{w^2} + \frac{2 E^{-I t H w - W L} d L \dot{A}}{H^2 w - W L W} - \frac{2 E^{-I t H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I t w} d \dot{A}^2}{3 w^2}$$

Hence, the general solution of **eqEpsb[[2]]** can be written as

$$\mathbf{sol2New} = u_2 \rightarrow \mathbf{FunctionAt}, B E^{I t w} + \dot{B} E^{-I t w} + \mathbf{sol2pNew} \cdot \cdot \mathbf{EvaluateE}$$

$$u_2 @ \mathbf{FunctionBt}, B E^{I t w} + \frac{A^2 E^{2 I t w} d}{3 w^2} - \frac{2 d L^2}{w^2} + \frac{2 A E^{I t H w - W L} d L}{H^2 w - W L W} - \frac{2 A E^{I t H w + W L} d L}{W H^2 w + W L} + \frac{E^{-2 I t w} d L^2}{-w^2 + 4 W^2} + \frac{E^{2 I t w} d L^2}{-w^2 + 4 W^2} - \frac{2 A d \dot{A}}{w^2} + \frac{2 E^{-I t H w - W L} d L \dot{A}}{H^2 w - W L W} - \frac{2 E^{-I t H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I t w} d \dot{A}^2}{3 w^2} + E^{-I t w} \dot{B} F$$

where B is a complex-valued constant.

Substituting **sol1** and **sol2New** into **eqEpsb[[3]]** yields

order3Eqb = $u_3^2 @ tD + w^2 u_3 @ tD - \# \& \bullet \checkmark$ eqEpsb@@3DD •. sol1 •. sol2New •• ExpandAll •• Reverse

$$\begin{aligned}
 & w^2 u_3 @ tD + u_3^2 @ tD == -A^3 E^{3Itw} a - 2ABE^{2Itw} d + 3A^2 E^{2Itw-Itw} aL + 3A^2 E^{2Itw+Itw} aL + \\
 & 2BE^{Itw-Itw} dL + 2BE^{Itw+Itw} dL - 6AE^{Itw} aL^2 - 3AE^{Itw-2Itw} aL^2 - 3AE^{Itw+2Itw} aL^2 + \\
 & 3E^{-Itw} aL^3 + 3E^{Itw} aL^3 + E^{-3Itw} aL^3 + E^{3Itw} aL^3 - \frac{2A^3 E^{3Itw} d^2}{3w^2} + \frac{2A^2 E^{2Itw-Itw} d^2 L}{3w^2} + \\
 & \frac{2A^2 E^{2Itw+Itw} d^2 L}{3w^2} + \frac{4AE^{Itw} d^2 L^2}{w^2} - \frac{4E^{-Itw} d^2 L^3}{w^2} - \frac{4E^{Itw} d^2 L^3}{w^2} - 2IAE^{Itw} mW - \\
 & 2IE^{-Itw} LmW + 2IE^{Itw} LmW - \frac{4A^2 E^{2Itw-Itw} d^2 L}{2wW - W^2} + \frac{4AE^{Itw} d^2 L^2}{2wW - W^2} + \frac{4AE^{Itw-2Itw} d^2 L^2}{2wW - W^2} + \\
 & \frac{4A^2 E^{2Itw+Itw} d^2 L}{2wW + W^2} - \frac{4AE^{Itw} d^2 L^2}{2wW + W^2} - \frac{4AE^{Itw+2Itw} d^2 L^2}{2wW + W^2} - \frac{2AE^{Itw+2Itw} d^2 L^2}{-w^2 + 4W^2} + \frac{2E^{Itw} d^2 L^3}{-w^2 + 4W^2} + \\
 & \frac{2E^{3Itw} d^2 L^3}{-w^2 + 4W^2} - \frac{2AE^{Itw} d^2 L^2}{-E^{2Itw} w^2 + 4E^{2Itw} W^2} + \frac{2E^{-Itw} d^2 L^3}{-E^{2Itw} w^2 + 4E^{2Itw} W^2} + \frac{2E^{Itw} d^2 L^3}{-E^{2Itw} w^2 + 4E^{2Itw} W^2} - \\
 & 3A^2 E^{Itw} a \dot{A} - 2Bd \dot{A} + 6AE^{-Itw} aL \dot{A} + 6AE^{Itw} aL \dot{A} - 6E^{-Itw} aL^2 \dot{A} - 3E^{-Itw-2Itw} aL^2 \dot{A} - \\
 & 3E^{-Itw+2Itw} aL^2 \dot{A} + \frac{10A^2 E^{Itw} d^2 \dot{A}}{3w^2} - \frac{4AE^{-Itw} d^2 L \dot{A}}{w^2} - \frac{4AE^{Itw} d^2 L \dot{A}}{w^2} + \frac{4E^{-Itw} d^2 L^2 \dot{A}}{w^2} + \\
 & 2IE^{-Itw} mW \dot{A} - \frac{4AE^{-Itw} d^2 L \dot{A}}{2wW - W^2} + \frac{4AE^{Itw} d^2 L \dot{A}}{2wW + W^2} - \frac{2E^{-Itw+2Itw} d^2 L^2 \dot{A}}{-w^2 + 4W^2} - \frac{2E^{-Itw} d^2 L^2 \dot{A}}{-E^{2Itw} w^2 + 4E^{2Itw} W^2} - \\
 & \frac{4AE^{Itw} d^2 L \dot{A}}{2E^{Itw-Itw} wW - E^{Itw-Itw} W^2} + \frac{4E^{-Itw} d^2 L^2 \dot{A}}{2E^{Itw-Itw} wW - E^{Itw-Itw} W^2} + \frac{4E^{Itw} d^2 L^2 \dot{A}}{2E^{Itw-Itw} wW - E^{Itw-Itw} W^2} + \\
 & \frac{4AE^{Itw} d^2 L \dot{A}}{2E^{Itw+Itw} wW + E^{Itw+Itw} W^2} - \frac{4E^{-Itw} d^2 L^2 \dot{A}}{2E^{Itw+Itw} wW + E^{Itw+Itw} W^2} - \frac{4E^{Itw} d^2 L^2 \dot{A}}{2E^{Itw+Itw} wW + E^{Itw+Itw} W^2} - \\
 & 3AE^{-Itw} a \dot{A}^2 + 3E^{-2Itw-Itw} aL \dot{A}^2 + 3E^{-2Itw+Itw} aL \dot{A}^2 + \frac{10AE^{-Itw} d^2 \dot{A}^2}{3w^2} + \frac{2E^{-2Itw-Itw} d^2 L \dot{A}^2}{3w^2} + \\
 & \frac{2E^{-2Itw+Itw} d^2 L \dot{A}^2}{3w^2} - \frac{4E^{-Itw} d^2 L \dot{A}^2}{2E^{Itw-Itw} wW - E^{Itw-Itw} W^2} + \frac{4E^{-Itw} d^2 L \dot{A}^2}{2E^{Itw+Itw} wW + E^{Itw+Itw} W^2} - \\
 & E^{-3Itw} a \dot{A}^3 - \frac{2E^{-3Itw} d^2 \dot{A}^3}{3w^2} - 2Ad \dot{B} + 2E^{-Itw-Itw} dL \dot{B} + 2E^{-Itw+Itw} dL \dot{B} - 2E^{-2Itw} d \dot{A} \dot{B}
 \end{aligned}$$

The particular solution of **order3Eqb** is given by

Hsol3p = Hu₃@tD . DSolve@order3Eqb@@1DD == #, u₃@tD, tD@@1DD . C@_D -> 0 . TrigToExp .
SimplifyL & .ž order3Eqb@@2DDL . Timing

$$\begin{aligned}
 & :64.803 \text{ Second, } \frac{A^3 E^{3Itw} d^2}{12 w^4} + \frac{A^3 E^{3Itw} a}{8 w^2} + \frac{2 A B E^{2Itw} d}{3 w^2} + \frac{A E^{Itw} d^2 L^2 H_1 - 2 ItwL}{w^4} + \\
 & \frac{3 I A E^{Itw} a L^2 H_1 + 2 twL}{2 w^2} - \frac{A E^{Itw} m H_1 + 2 twL}{2 w} - \frac{3 A E^{Itw} Hw - 2 WL}{4 Hw - WL W} + \frac{2 B E^{Itw} Hw - WL dL}{H_2 w - WL W} + \\
 & \frac{A E^{Itw} d^2 L^2 H_1 - 2 ItwL}{w^2 H_2 w - WL W} + \frac{3 A E^{Itw} Hw + 2 WL a L^2}{4 W Hw + WL} - \frac{2 B E^{Itw} Hw + WL dL}{W H_2 w + WL} + \frac{I A E^{Itw} d^2 L^2 H_1 + 2 twL}{w^2 W H_2 w + WL} + \\
 & \frac{A E^{Itw} Hw + 2 WL d^2 L^2}{W^2 Hw + WL H_2 w + WL} - \frac{3 A^2 E^{Itw} H_2 w + WL a L^2}{Hw + WL H_3 w + WL} - \frac{2 A^2 E^{Itw} H_2 w + WL d^2 L}{3 w^2 Hw + WL H_3 w + WL} - \frac{4 A^2 E^{Itw} H_2 w + WL d^2 L}{W Hw + WL H_2 w + WL H_3 w + WL} + \\
 & \frac{E^{-3Itw} a L^3}{w^2 - 9 W^2} + \frac{E^{3Itw} a L^3}{w^2 - 9 W^2} + \frac{A E^{Itw} Hw - 2 WL d^2 L^2}{2 Hw - WL W Hw^2 - 4 W^2 L} - \frac{A E^{Itw} Hw + 2 WL d^2 L^2}{2 W Hw + WL Hw^2 - 4 W^2 L} + \frac{3 E^{-Itw} a L^3}{w^2 - W^2} + \\
 & \frac{3 E^{Itw} a L^3}{w^2 - W^2} + \frac{2 I E^{Itw} L m W}{w^2 - W^2} + \frac{4 E^{-Itw} d^2 L^3}{w^2 H - w^2 + W^2 L} + \frac{4 E^{Itw} d^2 L^3}{w^2 H - w^2 + W^2 L} + \frac{2 I E^{-Itw} L m W}{-w^2 + W^2} - \frac{3 A^2 E^{Itw} H_2 w - WL a L^2}{3 w^2 - 4 w W + W^2} - \\
 & \frac{2 A^2 E^{Itw} H_2 w - WL d^2 L}{3 w^2 H_3 w^2 - 4 w W + W^2 L} + \frac{A E^{Itw} Hw - 2 WL d^2 L^2}{W^2 H_2 w^2 - 3 w W + W^2 L} - \frac{4 A^2 E^{Itw} H_2 w - WL d^2 L}{W H - 6 w^3 + 11 w^2 W - 6 w W^2 + W^3 L} - \\
 & \frac{2 E^{-Itw} d^2 L^3}{w^4 - 5 w^2 W^2 + 4 W^4} - \frac{2 E^{Itw} d^2 L^3}{w^4 - 5 w^2 W^2 + 4 W^4} - \frac{2 E^{-3Itw} d^2 L^3}{w^4 - 13 w^2 W^2 + 36 W^4} - \frac{2 E^{3Itw} d^2 L^3}{w^4 - 13 w^2 W^2 + 36 W^4} - \frac{2 B d \dot{A}}{w^2} + \\
 & \frac{5 A^2 E^{Itw} d^2 H_1 - 2 ItwL \dot{A}}{6 w^4} + \frac{E^{-Itw} d^2 L^2 H_1 + 2 ItwL \dot{A}}{w^4} - \frac{3 I E^{-Itw} a L^2 H - I + 2 twL \dot{A}}{2 w^2} - \\
 & \frac{E^{-Itw} m H - I + 2 twL \dot{A}}{2 w} + \frac{3 I A^2 E^{Itw} a H_1 + 2 twL \dot{A}}{4 w^2} + \frac{E^{-Itw} d^2 L^2 H_1 + 2 ItwL \dot{A}}{w^2 H_2 w - WL W} + \\
 & \frac{3 E^{-Itw} Hw + 2 WL a L^2 \dot{A}}{4 W Hw + WL} - \frac{4 A E^{-Itw} d^2 L \dot{A}}{W H - 2 w + WL H - w + WL Hw + WL} - \frac{4 A E^{Itw} d^2 L \dot{A}}{W H - 2 w + WL H - w + WL Hw + WL} - \\
 & \frac{I E^{-Itw} d^2 L^2 H - I + 2 twL \dot{A}}{w^2 W H_2 w + WL} + \frac{E^{-Itw} Hw + 2 WL d^2 L^2 \dot{A}}{W^2 Hw + WL H_2 w + WL} - \frac{4 A E^{-Itw} d^2 L \dot{A}}{W H - w + WL Hw + WL H_2 w + WL} - \\
 & \frac{4 A E^{Itw} d^2 L \dot{A}}{W H - w + WL Hw + WL H_2 w + WL} + \frac{E^{-Itw} Hw - 2 WL d^2 L^2 \dot{A}}{2 Hw - WL W Hw^2 - 4 W^2 L} - \frac{E^{-Itw} Hw + 2 WL d^2 L^2 \dot{A}}{2 W Hw + WL Hw^2 - 4 W^2 L} + \\
 & \frac{6 A E^{-Itw} a L \dot{A}}{w^2 - W^2} + \frac{6 A E^{Itw} a L \dot{A}}{w^2 - W^2} + \frac{4 A E^{-Itw} d^2 L \dot{A}}{w^2 H - w^2 + W^2 L} + \frac{4 A E^{Itw} d^2 L \dot{A}}{w^2 H - w^2 + W^2 L} + \frac{E^{-Itw} Hw - 2 WL d^2 L^2 \dot{A}}{W^2 H_2 w^2 - 3 w W + W^2 L} + \\
 & \frac{3 E^{-Itw} Hw + 2 ItwL a L^2 \dot{A}}{-4 w W + 4 W^2} + \frac{5 A E^{-Itw} d^2 H_1 + 2 ItwL \dot{A}^2}{6 w^4} - \frac{3 I A E^{-Itw} a H - I + 2 twL \dot{A}^2}{4 w^2} - \\
 & \frac{3 E^{-Itw} H_2 w + WL a L \dot{A}^2}{Hw + WL H_3 w + WL} - \frac{2 E^{-Itw} H_2 w + WL d^2 L \dot{A}^2}{3 w^2 Hw + WL H_3 w + WL} - \frac{4 E^{-Itw} H_2 w + WL d^2 L \dot{A}^2}{W Hw + WL H_2 w + WL H_3 w + WL} - \\
 & \frac{3 E^{-2Itw} ItwL a L \dot{A}^2}{3 w^2 - 4 w W + W^2} - \frac{2 E^{-2Itw} ItwL d^2 L \dot{A}^2}{3 w^2 H_3 w^2 - 4 w W + W^2 L} - \frac{4 E^{-2Itw} ItwL d^2 L \dot{A}^2}{W H - 6 w^3 + 11 w^2 W - 6 w W^2 + W^3 L} + \\
 & \frac{E^{-3Itw} d^2 \dot{A}^3}{12 w^4} + \frac{E^{-3Itw} a \dot{A}^3}{8 w^2} - \frac{2 A d \dot{B}}{w^2} + \frac{2 E^{-Itw} Hw - WL dL \dot{B}}{H_2 w - WL W} - \frac{2 E^{-Itw} Hw + WL dL \dot{B}}{W H_2 w + WL} + \frac{2 E^{-2Itw} d \dot{A} \dot{B}}{3 w^2} >
 \end{aligned}$$

Clearly, **sol3p** breaks down because it contains secular terms and small-divisor terms when $W \gg w$, $W \gg 0$, $W \gg 2w$, $W \gg w \cdot 2$, $W \gg 3w$, and $W \gg w \cdot 3$. As shown in the next section, the small-divisor terms arising from $W \gg 3w$ produce a

subharmonic resonance of order one-third and those arising from $W \gg W \cdot 3$ produce a superharmonic resonance of order three. Carrying out the expansion to higher order, one finds that other resonances may occur.

We note that the resonances that occur depend on the order of the nonlinearity. Quadratic nonlinearities produce (a) subharmonic resonances of order one-half and superharmonic resonances of order two at second order and (b) subharmonic resonances of order one-third and superharmonic resonances of order three at third order. On the other hand, cubic nonlinearities produce subharmonic resonances of order one-third and superharmonic resonances of order three at third order. For a given system and order of approximation, the resonances produced can be easily identified by carrying out a straightforward expansion as done above.

In the next three sections, we use the method of multiple scales, generalized method of averaging, and method of normal forms to determine second-order uniform expansions for the solutions of **FDuffingEq** that do not contain secular or small-divisor terms.

à 4.2 The Method of Multiple Scales

Ÿ 4.2.1 Preliminaries

For a uniform second-order expansion, we need three time scales, which we symbolize and list as follows:

```
Needs@"Utilities`Notation`"D
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = {T0, T1, T2};
```

The maximum order of the expansion is related, in general, to the number of time scales by

```
maxOrder = Length@timeScalesD - 1;
```

In what follows, we need the complex conjugates of A , z , and G , which we define by

```
conjugateRule = {Complex@0, n_D -> Complex@0, -nD, A ->  $\bar{A}$ ,  $\bar{A}$  -> A, z ->  $\bar{z}$ ,  $\bar{z}$  -> z, G ->  $\bar{G}$ ,  $\bar{G}$  -> G};
```

In terms of the time scales T_n , the time derivatives become

```
dt@1D@expr_D := SumAei D@expr, timeScales@@i + 1DDD, {i, 0, maxOrder<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD • ExpandL • ei • i > maxOrder -> 0;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```

displayRule =
  9Derivative@a__DAz_i_E@__D := SequenceFormATimes žž MapIndexedAD#1#2@1DD-1 &, 8a<E, z_iE,
  Derivative@a__D@AD@__D := SequenceFormATimes žž MapIndexedAD#1#2@1DD &, 8a<E, AE,
  Derivative@a__D@ĀD@__D := SequenceFormATimes žž MapIndexedAD#1#2@1DD &, 8a<E, ĀE,
  z_i@__D -> z_i, A@__D -> A, Ā@__D -> Ā=;

```

Using the time scales T_0 , T_1 , and T_2 , we transform **FDuffingEq** from an ordinary-differential equation into a partial-differential equation according to the rule

```

multiScalesRule =
  8u@tD -> u@T0, T1, T2D, Derivative@n_D@uD@tD := dt@nD@u@T0, T1, T2DD, t -> T0<;

```

We seek a second-order uniform expansion of the solution of **FDuffingEq** in the form

```

solRule = u -> ISumAei u_i@#1, #2, #3D, 8i, maxOrder + 1<E &M;

```

For a uniform expansion, we need to eliminate the secular and small-divisor terms. To accomplish this, we need to distinguish between primary and secondary resonances. They are treated separately beginning with the case of primary resonance.

§ 4.2.2 Primary Resonance

We scale the damping m and forcing F so that the damping term and resonance terms appear at the same order at which the secular and / or small-divisor terms generated by the nonlinearities appear. In this case, they appear at order ϵ^3 , and hence we scale the forcing and damping as

```

scaling = 9F -> e3 f, m -> e2 m=;

```

Substituting the **solRule** into **FDuffingEq**, using the **scaling** and **multiScalesRule**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```

eq422a =
  HFDuffingEq . scaling . multiScalesRule . solRule . ExpandAllL . en_*; n>maxOrder+1 -> 0;
eq422a . displayRule
  2 e3 m HD0u1L + e HD02u1L + e2 HD02u2L + e3 HD02u3L + 2 e2 HD0 D1u1L + 2 e3 HD0 D1u2L + e3 HD12u1L +
  2 e3 HD0 D2u1L + e w2 u1 + d e2 u12 + a e3 u13 + e2 w2 u2 + 2 d e3 u1 u2 + e3 w2 u3 == f e3 Cos@T0 WD

```

Equating coefficients of like powers of ϵ in **eq422a**, we obtain

```

eqEps = CoefficientList@Subtract žž eq422a, eD == 0 . Thread . Rest . TrigToExp;

```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```

eqOrder@i_D := HeqEps@@1, 1DD . u1 -> u1L - # & . žž eqEps@@iDD . Reverse

```

Using **eqOrder[i]** and the **displayRule**, we rewrite **eqEps** in a concise way as

Array@eqOrder, 3D •. displayRule

$$9D_0^2 u_1 + w^2 u_1 == 0, D_0^2 u_2 + w^2 u_2 == -2 HD_0 D_1 u_1 L - d u_1^2,$$

$$D_0^2 u_3 + w^2 u_3 == \frac{1}{2} H E^{-I T_0 W} + E^{I T_0 W} L f - 2 m HD_0 u_1 L - 2 HD_0 D_1 u_2 L - D_1^2 u_1 - 2 HD_0 D_2 u_1 L - a u_1^3 - 2 d u_1 u_2 =$$

The general solution of **eqOrder[1]** can be expressed as

sol1 = u1 -> FunctionA8T0, T1, T2<, A@T1, T2D Exp@I w T0D + A@T1, T2D Exp@- I w T0DE;

Using **sol1**, we rewrite **eqOrder[2]** as

order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule

$$D_0^2 u_2 + w^2 u_2 == -A^2 E^{2 I T_0 W} d - 2 A d \dot{A} - E^{-2 I T_0 W} d \dot{A}^2 - 2 I E^{I T_0 W} w HD_1 A L + 2 I E^{-I T_0 W} w HD_1 \dot{A} L$$

Eliminating the terms that produce secular terms in **order2Eq** demands that

SCond1 = Coefficient@order2Eq@@2DD, Exp@I w T0DD == 0
 $-2 I w A^{H1, 0L} @T_1, T_2D == 0$

Solving **SCond1** for $A^{H1, 0L} @T_1, T_2D$ yields

SCond1Rule = SolveASCond1, A^{H1, 0L}@T1, T2DE@@1DD
 $8 A^{H1, 0L} @T_1, T_2D \otimes 0 <$

Hence, $A = A @T_2D$.

Substituting the solvability condition into **order2Eq**, we have

order2Eqm = order2Eq •. SCond1Rule •. HSCond1Rule •. conjugateRuleL;
order2Eqm •. displayRule

$$D_0^2 u_2 + w^2 u_2 == -A^2 E^{2 I T_0 W} d - 2 A d \dot{A} - E^{-2 I T_0 W} d \dot{A}^2$$

The particular solution of **order2Eqm** can be obtained by using **DSolve**; the result is

sol2p = DSolve@order2Eqm, u2 – timeScales, timeScalesD@@1DD •. C@_D -> H0 &L •• TrigToExp ••
ExpandAll;
sol2p •. displayRule

$$: u_2 \otimes \frac{A^2 E^{2 I T_0 W} d}{3 w^2} - \frac{2 A d \dot{A}}{w^2} + \frac{E^{-2 I T_0 W} d \dot{A}^2}{3 w^2} >$$

Next, we use **sol2p[[1,2]]** directly to express u_2 in a pure function form as

sol2 = u2 -> Function@8T0, T1, T2<, sol2p@@1, 2DD •• Evaluated;

Substituting the first- and second-order solutions, **sol1** and **sol2**, into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
order3Eq •. displayRule
```

$$D_0^2 u_3 + w^2 u_3 = \frac{1}{2} E^{-i T_0 w} f + \frac{1}{2} E^{i T_0 w} f - A^3 E^{3 i T_0 w} a - \frac{2 A^3 E^{3 i T_0 w} d^2}{3 w^2} - \\ 2 I A E^{i T_0 w} m w - 3 A^2 E^{i T_0 w} a \dot{A} + \frac{10 A^2 E^{i T_0 w} d^2 \dot{A}}{3 w^2} + 2 I E^{-i T_0 w} m w \dot{A} - 3 A E^{-i T_0 w} a \dot{A}^2 + \\ \frac{10 A E^{-i T_0 w} d^2 \dot{A}^2}{3 w^2} - E^{-3 i T_0 w} a \dot{A}^3 - \frac{2 E^{-3 i T_0 w} d^2 \dot{A}^3}{3 w^2} - \frac{8 I A E^{2 i T_0 w} d H D_1 A L}{3 w} + \\ \frac{8 I E^{-2 i T_0 w} d \dot{A} H D_1 A L}{3 w} - E^{i T_0 w} H D_1^2 A L - E^{-i T_0 w} H D_1^2 \dot{A} L - 2 I E^{i T_0 w} w H D_2 A L + 2 I E^{-i T_0 w} w H D_2 \dot{A} L$$

To express the nearness of W to w , we introduce a detuning parameter $S = O(1)$ defined by

```
OmgRule = 9W -> w + e^2 s;
```

Using the **OmgRule**, we convert the small-divisor term arising from the excitation into a secular term. Then, eliminating the secular terms from u_3 demands that

```
expRule1 = Exp@a_D -> ExpAExpand@a •. OmgRuleD •. e^2 T_0 -> T_2 E;
SCond2 = Coefficient@order3Eq@@2DD •. expRule1, Exp@I w T_0 DD == 0;
SCond2 •. displayRule
```

$$\frac{1}{2} E^{i T_2 s} f - 2 I A m w - 3 A^2 a \dot{A} + \frac{10 A^2 d^2 \dot{A}}{3 w^2} - D_1^2 A - 2 I w H D_2 A L = 0$$

Using the fact that $A = A(T_2)$ from **SCond1Rule**, we rewrite **SCond2** as

```
SCond = SCond2 •. D@SCond1Rule, T_1D •. 9A -> HA@#2D &L, A -> I A@#2D &M=
```

$$\frac{1}{2} E^{i T_2 s} f - 2 I m w A(T_2) - 3 A(T_2) D^2 A(T_2) + \frac{10 d^2 A(T_2) D^2 \dot{A}(T_2)}{3 w^2} - 2 I w A(T_2) = 0$$

4.2.3 Secondary Resonances Due to Cubic Nonlinearities

In this case W is away from w , $2w$, and $w \cdot 2$ and small-divisor terms first appear at $O(\epsilon^3)$. We scale the damping m and forcing F so that the damping term and resonance terms appear at the same order as the cubic nonlinearity according to

```
scaling = 9F -> e f, m -> e^2 m=;
```

```
eq423a =
  HFDuffingEq •. scaling •. multiScalesRule •. solRule •• ExpandAllL •. en_; n>maxOrder+1 -> 0;
eq423a •. displayRule

2 e3 m HD0u1L + e HD02u1L + e2 HD02u2L + e3 HD02u3L + 2 e2 HD0 D1u1L + 2 e3 HD0 D1u2L + e3 HD12u1L +
  2 e3 HD0 D2u1L + e w2 u1 + d e2 u12 + a e3 u13 + e2 w2 u2 + 2 d e3 u1 u2 + e3 w2 u3 == f e COS@T0 WD
```

Equating coefficients of like powers of e , we obtain

```
eqEps = CoefficientList@Subtract @@ eq423a, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. f -> 0 •. u1 -> u1L - # & •• eqEps@@iDD •• Reverse
```

Using `eqOrder[i]` and `displayRule`, we rewrite `eqEps` in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm

D02u1 + w2 u1 ==  $\frac{1}{2} H E^{-I T_0 W} + E^{I T_0 W} L f$ 
D02u2 + w2 u2 == -2 HD0 D1u1L - d u12
D02u3 + w2 u3 == -2 m HD0u1L - 2 HD0 D1u2L - D12u1 - 2 HD0 D2u1L - a u13 - 2 d u1 u2
```

The particular solution of `eqOrder[1]` can be expressed as

```
sol1p = DSolve@eqOrder@1D, u1 @@ timeScales, timeScalesD@@1DD •. C@_D -> H0 &L •• Simplify

: u1@T0, T1, T2D @ -  $\frac{E^{-I T_0 W} H_1 + E^{2 I T_0 W} L f}{2 H - w^2 + W^2 L} >$ 
```

The general solution of `eqOrder[1]` can be written as

```
fRule = 9f @ -2 L I - w2 + W2M=;

sol1 = u1 -> FunctionA8T0, T1, T2<,
  A@T1, T2D Exp@I w T0D + A@T1, T2D Exp@- I w T0D + sol1p@@1, 2DD •. fRule •• Expand •• EvaluateE

u1 @ Function@8T0, T1, T2<, E-I T0 W L + EI T0 W L + EI T0 W A@T1, T2D + E-I T0 W A@T1, T2D
```

Substituting `sol1` into `eqOrder[2]` yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule

D02u2 + w2 u2 == -A2 E2 I T0 W d - 2 A EI T0 W - I T0 W d L - 2 A EI T0 W + I T0 W d L - 2 d L2 - E-2 I T0 W d L2 - E2 I T0 W d L2 -
  2 A d A - 2 E-I T0 W - I T0 W d L A - 2 E-I T0 W + I T0 W d L A - E-2 I T0 W d A2 - 2 I EI T0 W w HD1AL + 2 I E-I T0 W w HD1A L
```

¶ The Case $W \gg 3w$

To express the nearness of W to $3w$, we introduce a detuning parameter $S = \Omega H L$ defined by

$$\Omega m g R u l e = 9 W \rightarrow 3 w + \epsilon^2 s;$$

$$\exp R u l e 1 = 9 \text{Exp}@arg_D :> \text{ExpAExpand}@arg \cdot \Omega m g R u l e D \cdot \epsilon^2 T_0 \rightarrow T_2 E=;$$

Eliminating the secular terms from u_2 demands that

$$S \text{Cond}1 = \text{Coefficient}@order2Eq@@2DD \cdot \exp R u l e 1, \text{Exp}@I w T_0 DD == 0$$

$$-2 I w A^{H1, \theta L} @T_1, T_2 D == 0$$

$$S \text{Cond}1 R u l e = \text{SolveASCond}1, A^{H1, \theta L} @T_1, T_2 D E@@1DD$$

$$8 A^{H1, \theta L} @T_1, T_2 D \Re 0 <$$

whose complex conjugate is

$$ccS \text{Cond}1 R u l e = S \text{Cond}1 R u l e \cdot \text{conjugate} R u l e$$

$$: A^{H1, \theta L} @T_1, T_2 D \Re 0 >$$

Substituting the solvability condition, $S \text{Cond}1 R u l e$, and its complex conjugate, into $order2Eq$, we have

$$Horder2Eqm = order2Eq \cdot S \text{Cond}1 R u l e \cdot ccS \text{Cond}1 R u l e L \cdot \text{display} R u l e$$

$$D_0^2 u_2 + w^2 u_2 == -A^2 E^{2 I T_0 w} d - 2 A E^{I T_0 w - I T_0 W} d L - 2 A E^{I T_0 w + I T_0 W} d L - 2 d L^2 - \\ E^{-2 I T_0 W} d L^2 - E^{2 I T_0 W} d L^2 - 2 A d \dot{A} - 2 E^{-I T_0 w - I T_0 W} d L \dot{A} - 2 E^{-I T_0 w + I T_0 W} d L \dot{A} - E^{-2 I T_0 w} d \dot{A}^2$$

In order to efficiently use DSolve to determine the particular solution of $order2Eqm$, we first transform the partial-differential equation to an ordinary-differential equation. Then we solve for the particular solution of the resulting equation and obtain

$$\text{sol2p} =$$

$$Hu_2 @T_0 D \cdot \text{DSolve}@Horder2Eqm@@1DD \cdot u_2 \rightarrow Hu_2 @\#1D \&LL == \#, u_2 @T_0 D, T_0 D@@1DD \cdot C@_D \rightarrow 0 \cdot \cdot$$

$$\text{TrigToExp} \cdot \cdot \text{Expand} L \& \cdot \cdot \text{order2Eqm}@@2DD;$$

$$\text{sol2p} \cdot \cdot \text{display} R u l e$$

$$\frac{A^2 E^{2 I T_0 w} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 A E^{I T_0 H w - W L} d L}{H^2 w - W L W} + \frac{2 A E^{I T_0 H w + W L} d L}{W H^2 w + W L} + \frac{E^{-2 I T_0 W} d L^2}{H - w + 2 W L H w + 2 W L} + \\ \frac{E^{2 I T_0 W} d L^2}{H - w + 2 W L H w + 2 W L} - \frac{2 A d \dot{A}}{w^2} - \frac{2 E^{-I T_0 H w - W L} d L \dot{A}}{H^2 w - W L W} + \frac{2 E^{-I T_0 H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I T_0 w} d \dot{A}^2}{3 w^2}$$

which can be used directly to express u_2 in a pure function form as

$$\text{sol2} = u_2 \rightarrow \text{Function}@8T_0, T_1, T_2<, \text{sol2p} \cdot \cdot \text{Evaluated};$$

Substituting sol1 and sol2 into the third-order equation, $\text{eqOrder}[3]$, we obtain


```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Eliminating the secular terms from u_3 demands that

```
SCond2 =
```

```
Coefficient@order3Eq@@2DD •. expRule1 •. W -> 3 w, Exp@I w T0DD == 0 •. D@SCond1Rule, T1D;
SCond2 •. displayRule
```

$$-6 A a L^2 + \frac{12 A d^2 L^2}{5 w^2} - 2 I A m w - 3 A^2 a \dot{A} + \frac{10 A^2 d^2 \dot{A}}{3 w^2} - 3 E^{I T_2 s} a L \dot{A}^2 - \frac{2 E^{I T_2 s} d^2 L \dot{A}^2}{w^2} - 2 I w H D_2 A L == 0$$

§ The Case $W \gg \frac{1}{3}w$

To express the nearness of W to $\frac{1}{3}w$, we introduce a detuning parameter $S = O(1)$ defined by

$$\Omega m \text{Rule} = 9W \rightarrow \frac{1}{3} I w + \epsilon^2 s M;$$

```
expRule1 = 9Exp@arg_D := ExpAExpand@arg •. OmgRuleD •. \epsilon^2 T0 -> T2E;
```

Eliminating the secular terms from u_2 demands that

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule1, Exp@I w T0DD == 0
```

$$-2 I w A^{H1, 0L} @ T_1, T_2 D == 0$$

or

```
SCond1Rule = SolveASCond1, A^{H1, 0L} @ T_1, T_2 DE@@1DD
```

$$8 A^{H1, 0L} @ T_1, T_2 D \approx 0$$

whose complex conjugate is

```
ccSCond1Rule = SCond1Rule •. conjugateRule;
```

Substituting the solvability condition, **SCond1Rule**, and its complex conjugate, into **order2Eq**, we have

```
order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule;
```

```
order2Eqm •. displayRule
```

$$D_0^2 u_2 + w^2 u_2 == -A^2 E^{2 I T_0 w} d - 2 A E^{I T_0 w - I T_0 W} d L - 2 A E^{I T_0 w + I T_0 W} d L - 2 d L^2 - E^{-2 I T_0 W} d L^2 - E^{2 I T_0 W} d L^2 - 2 A d \dot{A} - 2 E^{-I T_0 w - I T_0 W} d L \dot{A} - 2 E^{-I T_0 w + I T_0 W} d L \dot{A} - E^{-2 I T_0 w} d \dot{A}^2$$

Again, we first transform the partial-differential equation into an ordinary-differential equation and then use **DSolve** to obtain the particular solution of **order2Eqm** as

Hsol2p =

```
Hu2@T0D • DSolve@Horder2Eqm@@1DD • u2 -> Hu2@#1D &LL == #, u2@T0D, T0D@@1DD • C@_D -> 0 ••
TrigToExpL & •ž order2Eqm@@2DDL • displayRule
```

$$\frac{A^2 E^{2 I T_0 w} d^2 L^2}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 A E^{I T_0 H w - W L} d L}{H^2 w - W L W} + \frac{2 A E^{I T_0 H w + W L} d L}{W H^2 w + W L} + \frac{E^{-2 I T_0 w} d L^2}{H - w + 2 W L H w + 2 W L} +$$

$$\frac{E^{2 I T_0 w} d L^2}{H - w + 2 W L H w + 2 W L} - \frac{2 A d \dot{A}}{w^2} - \frac{2 E^{-I T_0 H w - W L} d L \dot{A}}{H^2 w - W L W} + \frac{2 E^{-I T_0 H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I T_0 w} d \dot{A}^2}{3 w^2}$$

sol2 = u2 -> Function@8T0, T1, T2<, sol2p •• Evaluated;

Substituting **sol1** and **sol2** into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D • sol1 • sol2 •• ExpandAll;
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from u_3 demands that

SCond2 =

```
CoefficientAorder3Eq@@2DD • expRule1 • W ->  $\frac{1}{3} w$ , Exp@I w T0DE == 0 • D@SCond1Rule, T1D;
```

SCond2 •• displayRule

$$-6 A a L^2 - E^{I T_2 s} a L^3 + \frac{212 A d^2 L^2}{35 w^2} + \frac{18 E^{I T_2 s} d^2 L^3}{5 w^2} - 2 I A m w - 3 A^2 a \dot{A} + \frac{10 A^2 d^2 \dot{A}}{3 w^2} - 2 I w H D_2 A L == 0$$

§ 4.2.4 Secondary Resonances Due to Quadratic Nonlinearities

In this case W is away from w and small-divisor terms first appear at $O(\epsilon^2 L)$. We scale the damping $\tilde{m}$ and forcing F so that the damping term and resonance terms appear at the same order as the quadratic nonlinearity; that is,

scaling = 8F -> ϵf , $\tilde{m}$ -> ϵm ;

eq424a =

```
HFDuffingEq • scaling • multiScalesRule • solRule •• ExpandAllL • e^n_•; n>maxOrder+1 -> 0;
```

eq424a •• displayRule

$$2 \epsilon^2 m H D_0 u_1 L + 2 \epsilon^3 m H D_0 u_2 L + \epsilon H D_0^2 u_1 L + \epsilon^2 H D_0^2 u_2 L + \epsilon^3 H D_0^2 u_3 L +$$

$$2 \epsilon^3 m H D_1 u_1 L + 2 \epsilon^2 H D_0 D_1 u_1 L + 2 \epsilon^3 H D_0 D_1 u_2 L + \epsilon^3 H D_1^2 u_1 L + 2 \epsilon^3 H D_0 D_2 u_1 L +$$

$$\epsilon w^2 u_1 + d \epsilon^2 u_1^2 + a \epsilon^3 u_1^3 + \epsilon^2 w^2 u_2 + 2 d \epsilon^3 u_1 u_2 + \epsilon^3 w^2 u_3 == f \epsilon \cos@T_0 W D$$

Equating coefficients of like powers of ϵ , we obtain

```
eqEps = CoefficientList@Subtract žž eq424a, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```
eqOrder@i_D := HeqEps@@1, 1DD • f -> 0 • u1 -> u1L - # & •ž eqEps@@iDD •• Reverse
```

Using `eqOrder[i]` and `displayRule`, we rewrite `eqEps` in a concise way as

```
Array@eqOrder, 3D •. displayRule •. TableForm
```

$$\begin{aligned} D_0^2 u_1 + w^2 u_1 &= \frac{1}{2} H E^{-I T_0 W} + E^{I T_0 W} L f \\ D_0^2 u_2 + w^2 u_2 &= -2 m H D_0 u_1 L - 2 H D_0 D_1 u_1 L - d u_1^2 \\ D_0^2 u_3 + w^2 u_3 &= -2 m H D_0 u_2 L - 2 m H D_1 u_1 L - 2 H D_0 D_1 u_2 L - D_1^2 u_1 - 2 H D_0 D_2 u_1 L - a u_1^3 - 2 d u_1 u_2 \end{aligned}$$

The particular solution of `eqOrder[1]` can be expressed as

```
sol1p = DSolve@eqOrder@1D, u1 & timeScales, timeScalesD@@1DD •. C@_D -> H0 &L •. Simplify
```

$$: u_1 @ T_0, T_1, T_2 D \rightarrow - \frac{E^{-I T_0 W} H_1 + E^{2 I T_0 W} L f}{2 H - w^2 + W^2 L}$$

```
fRule = f -> -2 I - w^2 + W^2 M L;
```

The general solution of `eqOrder[1]` can be written in pur function form as

```
sol1 = u1 -> FunctionA8T0, T1, T2<,
```

```
A@T1, T2D Exp@I w T0D + A@T1, T2D Exp@- I w T0D + sol1p@@1, 2DD •. fRule •. Expand •. EvaluateE
```

$$u_1 @ T_0, T_1, T_2 <, E^{-I T_0 W} L + E^{I T_0 W} L + E^{I T_0 W} A@T_1, T_2 D + E^{-I T_0 W} \dot{A}@T_1, T_2 D D$$

Substituting `sol1` into `eqOrder[2]` yields

```
order2Eq = eqOrder@2D •. sol1 •. ExpandAll;
```

```
order2Eq •. displayRule
```

$$\begin{aligned} D_0^2 u_2 + w^2 u_2 &= -A^2 E^{2 I T_0 W} d - 2 A E^{I T_0 W} d L - 2 A E^{I T_0 W + I T_0 W} d L - 2 d L^2 - E^{-2 I T_0 W} d L^2 - \\ &E^{2 I T_0 W} d L^2 - 2 I A E^{I T_0 W} m w + 2 I E^{-I T_0 W} L m W - 2 I E^{I T_0 W} L m W - 2 A d \dot{A} - 2 E^{-I T_0 W} d L \dot{A} - \\ &2 E^{-I T_0 W + I T_0 W} d L \dot{A} + 2 I E^{-I T_0 W} m w \dot{A} - E^{-2 I T_0 W} d \dot{A}^2 - 2 I E^{I T_0 W} w H D_1 A L + 2 I E^{-I T_0 W} w H D_1 \dot{A} L \end{aligned}$$

§ The Case $W \gg 2w$

To express the nearness of W to $2w$, we introduce a detuning parameter $s = O(1)$ defined by

```
OmgRule = 8W -> 2 w + e s<;
```

```
sigRule = Solve@OmgRule •. Rule -> Equal, sD@@1DD
```

$$9s @ - \frac{2 w - W}{e} =$$

```
expRule2 = 8Exp@arg_D :> Exp@Expand@arg •. OmgRuleD •. e T0 -> T1D<;
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from u_2 demands that

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule2, Exp@I w T0DD == 0;
```

```
SCond1 •. displayRule
```

$$-2 I A m w - 2 E^{I T_1 s} d L \dot{A} - 2 I w H D_1 A L == 0$$

or

```
SCond1Rule = ISolveASCond1, AH1, 0L@T1, T2DE@@1DD •• ExpandAllM;
```

```
SCond1Rule •. displayRule
```

$$: D_1 A \otimes -A m + \frac{I E^{I T_1 s} d L \dot{A}}{w} >$$

Substituting **sigRule** into **SCond1Rule** yields

```
SCond1Rulem = SCond1Rule •. Exp@a_D :=> Exp@a •. sigRule •. T1 -> e T0 •• ExpandD;
```

```
SCond1Rulem •. displayRule
```

$$: D_1 A \otimes -A m + \frac{I E^{-2 I T_0 w + I T_0 W} d L \dot{A}}{w} >$$

whose complex conjugate is

```
ccSCond1Rulem = SCond1Rulem •. conjugateRule;
```

Substituting these conditions into **order2Eq** yields

```
order2Eqm = order2Eq •. SCond1Rulem •. ccSCond1Rulem •• ExpandAll;
```

```
order2Eqm •. displayRule
```

$$D_0^2 u_2 + w^2 u_2 == -A^2 E^{2 I T_0 w} d - 2 A E^{I T_0 w + I T_0 W} d L - 2 d L^2 - E^{-2 I T_0 W} d L^2 - \\ E^{2 I T_0 W} d L^2 + 2 I E^{-I T_0 W} L m W - 2 I E^{I T_0 W} L m W - 2 A d \dot{A} - 2 E^{-I T_0 w - I T_0 W} d L \dot{A} - E^{-2 I T_0 w} d \dot{A}^2$$

Transforming **order2Eqm** into an ordinary-differential equation, we can obtain the particular solution as

```
sol2p =
```

```
Hu2@T0D •. DSolve@Horder2Eqm@@1DD •. u2 -> Hu2@#1D &LL == #, u2@T0D, T0D@@1DD •. C@_D -> 0 ••
```

```
TrigToExpL & •ž order2Eqm@@2DD;
```

```
sol2p •. displayRule
```

$$\frac{A^2 E^{2 I T_0 w} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 I E^{-I T_0 W} L m W}{H - w + W L H w + W L} + \frac{2 I E^{I T_0 W} L m W}{H - w + W L H w + W L} + \frac{2 A E^{I T_0 H w + W L} d L}{W H^2 w + W L} + \\ \frac{E^{-2 I T_0 W} d L^2}{H - w + 2 W L H w + 2 W L} + \frac{E^{2 I T_0 W} d L^2}{H - w + 2 W L H w + 2 W L} - \frac{2 A d \dot{A}}{w^2} + \frac{2 E^{-I T_0 H w + W L} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I T_0 w} d \dot{A}^2}{3 w^2}$$

```
sol2 = u2 -> Function@8T0, T1, T2<, sol2p •• Evaluated;
```

Then, using **sol1** and **sol2** in **eqOrder[3]**, we have

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from U_3 demands that

$$\begin{aligned} \text{SCond2} = & \text{Coefficient}[\text{order3Eq}[\text{2DD}], \text{expRule2} \cdot W \rightarrow 2w, \text{Exp}[\text{I}wT_0\text{DD}] == 0 \cdot \text{D}[\text{SCond1Rule}, T_1D] \cdot \\ & \text{SCond1Rule} \cdot \text{HSCond1Rule} \cdot \text{conjugateRuleL} \cdot \text{ExpandAll}; \\ \text{SCond2} \cdot \text{displayRule} \\ & -6AaL^2 + Am^2 + \frac{5A^2d^2L^2}{2w^2} - 3A^2a\dot{A} + \frac{10A^2d^2\dot{A}}{3w^2} - \frac{8IE^{IT_1s}dLm\dot{A}}{3w} + \frac{E^{IT_1s}dLs\dot{A}}{w} - 2IwHD_2AL == 0 \end{aligned}$$

or

$$\begin{aligned} \text{SCond2Rule} = & \text{SolveASCond2}, A^{H_0, 1L}[T_1, T_2DE][1DD] \cdot \text{ExpandAll}; \\ \text{SCond2Rule} \cdot \text{displayRule} \\ & :D_2A \otimes - \frac{5IA^2d^2L^2}{4w^3} + \frac{3IAaL^2}{w} - \frac{IAM^2}{2w} - \frac{5IA^2d^2\dot{A}}{3w^3} - \frac{4E^{IT_1s}dLm\dot{A}}{3w^2} - \frac{IE^{IT_1s}dLs\dot{A}}{2w^2} + \frac{3IA^2a\dot{A}}{2w} > \end{aligned}$$

The two partial-differential equations, **SCond1Rule** and **SCond2Rule**, can be reconstituted to obtain an ordinary-differential equation governing A . The result is

$$\begin{aligned} H_2IwA^\epsilon == & H_2Iw\text{dt}[1D][A][T_1, T_2DD] \cdot \text{SCond1Rule} \cdot \text{SCond2Rule} \cdot \text{Expand} \cdot \text{Collect}[\#, eD \& LL] \cdot \\ & \text{displayRule} \\ & 2IwA^\epsilon = eH - 2IAMw - 2E^{IT_1s}dL\dot{A}L + \\ & e^2 \left\{ -6AaL^2 + Am^2 + \frac{5A^2d^2L^2}{2w^2} - 3A^2a\dot{A} + \frac{10A^2d^2\dot{A}}{3w^2} - \frac{8IE^{IT_1s}dLm\dot{A}}{3w} + \frac{E^{IT_1s}dLs\dot{A}}{w} \right\} \end{aligned}$$

4.2.5 First-Order Real-Valued System

As in Section 4.2.4, we determine a second-order uniform expansion of the solution of the **FDuffingEq** for the case of subharmonic resonance of order one-half. However, instead of treating the second-order form of this equation, we treat its corresponding first-order form; that is, we first transform it into a system of two real-valued first-order equations. To this end, we introduce the transformation

$$vRule = u^\epsilon[tD] \rightarrow v[tD];$$

In order that the influence of the nonlinearity and damping balance the subharmonic resonance, we scale the damping coefficient m and forcing amplitude F as

$$\text{scaling} = 8F \rightarrow e f, m \rightarrow e m \ll;$$

Using the **scaling** and **vRule**, we transform the **FDuffingEq** into the following system of two real-valued first-order equations:

$$\begin{aligned} \text{eq425a} = & 8vRule \cdot \text{Rule} \rightarrow \text{Equal}, \text{FDuffingEq} \cdot \text{scaling} \cdot vRule \cdot \text{D}[vRule, tD] < \\ & 8u^\epsilon[tD] == v[tD], w^2 u[tD] + d u[tD]^2 + a u[tD]^3 + 2em v[tD] + v^\epsilon[tD] == f e \text{Cos}[tWD] < \end{aligned}$$

To determine a second-order uniform expansion of the solution of [eq425a](#) using the method of multiple scales, we first transform these equations from ordinary-differential equations into partial-differential equations in terms of the three time scales T_0 , T_1 , and T_2 according to

```
multiScales =
  8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D, u_@tD -> dt@1D@u@T0, T1, T2DD, t -> T0<;
```

Next, we expand u and v in the form

```
solRule = 9u -> ISumAe^j u_j@#1, #2, #3D, 8j, 3<E &M, v -> ISumAe^j v_j@#1, #2, #3D, 8j, 3<E &M=;
```

Substituting [multiScales](#) and [solRule](#) into [eq425a](#), expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq425b = Heq425a . multiScales . solRule . ExpandAllL . e^{"n-"; n>3} -> 0;
```

Equating coefficients of like powers of ϵ in [eq425b](#) yields

```
eqEps = Thread@CoefficientList@Subtract @@ #, eD == 0D & . eq425b . Transpose . Rest .
  TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```
linearSys = #@@1DD & . eqEps@@1DD . f -> 0;
linearSys . displayRule

8D0u1 - v1, D0v1 + w^2 u1<

eqOrder@i_D :=
  HlinearSys . u_1 -> u1L == HlinearSys . u_1 -> u1L - H@@1DD & . eqEps@@iDDL . Thread
```

Using [eqOrder\[i\]](#) and the [displayRule](#), we rewrite [eqEps](#) in a concise way as

```
eqOrder@1D . displayRule . TableForm
eqOrder@2D . displayRule . TableForm
eqOrder@3D . displayRule . TableForm

D0u1 - v1 == 0
D0v1 + w^2 u1 == (1/2) H E^-I T0 W + E^I T0 W L f

D0u2 - v2 == -HD1u1L
D0v2 + w^2 u2 == -HD1v1L - d u1^2 - 2 m v1

D0u3 - v3 == -HD1u2L - D2u1
D0v3 + w^2 u3 == -HD1v2L - D2v1 - a u1^3 - 2 d u1 u2 - 2 m v2
```

§ First-Order Equations: Linear System

We seek a solution of the [linearSys](#) in the form $u_1 = P E^{I W T_0}$ and $v_1 = Q E^{I W T_0}$ and obtain

```
coefList = E-I w T0 linearSys •. 9u1 -> I P EI w # &M, v1 -> I Q EI w # &M= •• Expand
```

```
8 - Q + I P w, I Q w + P w2 <
```

The coefficient matrix of **coefList** is

```
coefMat = Outer@Coefficient, coefList, 8P, Q<D
```

```
88I w, -1<, 8w2, I w<<
```

whose adjoint is defined by

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

Hence, the right and left eigenvectors of **coefMat** are

```
rightVec = 81, c1< •. Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD
```

```
81, I w<
```

```
leftVec = 8c1, 1< •. Solve@Hhermitian@coefMatD.8c1, 1<L@@1DD == 0, c1D@@1DD
```

```
8 - I w, 1<
```

The complexconjugate of **leftVec** is

```
ccleftVec = leftVec •. conjugateRule
```

```
8I w, 1<
```

Therefore, the homogeneous solution of **eqOrder[1]** is given by

```
sol1h = # + H# •. conjugateRuleL & •ž I rightVec A@T1, T2D EI w T0 M
```

```
8EI T0 w A@T1, T2D + E-I T0 w A@T1, T2D, I EI T0 w w A@T1, T2D - I E-I T0 w w A@T1, T2D<
```

Transforming **eqOrder[1]** into ordinary-differential equations and using **DSolve**, we obtain the particular solution of **eqOrder[1]** as

```
sol1p = DSolve@eqOrder@1D •. 8u1 -> Hu1@#1D &L, v1 -> Hv1@#1D &L<, 8u1@T0D, v1@T0D<, T0D@@1DD •.
C@_D -> 0 •• TrigToExp •• Simplify
```

```
: u1@T0D @ -  $\frac{E^{-I T_0 w} H_1 + E^{2 I T_0 w} L f}{2 H - w^2 + W^2 L}$ , v1@T0D @ -  $\frac{I E^{-I T_0 w} H - 1 + E^{2 I T_0 w} L f W}{2 H - w^2 + W^2 L}$  >
```

Then, the general solution of the first-order equations can be expressed in pure function form as

```
fRule = f -> -2 I -w2 + W2 M L;
```

```

sol1 =
8u1 -> Function@8T0, T1, T2<, sol1h@@1DD + Hu1@T0D •. sol1p •. fRule •. ExpandL •. EvaluateD,
v1 -> Function@8T0, T1, T2<, sol1h@@2DD + Hv1@T0D •. sol1p •. fRule •. ExpandL •. EvaluateD<

8u1 ® Function@8T0, T1, T2<, E-I T0 W L + EI T0 W L + EI T0 W A@T1, T2D + E-I T0 W A@T1, T2DD,
v1 ® Function@8T0, T1, T2<, -I E-I T0 W L W + I EI T0 W L W + I EI T0 W w A@T1, T2D - I E-I T0 W w A@T1, T2DD<

```

Ÿ Second-Order Equations

Substituting the first-order solution, **sol1**, into the second-order equations, **eqOrder[2]**, we have

```

order2Eq = eqOrder@2D •. sol1 •. ExpandAll;
order2Eq •. displayRule

:D0u2 - v2 == -EI T0 W HD1AL - E-I T0 W HD1AŁ,
D0v2 + w2 u2 == -A2 E2 I T0 W d - 2 A EI T0 W - I T0 W d L - 2 A EI T0 W + I T0 W d L - 2 d L2 - E-2 I T0 W d L2 -
E2 I T0 W d L2 - 2 I A EI T0 W m w + 2 I E-I T0 W L m W - 2 I EI T0 W L m W - 2 A d A - 2 E-I T0 W - I T0 W d L A -
2 E-I T0 W + I T0 W d L A + 2 I E-I T0 W m w A - E-2 I T0 W d A2 - I EI T0 W w HD1AL + I E-I T0 W w HD1AŁ>

```

To express quantitatively the nearness of the subharmonic resonance of order one-half, we introduce the detuning parameter s defined by

```

OmgRule = 8W -> 2 w + e s<;

```

To convert the terms that produce small-divisor terms into terms that produce secular terms, we use the rule

```

expRule1 = Exp@arg_D :> Exp@Expand@arg •. OmgRuleD •. e T0 -> T1D;

```

To eliminate the terms that produce secular terms in u_2 and v_2 (i.e., the solvability condition of **order2Eq**), we first determine the vector proportional to $E^{I W T_0}$ in the right-hand sides of **order2Eq** and obtain

```

STerms1 = Coefficient@#@2DD •. expRule1, Exp@I w T0DD & •. Ź order2Eq
8-AH1,0L@T1, T2D, -2 I m w A@T1, T2D - 2 EI T1 s d L A@T1, T2D - I w AH1,0L@T1, T2D<

```

Then, the solvability condition of **order2Eq** demands that **STerms1** be orthogonal to the solution of the adjoint; that is, **ccleftVec**. The result is

```

SCond1 = SolveAccleftVec.STerms1 == 0, AH1,0L@T1, T2DE@@1DD
:AH1,0L@T1, T2D ® -  $\frac{m w A@T1, T2D - I E^{I T1 s} d L A@T1, T2D}{w}$  >

```

whose complex conjugate is

```

ccSCond1 = SCond1 •. conjugateRule
:AH1,0L@T1, T2D ® -  $\frac{I E^{-I T1 s} d L A@T1, T2D + m w A@T1, T2D}{w}$  >

```


Next, we use the solvability condition to eliminate $D_1 A$ and its complex conjugate from **order2Eq** and then find their particular solution. To simplify the resulting equations using *Mathematica*, we need to replace T_1 with $e T_0$. To this end, we express the detuning parameter S in terms of w and W as

```
sigRule = Solve@OmgRule /. Rule -> Equal, sD@@1DD
```

$$9s \approx -\frac{2w - W}{e}$$

and define the rule

```
expRule2 = Exp@a_D := Exp@a /. T1 -> e T0 /. sigRule /. ExpandD;
```

This rule enables us to rewrite **SCond1** and **ccSCond1** as

```
SCond1f = Join@SCond1, ccSCond1D /. expRule2;
```

```
SCond1f /. displayRule
```

$$: D_1 A \approx -\frac{A m w - I E^{-2 I T_0 w + I T_0 W} d L \dot{A}}{w}, D_1 \bar{A} \approx -\frac{I A E^{2 I T_0 w - I T_0 W} d L + m w \dot{A}}{w} >$$

With this form of the solvability condition, **order2Eq** becomes

```
order2Eqm = order2Eq /. SCond1f /. ExpandAll;
```

```
order2Eqm /. displayRule
```

$$: D_0 u_2 - v_2 == A E^{I T_0 w} m + \frac{I A E^{I T_0 w - I T_0 W} d L}{w} + E^{-I T_0 w} m \dot{A} - \frac{I E^{-I T_0 w + I T_0 W} d L \dot{A}}{w},$$

$$D_0 v_2 + w^2 u_2 == -A^2 E^{2 I T_0 w} d - A E^{I T_0 w - I T_0 W} d L - 2 A E^{I T_0 w + I T_0 W} d L - 2 d L^2 -$$

$$E^{-2 I T_0 W} d L^2 - E^{2 I T_0 W} d L^2 - I A E^{I T_0 w} m w + 2 I E^{-I T_0 W} L m W - 2 I E^{I T_0 W} L m W -$$

$$2 A d \dot{A} - 2 E^{-I T_0 w - I T_0 W} d L \dot{A} - E^{-I T_0 w + I T_0 W} d L \dot{A} + I E^{-I T_0 w} m w \dot{A} - E^{-2 I T_0 w} d \dot{A}^2 >$$

Next, we use the method of undetermined coefficients to determine the particular solution of **order2Eqm**. To accomplish this, we first identify the form of their nonhomogeneous terms, which we will refer to as possible terms, and then seek the solution as a linear combination of them. Substituting this solution into **order2Eqm** and equating the coefficient of each possible term to zero, we obtain a set of pairs of algebraic equations that need to be solved for the unknown coefficients. The solutions of these pairs are unique except for the coefficients of $E^{I w T_0}$ and $E^{-I w T_0}$. To determine a unique solution corresponding to these terms, we use two different approaches.

Approach 1:

In the first approach, we replace w with another symbol, say w_0 , find the solution of the system of algebraic equations, and then take the limit as w_0 tends to w . To accomplish this, we define

```
collectForm = 9 E^{I T_0 w} A@T1, T2D, E^{-I T_0 w} A@T1, T2D, E^{I T_0 W} L, E^{-I T_0 W} L=;
```

Then, the form of the possible terms on the right-hand sides of **order2Eqm** can be determined as

```
possibleTerms =
HcollectForm •. w -> w0L~Join~HOuter@Times, collectForm, collectFormD •• Flatten •• UnionL
: EI T0 w0 A@T1, T2D, E-I T0 w0 A@T1, T2D, EI T0 w L, E-I T0 w L, L2, E-2 I T0 w L2,
E2 I T0 w L2, EI T0 w - I T0 w L A@T1, T2D, EI T0 w + I T0 w L A@T1, T2D, E2 I T0 w A@T1, T2D2,
E-I T0 w - I T0 w L A@T1, T2D, E-I T0 w + I T0 w L A@T1, T2D, A@T1, T2D A@T1, T2D, E-2 I T0 w A@T1, T2D2>
```

Next, we associate with each possible term an undetermined coefficient in u_2 by

```
symbolList1 = Table@Gi, 8i, Length@possibleTermsD<D
8G1, G2, G3, G4, G5, G6, G7, G8, G9, G10, G11, G12, G13, G14<
```

and an undetermined coefficient in v_2 by

```
symbolList2 = symbolList1 •. G -> L
8L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, L11, L12, L13, L14<
```

Then, we seek a particular solution of **order2Eqm** in the form

```
sol2Rule = 8u2 -> Function@8T0, T1, T2<, symbolList1.possibleTerms •• Evaluated,
v2 -> Function@8T0, T1, T2<, symbolList2.possibleTerms •• Evaluated<;
```

Substituting **sol2Rule** into **order2Eqm** and replacing $E^{\pm I w T_0}$ on the right-hand sides of **order2Eqm** by $E^{\pm I w_0 T_0}$, we have

```
order2Eqf = Subtract @@ # & •• order2Eqm •. sol2Rule •.
8Exp@I w T0D -> Exp@I w0 T0D, Exp@- I w T0D -> Exp@- I w0 T0D< •• Expand;
```

Applying directly the function **Coefficient** to collect the coefficients of all **possibleTerms** could result in extra terms that are functions of T_0 , which we eliminate. The result is

```
algEqs = Flatten@Coefficient@order2Eqf, #D & •• possibleTermsD == 0 •. Exp@_ T0D -> 0 •• Thread
9 - m - L1 + I G1 w0 == 0, I m w + w2 G1 + I L1 w0 == 0, - m - L2 - I G2 w0 == 0,
- I m w + w2 G2 - I L2 w0 == 0, I W G3 - L3 == 0, 2 I m W + w2 G3 + I W L3 == 0, - I W G4 - L4 == 0,
- 2 I m W + w2 G4 - I W L4 == 0, - L5 == 0, 2 d + w2 G5 == 0, - 2 I W G6 - L6 == 0,
d + w2 G6 - 2 I W L6 == 0, 2 I W G7 - L7 == 0, d + w2 G7 + 2 I W L7 == 0, -  $\frac{I d}{w}$  + I w G8 - I W G8 - L8 == 0,
d + w2 G8 + I w L8 - I W L8 == 0, I w G9 + I W G9 - L9 == 0, 2 d + w2 G9 + I w L9 + I W L9 == 0,
2 I w G10 - L10 == 0, d + w2 G10 + 2 I w L10 == 0, - I w G11 - I W G11 - L11 == 0,
2 d + w2 G11 - I w L11 - I W L11 == 0,  $\frac{I d}{w}$  - I w G12 + I W G12 - L12 == 0, d + w2 G12 - I w L12 + I W L12 == 0,
- L13 == 0, 2 d + w2 G13 == 0, - 2 I w G14 - L14 == 0, d + w2 G14 - 2 I w L14 == 0=
```

Using the function **Solve**, we directly solve the system of algebraic equations **algEqs** for these undetermined coefficients and obtain

```
var = Join@symbolList1, symbolList2D;
```

```
coefs = Solve@algEqs, varD@@1DD;
```

Substituting **coefs** into **sol2Rule** and changing w_0 back to w , we obtain the solution of the second-order equations in pure function form as

```
sol2a = sol2Rule •. Function@8T0, T1, T2<, b_D :>
```

```
Function@8T0, T1, T2< •. Evaluate, b •. coefs •. w0 -> w •. Evaluated;
```

```
sol2a •. displayRule
```

$$\begin{aligned}
 :u_2 @ \text{FunctionB8T}_0, T_1, T_2<, & -\frac{2 d I^2}{w^2} - \frac{E^{-2 I T_0} w d I^2}{w^2 - 4 W^2} - \frac{E^{2 I T_0} w d I^2}{w^2 - 4 W^2} + \frac{2 I E^{-I T_0} w L m W}{w^2 - W^2} - \\
 & \frac{2 I E^{I T_0} w L m W}{w^2 - W^2} - \frac{I E^{I T_0} w m A}{2 w} - \frac{E^{I T_0} w - I T_0 W d L A}{w W} + \frac{2 E^{I T_0} w + I T_0 W d L A}{W H^2 w + W L} + \frac{E^{2 I T_0} w d A^2}{3 w^2} + \\
 & \frac{I E^{-I T_0} w m A}{2 w} - \frac{E^{-I T_0} w + I T_0 W d L A}{w W} + \frac{2 E^{-I T_0} w - I T_0 W d L A}{W H^2 w + W L} - \frac{2 d A \dot{A}}{w^2} + \frac{E^{-2 I T_0} w d \dot{A}^2}{3 w^2} F, \\
 v_2 @ \text{FunctionB8T}_0, T_1, T_2<, & \frac{2 I E^{-2 I T_0} w d I^2 W}{w^2 - 4 W^2} - \frac{2 I E^{2 I T_0} w d I^2 W}{w^2 - 4 W^2} + \frac{2 E^{-I T_0} w L m W^2}{w^2 - W^2} + \\
 & \frac{2 E^{I T_0} w L m W^2}{w^2 - W^2} - \frac{1}{2} E^{I T_0} w m A - \frac{I E^{I T_0} w - I T_0 W d L A}{W} + \frac{2 I E^{I T_0} w + I T_0 W d L H w + W L A}{W H^2 w + W L} + \frac{2 I E^{2 I T_0} w d A^2}{3 w} - \\
 & \frac{1}{2} E^{-I T_0} w m A + \frac{I E^{-I T_0} w + I T_0 W d L A}{W} - \frac{2 I E^{-I T_0} w - I T_0 W d L H w + W L A}{W H^2 w + W L} - \frac{2 I E^{-2 I T_0} w d \dot{A}^2}{3 w} F>
 \end{aligned}$$

Approach 2:

As discussed above, solutions of the algebraic equations resulting from applying the method of undetermined coefficients are unique except those corresponding to $E^{I w T_0}$ and $E^{-I w T_0}$. To render them unique, we demand that they be orthogonal to solutions of the corresponding adjoint problems; that is, we demand that the coefficient vector of $E^{I w T_0}$ be $\delta u_2, v_2$ be orthogonal to **ccleftVec**. To accomplish this, we first determine the form of all possible terms on the right-hand sides of **order2Eqm** and then identify those among them corresponding to the resonance terms and those corresponding to nonresonance terms.

```
possibleTerms =
```

```
collectForm~Join~HOuter@Times, collectForm, collectFormD •. Flatten •. UnionL
```

$$\begin{aligned}
 :E^{I T_0} w A @ T_1, T_2 D, E^{-I T_0} w \dot{A} @ T_1, T_2 D, E^{I T_0} w L, E^{-I T_0} w L, L^2, E^{-2 I T_0} w L^2, \\
 E^{2 I T_0} w L^2, E^{I T_0} w - I T_0 W L A @ T_1, T_2 D, E^{I T_0} w + I T_0 W L A @ T_1, T_2 D, E^{2 I T_0} w A @ T_1, T_2 D^2, \\
 E^{-I T_0} w - I T_0 W L \dot{A} @ T_1, T_2 D, E^{-I T_0} w + I T_0 W L \dot{A} @ T_1, T_2 D, A @ T_1, T_2 D \dot{A} @ T_1, T_2 D, E^{-2 I T_0} w \dot{A} @ T_1, T_2 D^2 >
 \end{aligned}$$

Next, we identify the resonance terms among them by

```
RT = l# •. 8a_ •; a != 0 -> 1< & •ž l E^{-I w T_0} possibleTerms •. expRule1 •. Exp@_ T_0 + _ .D -> 0MM
possibleTerms •. Union •. Rest
```

$$8 E^{I T_0} w A @ T_1, T_2 D, E^{-I T_0} w + I T_0 W L \dot{A} @ T_1, T_2 D <$$

Hence, the nonresonance part of **possibleTerms** is the complement of **RT**; that is,

```
NRT = Complement@possibleTerms, Join@RT, RT •. conjugateRuleDD
:E-I T0 W L, EI T0 W L, L2, E-2 I T0 W L2, E2 I T0 W L2, EI T0 W + I T0 W L A@T1, T2D,
E2 I T0 W A@T1, T2D2, E-I T0 W - I T0 W L  $\dot{A}$ @T1, T2D, A@T1, T2D  $\dot{A}$ @T1, T2D, E-2 I T0 W  $\dot{A}$ @T1, T2D2>
```

Next, we associate with each possible resonance term an undetermined coefficient by using the rule

```
RTsymbolList = Table@Gj, 8j, Length@RTD<D
8G1, G2<
```

and we associate with each possible nonresonance term an undetermined coefficient by using the rule

```
NRTsymbolList@i_D = Table@Li,j, 8j, Length@NRTD<D
8Li,1, Li,2, Li,3, Li,4, Li,5, Li,6, Li,7, Li,8, Li,9, Li,10<
```

Hence, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm** are

```
coefRT = Coefficient@#@@@2DD, RTD & •ž order2Eqm •. Exp@_ T0 + _ .D -> 0
99m, -  $\frac{I d}{w}$  =, 8 - I m w, - d<=
```

It follows from

```
ccleftVec.coefRT === 80, 0<
True
```

that **coefRT** is orthogonal to the solution of the adjoint, **ccleftVec**, as it should.

Imposing the condition that $\{u_2, v_2\}$ is orthogonal to **ccleftVec**, we seek u_2 and v_2 in pure function form as

```
sol2Form = 8
u2 -> Function@8T0, T1, T2<, RTsymbolList.RT +
HRTsymbolList.RT •. conjugateRuleL + NRTsymbolList@1D.NRT •• Evaluated, v2 ->
Function@8T0, T1, T2<, -I w RTsymbolList.RT + H - I w RTsymbolList.RT •. conjugateRuleL +
NRTsymbolList@2D.NRT •• Evaluated<;
```

Substituting **sol2Form** into **order2Eqm** and collecting coefficients of **NRT**, we obtain the following set of algebraic equations:

```
algEqs1 = HCoefficient@Subtract žž #, NRTD & •ž Horder2Eqm •. sol2FormL •. Exp@_ T0D -> 0 ••
FlattenL == 0 •• Thread
```

$$\begin{aligned}
&8 - I W L_{1,1} - L_{2,1} == 0, I W L_{1,2} - L_{2,2} == 0, -L_{2,3} == 0, -2 I W L_{1,4} - L_{2,4} == 0, \\
&2 I W L_{1,5} - L_{2,5} == 0, I W L_{1,6} + I W L_{1,6} - L_{2,6} == 0, 2 I W L_{1,7} - L_{2,7} == 0, \\
&-I W L_{1,8} - I W L_{1,8} - L_{2,8} == 0, -L_{2,9} == 0, -2 I W L_{1,10} - L_{2,10} == 0, -2 I m W + w^2 L_{1,1} - I W L_{2,1} == 0, \\
&2 I m W + w^2 L_{1,2} + I W L_{2,2} == 0, 2 d + w^2 L_{1,3} == 0, d + w^2 L_{1,4} - 2 I W L_{2,4} == 0, \\
&d + w^2 L_{1,5} + 2 I W L_{2,5} == 0, 2 d + w^2 L_{1,6} + I W L_{2,6} + I W L_{2,6} == 0, d + w^2 L_{1,7} + 2 I W L_{2,7} == 0, \\
&2 d + w^2 L_{1,8} - I W L_{2,8} - I W L_{2,8} == 0, 2 d + w^2 L_{1,9} == 0, d + w^2 L_{1,10} - 2 I W L_{2,10} == 0 <
\end{aligned}$$

Solving **algEqs1** for the undetermined coefficients **NRTsymbolList** yields

```
coef1 = Solve@algEqs1, Array@NRTsymbolList, 2D •• FlattenD@@1DD
```

$$\begin{aligned}
&:L_{2,9} \otimes 0, L_{1,3} \otimes -\frac{2d}{w^2}, L_{1,9} \otimes -\frac{2d}{w^2}, L_{2,3} \otimes 0, L_{1,1} \otimes \frac{2ImW}{w^2 - W^2}, L_{1,4} \otimes -\frac{d}{w^2 - 4W^2}, \\
&L_{1,2} \otimes -\frac{2ImW}{w^2 - W^2}, L_{1,5} \otimes -\frac{d}{w^2 - 4W^2}, L_{2,1} \otimes \frac{2mW^2}{w^2 - W^2}, L_{1,7} \otimes \frac{d}{3w^2}, L_{2,2} \otimes \frac{2mW^2}{w^2 - W^2}, \\
&L_{2,5} \otimes -\frac{2IdW}{w^2 - 4W^2}, L_{1,10} \otimes \frac{d}{3w^2}, L_{2,10} \otimes -\frac{2Id}{3w}, L_{2,4} \otimes \frac{2IdW}{w^2 - 4W^2}, L_{2,7} \otimes \frac{2Id}{3w}, \\
&L_{2,6} \otimes \frac{2IdHw + WL}{WH^2w + WL}, L_{1,6} \otimes \frac{2d}{WH^2w + WL}, L_{1,8} \otimes \frac{2d}{2wW + W^2}, L_{2,8} \otimes -\frac{2IdHw + WL}{2wW + W^2} >
\end{aligned}$$

Substituting **sol2Form** into either of the two **order2Eqm** and collecting the coefficients of **RT**, we obtain the following set of algebraic equations:

```
algEqs2 =
```

```
Coefficient@Subtract žž order2Eqm@@1DD •. sol2Form, RTD == 0 •. Exp@_ T0D -> 0 •• Thread
```

$$9G_1 \otimes -\frac{Im}{2w}, G_2 \otimes -\frac{d}{wW} + I W G_2 == 0 =$$

Solving **algEqs2** for the undetermined coefficients **RTsymbolList** yields

```
coef2a = Solve@algEqs2, RTsymbolListD@@1DD
```

$$9G_1 \otimes -\frac{Im}{2w}, G_2 \otimes -\frac{d}{wW} =$$

```
coef2 = Join@coef2a, coef2a •. conjugateRuleD
```

$$9G_1 \otimes -\frac{Im}{2w}, G_2 \otimes -\frac{d}{wW}, \dot{G}_1 \otimes \frac{Im}{2w}, \dot{G}_2 \otimes -\frac{d}{wW} =$$

Substituting **coef1** and **coef2** into **sol2Form**, we obtain the solution of the second-order equations in pure function form as

```
sol2b = sol2Form •. Function@8T0, T1, T2<, b_D :=
  Function@8T0, T1, T2< •. Evaluate, b •. coef1 •. coef2 •. Expand •. EvaluateD;
sol2b •. displayRule
```

$$\begin{aligned}
& :u_2 \otimes \text{FunctionB8T}_0, T_1, T_2<, -\frac{2dL^2}{w^2} - \frac{E^{-2IT_0W}dL^2}{w^2 - 4W^2} - \frac{E^{2IT_0W}dL^2}{w^2 - 4W^2} + \frac{2IE^{-IT_0W}LmW}{w^2 - W^2} - \\
& \frac{2IE^{IT_0W}LmW}{w^2 - W^2} - \frac{IE^{IT_0W}mA}{2w} - \frac{E^{IT_0W-IT_0W}dLA}{wW} + \frac{2E^{IT_0W+IT_0W}dLA}{WH^2w + WL} + \frac{E^{2IT_0W}dA^2}{3w^2} + \\
& \frac{IE^{-IT_0W}mA}{2w} - \frac{E^{-IT_0W+IT_0W}dLA}{wW} + \frac{2E^{-IT_0W-IT_0W}dLA}{2wW + W^2} - \frac{2dAA}{w^2} + \frac{E^{-2IT_0W}dA^2}{3w^2} F, \\
& v_2 \otimes \text{FunctionB8T}_0, T_1, T_2<, \frac{2IE^{-2IT_0W}dL^2W}{w^2 - 4W^2} - \frac{2IE^{2IT_0W}dL^2W}{w^2 - 4W^2} + \frac{2E^{-IT_0W}LmW^2}{w^2 - W^2} + \\
& \frac{2E^{IT_0W}LmW^2}{w^2 - W^2} - \frac{1}{2}E^{IT_0W}mA - \frac{IE^{IT_0W-IT_0W}dLA}{W} + \frac{2IE^{IT_0W+IT_0W}dLA}{2w + W} + \\
& \frac{2IE^{IT_0W+IT_0W}dLwA}{WH^2w + WL} + \frac{2IE^{2IT_0W}dA^2}{3w} - \frac{1}{2}E^{-IT_0W}mA + \frac{IE^{-IT_0W+IT_0W}dLA}{W} - \\
& \frac{2IE^{-IT_0W-IT_0W}dLwA}{2wW + W^2} - \frac{2IE^{-IT_0W-IT_0W}dLwA}{2wW + W^2} - \frac{2IE^{-2IT_0W}dA^2}{3w} F>
\end{aligned}$$

These two approaches generate the same results because

```
Subtract žž Hu2@T0, T1, T2D •. 8sol2a, sol2b<L •. Simplify
```

```
0
```

```
Subtract žž Hv2@T0, T1, T2D •. 8sol2a, sol2b<L •. Simplify
```

```
0
```

The second approach is more convenient to use for a general system of equations. Hence, we will adopt this approach in the remainder of this book.

Ÿ Third-Order Equations

Substituting the first- and second-order solutions into the third-order equations, **eqOrder[3]**, yields

```
order3Eq = eq0Order@3D •. sol1 •. sol2b •. ExpandAll;
```

Using the **expRule1** to convert the terms that produce small-divisor terms into terms that produce secular terms in the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, we have

```
STerms2 = Coefficient@@@2DD •. expRule1 •. W -> 2 w, Exp@I w T0DD & •ž order3Eq;
STerms2 •. displayRule
```

$$\begin{aligned} &: \frac{I m H D_1 A L}{2 w} + \frac{E^{I T_1 s} d L H D_1 \dot{A} L}{2 w^2} - D_2 A, -6 A a L^2 + A m^2 + \frac{9 A d^2 L^2}{2 w^2} - 3 A^2 a \dot{A} + \\ &\frac{10 A^2 d^2 \dot{A}}{3 w^2} - \frac{14 I E^{I T_1 s} d L m \dot{A}}{3 w} + \frac{1}{2} m H D_1 A L - \frac{I E^{I T_1 s} d L H D_1 \dot{A} L}{2 w} - I w H D_2 A L > \end{aligned}$$

Then, the solvability condition demands that **STerms2** be orthogonal to the solution of the adjoint **cleftVec**. The result is

```
SCond2 = SolveAccleftVec.STerms2 == 0, AH0,1L@T1, T2DE@@1DD •• ExpandAll;
SCond2 •. displayRule
```

$$: D_2 A \text{®} - \frac{9 I A d^2 L^2}{4 w^3} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} - \frac{7 E^{I T_1 s} d L m \dot{A}}{3 w^2} + \frac{3 I A^2 a \dot{A}}{2 w} >$$

The two partial-differential equations, **SCond1** and **SCond2**, can be reconstituted to obtain an ordinary-differential equation governing A . The result is

```
H2 I w Aε == H2 I w dt@1D@A@T1, T2DD •. SCond1 •. SCond2 •• Expand •• Collect@#, eD &LL •.
displayRule
```

```
2 I w Aε ==
```

$$e H - 2 I A m w - 2 E^{I T_1 s} d L \dot{A} L + e^2 \left\{ \begin{aligned} &-6 A a L^2 + A m^2 + \frac{9 A d^2 L^2}{2 w^2} - 3 A^2 a \dot{A} + \frac{10 A^2 d^2 \dot{A}}{3 w^2} - \frac{14 I E^{I T_1 s} d L m \dot{A}}{3 w} \end{aligned} \right\} \dot{z}$$

§ 4.2.6 First-Order Complex-Valued System

In this section, we first transform **FDuffingEq** into a single first-order complex-valued equation using the transformation

```
transformRule = 9u -> I z + žM, uε -> I w I z - žM=;
```

```
gRule = g -> - 2 e m uε - d u2 - a u3 + e f Cos@W tD;
```

then **FDuffingEq** becomes

```
eq426a = zε == I w z - \frac{I}{2 w} g •. gRule •. transformRule •• ExpandAll
```

```
zε == - e z m + \frac{I d z^2}{2 w} + \frac{I a z^3}{2 w} + I z w -
```

$$\frac{I f e \text{Cos@t WD}}{2 w} + e m z + \frac{I d z \dot{z}}{w} + \frac{3 I a z^2 \dot{z}}{2 w} + \frac{I d z^2}{2 w} + \frac{3 I a z \dot{z}^2}{2 w} + \frac{I a z^3}{2 w}$$

To determine a second-order uniform expansion of the solution of **eq426a** using the method of multiple scales, we first introduce

```
multiScales = 9z -> z@T0, T1, T2D, ž -> ž@T0, T1, T2D, zε -> dt@1D@z@T0, T1, T2DD, t -> T0=;
```

and then expand z and $\dot{z}$ in the form

```
solRule =
  9z -> I Sum Ae^j z_j@#1, #2, #3D, 8j, 1, 3<E &M,  $\dot{z}$  -> I Sum Ae^j  $\dot{z}_j$ @#1, #2, #3D, 8j, 1, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq426a**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq426b = Heq426a . multiScales . solRule . ExpandAllL . e^N_>N>3 -> 0;
```

Equating coefficients of like powers of ϵ yields

```
eqEps = CoefficientList@Subtract  $\check{z}$  eq426b, eD == 0 . Thread . Rest . TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```
eqOrder@i_D := HeqEps@@1, 1DD . f -> 0 . z_1 -> z_iL - # & .  $\check{z}$  eqEps@@iDD . Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D . displayRule . TableForm
```

$$\begin{aligned} D_0 z_1 - I w z_1 &= -\frac{I H E^{-I T_0 W} f}{4 w} \\ D_0 z_2 - I w z_2 &= -H D_1 z_1 L - m z_1 + \frac{I d z_1^2}{2 w} + m \dot{z}_1 + \frac{I d z_1 \dot{z}_1}{w} + \frac{I d \dot{z}_1^2}{2 w} \\ D_0 z_3 - I w z_3 &= -H D_1 z_2 L - D_2 z_1 + \frac{I d z_1^3}{2 w} - m z_2 + \frac{I d z_1 \dot{z}_2}{w} + \frac{3 I d z_1^2 \dot{z}_1}{2 w} + \frac{I d z_1 \dot{z}_1^2}{w} + \frac{3 I d z_1 \dot{z}_1^2}{2 w} + \frac{I d \dot{z}_1^3}{2 w} + m \dot{z}_2 + \frac{I d \dot{z}_1 \dot{z}_2}{w} \end{aligned}$$

The general solution of the first-order equation, **eqOrder[1]**, can be expressed in pure function form as

```
sol1Form =
  HDSolve@eqOrder@1D, z_1@T_0, T_1, T_2D, timeScalesD@@1, 1, 2DD . C@1D -> A . ExpandL .
  Exp@a_D := Exp@a . ExpandD . c1_ Exp@x_D + c2_ Exp@x_D := Factor@c1 + c2D Exp@xD
```

$$\frac{E^{I T_0 W} f}{4 w H w - W L} + \frac{E^{-I T_0 W} f}{4 w H w + W L} + E^{I T_0 W} A@T_1, T_2D$$

```
sol1 = 9z_1 -> Function@8T_0, T_1, T_2<, sol1Form . Evaluated,
   $\dot{z}_1$  -> Function@8T_0, T_1, T_2<, sol1Form . conjugateRule . Evaluated=
```

$$z_1 \otimes \text{FunctionB8}T_0, T_1, T_2<, \frac{E^{I T_0 W} f}{4 w H w - W L} + \frac{E^{-I T_0 W} f}{4 w H w + W L} + E^{I T_0 W} A@T_1, T_2Df,$$

$$\dot{z}_1 \otimes \text{FunctionB8}T_0, T_1, T_2<, \frac{E^{-I T_0 W} f}{4 w H w - W L} + \frac{E^{I T_0 W} f}{4 w H w + W L} + E^{-I T_0 W} \dot{A}@T_1, T_2Df>$$

Substituting the first-order solution into the second-order equation, **eqOrder[2]**, we have


```
order2Eq = Expand • ž HeqOrder@2D •. sol1L;
order2Eq •. displayRule
```

$D_0 z_2 - I w z_2 ==$

$$\begin{aligned}
& -A E^{I T_0 w} m + \frac{I A^2 E^{2 I T_0 w} d}{2 w} + \frac{I f^2 d}{16 w^3 H w - W L^2} + \frac{I E^{-2 I T_0 w} f^2 d}{32 w^3 H w - W L^2} + \frac{I E^{2 I T_0 w} f^2 d}{32 w^3 H w - W L^2} + \frac{I A E^{I T_0 w - I T_0 w} f d}{4 w^2 H w - W L} + \\
& \frac{I A E^{I T_0 w + I T_0 w} f d}{4 w^2 H w - W L} + \frac{E^{-I T_0 w} f m}{4 w H w - W L} - \frac{E^{I T_0 w} f m}{4 w H w - W L} + \frac{I f^2 d}{16 w^3 H w + W L^2} + \frac{I E^{-2 I T_0 w} f^2 d}{32 w^3 H w + W L^2} + \frac{I E^{2 I T_0 w} f^2 d}{32 w^3 H w + W L^2} + \\
& \frac{I A E^{I T_0 w - I T_0 w} f d}{4 w^2 H w + W L} + \frac{I A E^{I T_0 w + I T_0 w} f d}{4 w^2 H w + W L} - \frac{E^{-I T_0 w} f m}{4 w H w + W L} + \frac{E^{I T_0 w} f m}{4 w H w + W L} + \frac{I f^2 d}{8 w^3 H w - W L H w + W L} + \\
& \frac{I E^{-2 I T_0 w} f^2 d}{16 w^3 H w - W L H w + W L} + \frac{I E^{2 I T_0 w} f^2 d}{16 w^3 H w - W L H w + W L} + E^{-I T_0 w} m \dot{A} + \frac{I A d \dot{A}}{w} + \frac{I E^{-I T_0 w - I T_0 w} f d \dot{A}}{4 w^2 H w - W L} + \\
& \frac{I E^{-I T_0 w + I T_0 w} f d \dot{A}}{4 w^2 H w - W L} + \frac{I E^{-I T_0 w - I T_0 w} f d \dot{A}}{4 w^2 H w + W L} + \frac{I E^{-I T_0 w + I T_0 w} f d \dot{A}}{4 w^2 H w + W L} + \frac{I E^{-2 I T_0 w} d \dot{A}^2}{2 w} - E^{I T_0 w} H D_1 A L
\end{aligned}$$

Eliminating the terms that lead to secular terms in z_2 from **order2Eq** yields

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule1, Exp@I w T_0DD == 0
```

$$-m A @ T_1, T_2 D + \frac{I E^{I T_1 s} f d \dot{A} @ T_1, T_2 D}{4 w^2 H w - W L} + \frac{I E^{I T_1 s} f d \dot{A} @ T_1, T_2 D}{4 w^2 H w + W L} - A^{H1, 0L} @ T_1, T_2 D == 0$$

or

```
SCond1Rule =
```

```
ISolveASCond1, A^{H1, 0L} @ T_1, T_2 DE@@1DD •• ExpandAllM •. c1_m + c2_m := Factor@c1 + c2D m
```

$$:A^{H1, 0L} @ T_1, T_2 D @ -m A @ T_1, T_2 D + \frac{I E^{I T_1 s} f d \dot{A} @ T_1, T_2 D}{2 w^3 - 2 w W^2} >$$

With this solvability condition, we rewrite **order2Eq** as

```
order2Eqm = Expand • ž
```

```
Horder2Eq •. SCond1Rule •. Exp@arg_D := Exp@arg •. T_1 -> e T_0 •. sigRule •• ExpandDL;
```

There are at least two approaches we can use to solve for the particular solution of **order2Eqm**. In the first approach, we transform **order2Eqm** into an ordinary-differential equation and then use **DSolve** and **Map** to determine the following particular solution:

Hsol12a =

Hz₂@T₀D •. DSolve@Horder2Eqm@@1DD •. z₂ -> Hz₂@#1D &LL == #, z₂@T₀D, T₀D@@1DD •. C@_D ->
0L & •z order2Eqm@@2DD •. Exp@a_D := Exp@Expand@aDD •• ExpandL •• Timing

$$\begin{aligned}
 &: 6.629 \text{ Second, } -\frac{f^2 d}{16 w^4 H w - W L^2} - \frac{I E^{I T_0 W} f m}{4 w H w - W L^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 H w - 2 W L H w - W L^2} - \frac{f^2 d}{16 w^4 H w + W L^2} - \\
 &\frac{I E^{-I T_0 W} f m}{4 w H w + W L^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 H w - 2 W L H w + W L^2} - \frac{f^2 d}{8 w^4 H w - W L H w + W L} + \frac{I E^{-I T_0 W} f m}{4 w H w - W L H w + W L} + \\
 &\frac{I E^{I T_0 W} f m}{4 w H w - W L H w + W L} - \frac{E^{2 I T_0 W} f^2 d}{16 w^3 H w - 2 W L H w - W L H w + W L} - \frac{E^{-2 I T_0 W} f^2 d}{32 w^3 H w - W L^2 H w + 2 W L} - \\
 &\frac{E^{-2 I T_0 W} f^2 d}{32 w^3 H w + W L^2 H w + 2 W L} - \frac{E^{-2 I T_0 W} f^2 d}{16 w^3 H w - W L H w + W L H w + 2 W L} - \frac{E^{I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L W} + \\
 &\frac{E^{I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L W} - \frac{E^{I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 W H w + W L} + \frac{E^{I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 W H w + W L} + \\
 &\frac{E^{2 I T_0 W} d A@T_1, T_2 D^2}{2 w^2} + \frac{I E^{-I T_0 W} m A@T_1, T_2 D}{2 w} - \frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L H_2 w - W L} - \\
 &\frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H_2 w - W L H w + W L} - \frac{E^{-I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L H_2 w + W L} - \frac{E^{-I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w + W L H_2 w + W L} + \\
 &\frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{2 w H_2 w - W L H w^2 - W^2 L} - \frac{d A@T_1, T_2 D A@T_1, T_2 D}{w^2} - \frac{E^{-2 I T_0 W} d A@T_1, T_2 D^2}{6 w^2} >
 \end{aligned}$$

In the second approach, we multiply **order2Eqm** by an integrating factor and then use the function **Integrate**. To reduce the computation time, we define first the rule

intRule = int@a_ + b_, arg_D -> int@a, argD + int@b, argD;

Then, using this rule and an integrating factor, we find a particular solution of **order2Eqm** as

```
Hsol2b = HHint@order2Eqm@@2DD * Exp@-I T0 wD •• Expand, T0D ••. intRule •. int -> IntegrateL *
Exp@I T0 wD •• ExpandL •. Exp@a_D :-> Exp@Expand@aDDL •• Timing
```

$$\begin{aligned}
& :0.841 \text{ Second, } -\frac{f^2 d}{16 w^4 H w - W L^2} - \frac{I E^{I T_0 W} f m}{4 w H w - W L^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 H w - 2 W L H w - W L^2} - \frac{f^2 d}{16 w^4 H w + W L^2} - \\
& \frac{I E^{-I T_0 W} f m}{4 w H w + W L^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 H w - 2 W L H w + W L^2} - \frac{f^2 d}{8 w^4 H w - W L H w + W L} + \frac{I E^{-I T_0 W} f m}{4 w H w - W L H w + W L} + \\
& \frac{I E^{I T_0 W} f m}{4 w H w - W L H w + W L} - \frac{E^{2 I T_0 W} f^2 d}{16 w^3 H w - 2 W L H w - W L H w + W L} - \frac{E^{-2 I T_0 W} f^2 d}{32 w^3 H w - W L^2 H w + 2 W L} - \\
& \frac{E^{-2 I T_0 W} f^2 d}{32 w^3 H w + W L^2 H w + 2 W L} - \frac{E^{-2 I T_0 W} f^2 d}{16 w^3 H w - W L H w + W L H w + 2 W L} - \frac{E^{I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L W} + \\
& \frac{E^{I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L W} - \frac{E^{I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 W H w + W L} + \frac{E^{I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 W H w + W L} + \\
& \frac{E^{2 I T_0 W} d A@T_1, T_2 D^2}{2 w^2} + \frac{I E^{-I T_0 W} m A@T_1, T_2 D}{2 w} - \frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L H_2 w - W L} - \\
& \frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{4 w^2 H_2 w - W L H w + W L} - \frac{E^{-I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w - W L H_2 w + W L} - \frac{E^{-I T_0 W - I T_0 W} f d A@T_1, T_2 D}{4 w^2 H w + W L H_2 w + W L} + \\
& \frac{E^{-I T_0 W + I T_0 W} f d A@T_1, T_2 D}{2 w H_2 w - W L H w^2 - W L^2} - \frac{d A@T_1, T_2 D A@T_1, T_2 D}{w^2} - \frac{E^{-2 I T_0 W} d A@T_1, T_2 D^2}{6 w^2} >
\end{aligned}$$

The CPU time is decreased by a factor of

```
%%%@1DD • %@@1DD
```

```
7.88228
```

```
sol2a === sol2b
```

```
True
```

Then, the particular solution of **order2Eqm** can be expressed in pure function form as

```
sol2 = 9z2 -> Function@8T0, T1, T2<, sol2b •• EvaluateD,
z2 -> Function@8T0, T1, T2<, sol2b •. conjugateRule •• EvaluateD=;
```

Substituting the first- and second-order solutions into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = Expand • z HeqOrder@3D •. sol1 •. sol2L;
```

Eliminating the terms that produce secular terms in z_3 from **order3Eq** yields

```

SCond2Rule =
  ISolveACoefficient@order3Eq@@2DD . expRule1, Exp@I w T0DD == 0, A^H0,1L@T1, T2DE@@1DD .
  SCond1Rule . HSCond1Rule . conjugateRuleL . ExpandAllM .
  9c1_m + c2_m := Factor@c1 + c2D m, c1_f^2 d^2 + c2_f^2 d^2 := Factor@c1 + c2D f^2 d^2,
  c1_f^2 a + c2_f^2 a := Factor@c1 + c2D f^2 a;
SCond2Rule . displayRule

```

$$:D_2A @ - \frac{I A m^2}{2 w} + \frac{3 I A f^2 a}{4 w H w - W L^2 H w + W L^2} - \frac{I A f^2 d^2 H 5 w + 2 W L}{4 w^3 H w - W L^2 H w + W L^2 H 2 w + W L} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} - \frac{E^{I T_1 s} f d m H 2 w^3 + w^2 W - 6 w W^2 - W^3 L \dot{A}}{4 w^2 H w - W L^2 W H w + W L^2} >$$

or

```

fRule = 9f -> 2 I w^2 - W^2 M L;
SCond2Rule . fRule . W -> 2 w . displayRule

```

$$:D_2A @ - \frac{9 I A d^2 L^2}{4 w^3} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} - \frac{7 E^{I T_1 s} d L m \dot{A}}{3 w^2} + \frac{3 I A^2 a \dot{A}}{2 w} >$$

which is in agreement with that obtained in the preceding section.

4.2.7 The Function **MMS1**

Collecting the steps described in Section 4.2.5 for a system of two real-valued first-order equations, we can build a function named **MMS1** (Method of Multiple Scales for 1DOF system) specifically for **FDuffingEq**. A more general function (a Package) can be similarly created by considering as arguments the governing equation, symbols for the dependent variable, independent variable, excitation amplitudes and frequencies, and all other related quantities which allow the program to identify their respective meanings. We then use **MMS1** to solve for different resonance cases.

```

MMS1@scaling_List, ResonanceCond: 8__Equal<D :=
  ModuleA8<,
    vRule = u^e@tD -> v@tD;
    OmgRule = Solve@ResonanceCond, WD@@1DD;
    eqa = 8vRule . Rule -> Equal, FDuffingEq . scaling . vRule . D@vRule, tD<;
    multiScales =
      8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D, u^e@tD -> dt@1D@u@T0, T1, T2DD, t -> T0<;
    solRule = 9u -> ISumAe^j u_j@#1, #2, #3D, 8j, 3<E &M,
    v -> ISumAe^j v_j@#1, #2, #3D, 8j, 3<E &M;
    eqb = Heqa . multiScales . solRule . TrigToExp . ExpandAllL . e^n_*; n>3 -> 0;
    eqEps =
      Thread@CoefficientList@Subtract @@ #, eD == 0D & . eqb . Transpose . Rest;

```

H* First-Order Problem *L

```

linearSys = #@@1DD & . ž eqEps@@1DD . f -> 0;
eqOrder@i_D :=
HlinearSys . u1 -> u1L == HlinearSys . u1 -> u1L - H@@1DD & . ž eqEps@@iDDL . Thread;
coefList = E-I w T0 linearSys . 9u1 -> IP EI w # &M, v1 -> IQ EI w # &M= . Expand;
coefMat = Outer@Coefficient, coefList, 8P, Q<D;
hermitian@mat_?MatrixQD := mat . conjugateRule . Transpose;
rightVec = 81, c1< . Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD;
leftVec = 8c1, 1< . Solve@Hhermitian@coefMatD.8c1, 1<L@@1DD == 0, c1D@@1DD;
ccleftVec = leftVec . conjugateRule;
sol1h = # + H# . conjugateRuleL & . ž I rightVec A@T1, T2D EI w T0 M;
sol1p =
DSolve@eqOrder@1D . 8u1 -> Hu1@#1D &L, v1 -> Hv1@#1D &L<, 8u1@T0D, v1@T0D<, T0D@@1DD .
C@_D -> 0 . TrigToExp . Simplify;
fRule = f -> -2 I - w2 + W2M L;
sol1 =
8u1 -> Function@8T0, T1, T2<, sol1h@@1DD + Hu1@T0D . sol1p . fRule . ExpandL . Evaluated,
v1 -> Function@8T0, T1, T2<,
sol1h@@2DD + Hv1@T0D . sol1p . fRule . ExpandL . Evaluated<;

H* Second-Order Problem *L
order2Eq = eqOrder@2D . sol1 . ExpandAll;
expRule1 =
Exp@arg_D :> Exp@Expand@arg . OmgRuleD . en . T0 :> timeScales@@n + 1DDD;
STerms1 = Coefficient@#@@2DD . expRule1, Exp@I w T0DD & . ž order2Eq;
SCond1 = SolveAccleftVec.STerms1 == 0, AH1, 0L@T1, T2DE@@1DD;
ccSCond1 = SCond1 . conjugateRule;
sigRule = Solve@OmgRule . Rule -> Equal, sD@@1DD;
expRule2 = Exp@a_D :> ExpAa . 9T1 -> e T0, T2 -> e2 T0= . sigRule . ExpandE;
SCond1f = Join@SCond1, ccSCond1D . expRule2;
order2Eqm = order2Eq . SCond1f . ExpandAll;
collectForm = JoinA9EIT0 w A@T1, T2D, E-IT0 w Ā@T1, T2D=,
IfAHF . f . scalingL === e, 9EIT0 w L, E-IT0 w L=, 8<EE;
possibleTerms = collectForm~Join~
HOuter@Times, collectForm, collectFormD . Flatten . UnionL;
RT = I# . 8a_ .; a != 0 -> 1< & . ž I E-I w T0 possibleTerms . expRule1 .
Exp@_ T0 + _ .D -> 0MM possibleTerms . Union . Rest;
NRT = Complement@possibleTerms, Join@RT, RT . conjugateRuleDD;
RTsymbolList = Table@Gj, 8j, Length@RTD<D;
NRTsymbolList@i_D = Table@Li,j, 8j, Length@NRTD<D;
sol2Form = 8
u2 -> Function@8T0, T1, T2<,
RTsymbolList.RT + HRTsymbolList.RT . conjugateRuleL + NRTsymbolList@1D.NRT .
Evaluated, v2 -> Function@8T0, T1, T2<, -I w RTsymbolList.RT +

```

```

H-I w RTsymbolList.RT •. conjugateRuleL + NRTsymbolList@2D.NRT •• EvaluateD<;
algEqs1 = HCoefficient@Subtract žž #, NRTD & •ž Horder2Eqm •. sol2FormL •.
Exp@_ T0D -> 0 •• FlattenL == 0 •• Thread;
coef1 = Solve@algEqs1, Array@NRTsymbolList, 2D •• FlattenD@@1DD;
algEqs2 =
Coefficient@Subtract žž order2Eqm@@1DD •. sol2Form, RTD == 0 •. Exp@_ T0D -> 0 •• Thread;
coef2a = Solve@algEqs2, RTsymbolListD@@1DD;
coef2 = Join@coef2a, coef2a •. conjugateRuleD;
sol2 = sol2Form •. Function@8T0, T1, T2<, b_D :>
Function@8T0, T1, T2< •• Evaluate, b •. coef1 •. coef2 •• Expand •• EvaluateD;

H* Third-Order Problem *L
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
STerms2 =
Coefficient@#@@2DD •. expRule1 •. H0mgRule •. e -> 0L, Exp@I w T0DD & •ž order3Eq;
SCond2 = SolveAccleftVec.STerms2 == 0, AH0,1L@T1, T2DE@@1DD •• ExpandAll;

H* Reconstitution *L
moduEq = 2 I w Aε == H2 I w dt@1D@ A@T1, T2DD •. SCond1 •. SCond2 •• Collect@#, eD &L;
Print@"The second-order approximate solution:"D;
Print@
u@tD == Hu žž timeScales •. solRule •. e^3 -> 0 •. sol1 •. sol2 •. displayRuleLD;
IfAHF • f •. scalingL == e, PrintA"where\n L==f•H2Hw2-W2LL"E
E;
Print@"\nThe modulation equations:"D;
Print@moduEq •. displayRuleD
E

```

Ÿ Primary Resonance: $W \gg w$

scaling1 = 9F -> $e^3 f$, m -> $e^2 m$;

ResonanceCond1 = 9W == w + $e^2 s$;

MMS1@scaling1, ResonanceCond1D •• Timing

The second-order approximate solution:

$$u@tD == e^{i A E^{i T_0 w} + E^{-i T_0 w} \dot{A} M + e^2} \left\{ \frac{A^2 E^{2 i T_0 w} d}{3 w^2} - \frac{2 A d \dot{A}}{w^2} + \frac{E^{-2 i T_0 w} d \dot{A}^2}{3 w^2} \right\}$$

The modulation equations:

$$2 i w A^e == 2 i e^2 w \left\{ -A m - \frac{I E^{i T_2 s} f}{4 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} \right\}$$

81.513 Second, Null<

• Subharmonic Resonance of Order 1/2: $W \gg 2w$

scaling2 = 8F -> e f, m -> e m<;

ResonanceCond2 = 8W == 2 w + e s<;

MMS1@scaling2, ResonanceCond2D •• Timing

The second-order approximate solution:

u@tD ==

$$e^{i A E^{i T_0 w} + E^{-i T_0 w} L + E^{i T_0 w} L + E^{-i T_0 w} \dot{A} M + e^2} \left\{ \frac{A^2 E^{2 i T_0 w} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{I A E^{i T_0 w} m}{2 w} - \frac{A E^{i T_0 w - i T_0 w} d L}{w W} + \right. \\ \left. \frac{2 A E^{i T_0 w + i T_0 w} d L}{W H^2 w + W L} - \frac{E^{-2 i T_0 w} d L^2}{w^2 - 4 W^2} - \frac{E^{2 i T_0 w} d L^2}{w^2 - 4 W^2} + \frac{2 I E^{-i T_0 w} L m W}{w^2 - W^2} - \frac{2 I E^{i T_0 w} L m W}{w^2 - W^2} - \right. \\ \left. \frac{2 A d \dot{A}}{w^2} + \frac{I E^{-i T_0 w} m \dot{A}}{2 w} - \frac{E^{-i T_0 w + i T_0 w} d L \dot{A}}{w W} + \frac{2 E^{-i T_0 w - i T_0 w} d L \dot{A}}{2 w W + W^2} + \frac{E^{-2 i T_0 w} d \dot{A}^2}{3 w^2} \right\}$$

where

$$L == f \cdot H^2 w^2 - W^2 L L$$

The modulation equations:

$$2 i w A^e == -2 i e^{i A m w} - I E^{i T_1 s} d L \dot{A} M + \\ 2 i e^2 w \left\{ -\frac{9 I A d^2 L^2}{4 w^3} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} - \frac{7 E^{i T_1 s} d L m \dot{A}}{3 w^2} + \frac{3 I A^2 a \dot{A}}{2 w} \right\}$$

88.562 Second, Null<

• Subharmonic Resonance of Order 1/3: $W \gg 3w$

scaling3 = 9F -> e f, m -> e^2 m<;

ResonanceCond3 = 9W == 3 w + e^2 s<;

MMS1@scaling3, ResonanceCond3D •• Timing

The second-order approximate solution:

$$u@tD == e \cdot I A E^{I T_0 W} + E^{-I T_0 W} L + E^{I T_0 W} L + E^{-I T_0 W} \dot{A} M + e^2 \left\{ \begin{aligned} & \frac{A^2 E^{2 I T_0 W} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 A E^{I T_0 W - I T_0 W} d L}{H 2 w - W L W} + \frac{2 A E^{I T_0 W + I T_0 W} d L}{W H 2 w + W L} - \frac{E^{-2 I T_0 W} d L^2}{w^2 - 4 W^2} - \\ & \frac{E^{2 I T_0 W} d L^2}{w^2 - 4 W^2} - \frac{2 A d \dot{A}}{w^2} - \frac{2 E^{-I T_0 W + I T_0 W} d L \dot{A}}{H 2 w - W L W} + \frac{2 E^{-I T_0 W - I T_0 W} d L \dot{A}}{W H 2 w + W L} + \frac{E^{-2 I T_0 W} d \dot{A}^2}{3 w^2} \end{aligned} \right\}$$

where

$$L = f \cdot H 2 H w^2 - W^2 L L$$

The modulation equations:

$$2 I w A^\epsilon ==$$

$$2 I e^2 w \left\{ \begin{aligned} & -A m - \frac{6 I A d^2 L^2}{5 w^3} + \frac{3 I A a L^2}{w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} + \frac{I E^{I T_2 s} d^2 L \dot{A}^2}{w^3} + \frac{3 I E^{I T_2 s} a L \dot{A}^2}{2 w} \end{aligned} \right\}$$

87.03 Second, Null<

Y Superharmonic Resonance of Order 2: 2W » W

scaling4 = 8F -> e f, m -> e m<;

ResonanceCond4 = 82 W == w + e s<;

MMS1@scaling4, ResonanceCond4D •• Timing

The second-order approximate solution:

$$u@tD == e \cdot I A E^{I T_0 W} + E^{-I T_0 W} L + E^{I T_0 W} L + E^{-I T_0 W} \dot{A} M + e^2 \left\{ \begin{aligned} & \frac{A^2 E^{2 I T_0 W} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{I A E^{I T_0 W} m}{2 w} + \frac{2 A E^{I T_0 W + I T_0 W} d L}{W H 2 w + W L} - \\ & \frac{E^{-2 I T_0 W} d L^2}{2 w H w + 2 W L} - \frac{E^{2 I T_0 W} d L^2}{2 w H w + 2 W L} + \frac{2 I E^{-I T_0 W} L m W}{w^2 - W^2} - \frac{2 I E^{I T_0 W} L m W}{w^2 - W^2} - \frac{2 A E^{I T_0 W - I T_0 W} d L}{2 w W - W^2} - \\ & \frac{2 A d \dot{A}}{w^2} + \frac{I E^{-I T_0 W} m \dot{A}}{2 w} + \frac{2 E^{-I T_0 W - I T_0 W} d L \dot{A}}{W H 2 w + W L} - \frac{2 E^{-I T_0 W + I T_0 W} d L \dot{A}}{2 w W - W^2} + \frac{E^{-2 I T_0 W} d \dot{A}^2}{3 w^2} \end{aligned} \right\}$$

where

$$L = f \cdot H 2 H w^2 - W^2 L L$$

The modulation equations:

$$2 I w A^\epsilon == -I e H - I E^{I T_1 s} d L^2 + 2 A m w L +$$

$$2 I e^2 w \left\{ \begin{aligned} & -\frac{46 I A d^2 L^2}{15 w^3} + \frac{13 E^{I T_1 s} d L^2 m}{12 w^2} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} \end{aligned} \right\}$$

88.993 Second, Null<

Ψ Superharmonic Resonance of Order 3: $3W \gg w$

scaling5 = 9F -> e f, m -> e² m;

ResonanceCond5 = 93 W == w + e² s=;

MMS1@scaling5, ResonanceCond5D •• Timing

The second-order approximate solution:

$$u(tD) == e \left\{ \begin{aligned} & I A E^{I T_0 W} + E^{-I T_0 W} L + E^{I T_0 W} L + E^{-I T_0 W} \dot{A} M + \\ & e^2 \left\{ \begin{aligned} & \frac{A^2 E^{2 I T_0 W} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 A E^{I T_0 W - I T_0 W} d L}{H^2 w - W L W} + \frac{2 A E^{I T_0 W + I T_0 W} d L}{W H^2 w + W L} - \frac{E^{-2 I T_0 W} d L^2}{w^2 - 4 W^2} - \right. \\ & \left. \frac{E^{2 I T_0 W} d L^2}{w^2 - 4 W^2} - \frac{2 A d \dot{A}}{w^2} - \frac{2 E^{-I T_0 W + I T_0 W} d L \dot{A}}{H^2 w - W L W} + \frac{2 E^{-I T_0 W - I T_0 W} d L \dot{A}}{W H^2 w + W L} + \frac{E^{-2 I T_0 W} d \dot{A}^2}{3 w^2} \right\} \end{aligned} \right. \end{aligned} \right.$$

where

$$L = f \cdot H^2 H w^2 - W^2 L L$$

The modulation equations:

$$2 I w A^e == \left\{ \begin{aligned} & 2 I e^2 w \left\{ \begin{aligned} & - A m - \frac{106 I A d^2 L^2}{35 w^3} - \frac{9 I E^{I T_2 s} d^2 L^3}{5 w^3} + \frac{3 I A a L^2}{w} + \frac{I E^{I T_2 s} a L^3}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} \right\} \end{aligned} \right. \end{aligned} \right.$$

87.08 Second, Null<

Ψ Slowly-Modulated Load: $W \gg 0$

scaling6 = 8F -> e f, m -> e m<;

ResonanceCond6 = 8W == e s<;

MMS1@scaling6, ResonanceCond6D •• Timing

The second-order approximate solution:

u@tD ==

$$e^{i A E^{i T_0 W} + E^{-i T_0 W} L + E^{i T_0 W} L + E^{-i T_0 W} \dot{A} m} + e^2 \left\{ \frac{A^2 E^{2 i T_0 W} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{I A E^{i T_0 W} m}{2 w} - \frac{A E^{i T_0 W - i T_0 W} d L}{w H^2 w - W L} - \frac{A E^{i T_0 W + i T_0 W} d L}{w H^2 w + W L} - \frac{E^{-2 i T_0 W} d L^2}{w^2 - 4 W^2} - \frac{E^{2 i T_0 W} d L^2}{w^2 - 4 W^2} + \frac{2 I E^{-i T_0 W} L m W}{w^2 - W^2} - \frac{2 I E^{i T_0 W} L m W}{w^2 - W^2} - \frac{2 A d \dot{A}}{w^2} + \frac{I E^{-i T_0 W} m \dot{A}}{2 w} - \frac{E^{-i T_0 W + i T_0 W} d L \dot{A}}{w H^2 w - W L} - \frac{E^{-i T_0 W - i T_0 W} d L \dot{A}}{w H^2 w + W L} + \frac{E^{-2 i T_0 W} d \dot{A}^2}{3 w^2} \right\} \frac{y}{z}$$

where

$$L = f \cdot H^2 H w^2 - W^2 L L$$

The modulation equations:

$$2 I w A^e = -2 I A E^{-i T_1 s} e H - I d L - I E^{2 i T_1 s} d L + E^{i T_1 s} m w L + 2 I e^2 w \left\{ -\frac{3 I A d^2 L^2}{w^3} - \frac{3 I A E^{-2 i T_1 s} d^2 L^2}{2 w^3} - \frac{3 I A E^{2 i T_1 s} d^2 L^2}{2 w^3} + \frac{3 I A a L^2}{w} + \frac{3 I A E^{-2 i T_1 s} a L^2}{2 w} + \frac{3 I A E^{2 i T_1 s} a L^2}{2 w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} \right\} \frac{y}{z}$$

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à 4.3 The Generalized Method of Averaging

To apply either the method of averaging or the generalized method of averaging to **FDuffingEq**, we need first to use the method of variation of parameters to transform it into a system of two first-order equations governing the amplitude and phase. To this end, we introduce the following transformation:

states = 8a@tD, f@tD<;

transformRule = 8u -> a@tD Cos@f@tDD + L Cos@Y@tDD, u^e -> - w a@tD Sin@f@tDD - W L Sin@Y@tDD<;

gRule = g -> -2 e m u^e - e d u² - e² a u³;

After some algebraic manipulations (Nayfeh, 1973, 1981), we transform **FDuffingEq** into two first-order equations as

eq43a =

$$D@states, tD == \int_k^9 - \frac{g}{w} \sin@f@tDD, w - \frac{g}{a@tD w} \cos@f@tDD = \cdot. gRule \cdot. transformRule \cdot \cdot Expand \cdot \cdot TrigReduce \cdot \cdot Expand \cdot \cdot Thread;$$

We seek approximate solutions to **eq43a** in the form

basicTerms = 8h@tD, j@tD, Y@tD<;

```
solRule = 9a -> I EvaluateAh@tD + SumAej aj žž basicTerms, 8j, 2<E •. t -> #E &M,
f -> I EvaluateAj@tD + SumAej fj žž basicTerms, 8j, 2<E •. t -> #E &M=
```

```
8a @ Hh@#1D + e a1@h@#1D, j@#1D, Y@#1DD + e2 a2@h@#1D, j@#1D, Y@#1DD &L,
f @ Hj@#1D + e f1@h@#1D, j@#1D, Y@#1DD + e2 f2@h@#1D, j@#1D, Y@#1DD &L<
```

where $h(t)$ and $j(t)$ are expanded in power series in e as

```
basicDRule = D@basicTerms@@81, 2<DD, tD ->
9SumAei Ai žž basicTerms, 8i, 2<E, w + SumAei Fi žž basicTerms, 8i, 2<E= •• Thread
8he@tD @ e A1@h@tD, j@tD, Y@tDD + e2 A2@h@tD, j@tD, Y@tDD,
je@tD @ w + e F1@h@tD, j@tD, Y@tDD + e2 F2@h@tD, j@tD, Y@tDD<
```

The functions a_1 , a_2 , f_1 , and f_2 are fast varying functions of j , while it follows from **basicDRule** that h , and hence the A_n and F_n are slowly varying functions of t .

To the second approximation, we differentiate $a(t)$ and $f(t)$ once with respect to t , use **solRule** and **basicDRule**, expand the result for small e , discard terms of order higher than e^2 , and obtain

```
eq43bLHS =
CoefficientListAExpand@#@@1DD •. solRule •. basicDRuleD •. en_•;n>2 -> 0, eE & •ž eq43a;
```

Next, we substitute **solRule** into the right-hand sides of **eq43a**, expand the result for small e , keep terms up to $O(e^2)$, and rewrite their right-hand sides as

```
eq43bRHS =
CoefficientList@Series@#@@2DD •. solRule, 8e, 0, 2<D •• Normal •• Expand, eD & •ž eq43a;
```

Equating coefficients of like powers of e in **eq43bLHS** and **eq43bRHS**, we obtain

```
eqEps = MapThread@Equal, 8eq43bLHS, eq43bRHS<, 2D •• Transpose •• Rest;
```

Next, we use the method of separation of variables to separate fast and slowly varying terms in the first-order equations, **eqEps[[1]]**. We first introduce the rules

```
SVT@j_D := 8Aj žž basicTerms, Fj žž basicTerms<
solVar@j_D := 8aj, fj<
psiRule = 8j@tD -> w t + b@tD, Y@tD -> W t<;
betaRule = Solve@psiRule •. Rule -> Equal, 8b@tD, W<D@@1DD
9b@tD @ - t w + j@tD, W @  $\frac{Y@tD}{t}$  =
```

For the case of subharmonic resonance of $O(\frac{1}{2})$, we define

```
OmgRule = 8W -> 2 w + e s<;
```

```
sigRule = Solve@OmgRule •. Rule -> Equal, sD@@1DD
```

$$9s \otimes - \frac{2w - W}{e} =$$

```
expRule1 = f@a_D := f@Expand@a •. psiRule •. OmgRuleD •. e t -> t1D;
```

```
expRule2 = f@a_D := f@Expand@a •. t1 -> e t •. sigRule •. betaRuleDD;
```

Using these rules, we find that the slowly varying parts of **eqEps[[1]]** are given by

```
SVT1Rule =
```

```
Table@Solve@eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD •. expRule1 •. f@_ t + _ .D -> 0 •.
expRule2, SVT@1D@@iDDD @@1DD, 8i, 2<D •• Flatten
```

$$9A_1@h@tD, j@tD, Y@tDD \otimes - \frac{2mwh@tD - dL \sin 2j@tD - Y@tDD h@tD}{2w},$$

$$F_1@h@tD, j@tD, Y@tDD \otimes \frac{dL \cos 2j@tD - Y@tDD}{2w} =$$

whereas the fast varying parts are given by

```
FVT1 =
```

```
Table@Subtract žž eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD, 8i, 2<D •. SVT1Rule •• Expand
```

$$: \frac{LmW \cos j@tD - Y@tDD}{w} - \frac{LmW \cos j@tD + Y@tDD}{w} - \frac{dL^2 \sin j@tDD}{2w} -$$

$$\frac{dL^2 \sin j@tD - 2Y@tDD}{4w} - \frac{dL^2 \sin j@tD + 2Y@tDD}{4w} - m \cos 2j@tDD h@tD -$$

$$\frac{dL \sin 2j@tD + Y@tDD h@tD}{2w} - \frac{d \sin j@tDD h@tD^2}{4w} - \frac{d \sin 3j@tDD h@tD^2}{4w},$$

$$- \frac{dL \cos Y@tDD}{w} - \frac{dL \cos 2j@tD + Y@tDD}{2w} + m \sin 2j@tDD - \frac{dL^2 \cos j@tDD}{2wh@tD} -$$

$$\frac{dL^2 \cos j@tD - 2Y@tDD}{4wh@tD} - \frac{dL^2 \cos j@tD + 2Y@tDD}{4wh@tD} - \frac{LmW \sin j@tD - Y@tDD}{wh@tD} +$$

$$\frac{LmW \sin j@tD + Y@tDD}{wh@tD} - \frac{3d \cos j@tDD h@tD}{4w} - \frac{d \cos 3j@tDD h@tD}{4w} >$$

To determine a particular solution corresponding to these fast varying terms, we use the method of undetermined coefficients. To accomplish this, we first determine the possible forms of the terms in **FVT1** as follows:

FVT1Forms =

HCases@#, HCos $\vec{E}$ SinL@a_D -> 8Cos@aD, Sin@aD<, InfinityD •• Flatten •• UnionL & •ž FVT1

```
88Cos@j@tDD, Cos@2 j@tDD, Cos@3 j@tDD, Cos@j@tD - 2 Y@tDD,
Cos@j@tD - Y@tDD, Cos@j@tD + Y@tDD, Cos@2 j@tD + Y@tDD, Cos@j@tD + 2 Y@tDD,
Sin@j@tDD, Sin@2 j@tDD, Sin@3 j@tDD, Sin@j@tD - 2 Y@tDD, Sin@j@tD - Y@tDD,
Sin@j@tD + Y@tDD, Sin@2 j@tD + Y@tDD, Sin@j@tD + 2 Y@tDD<,
8Cos@j@tDD, Cos@2 j@tDD, Cos@3 j@tDD, Cos@j@tD - 2 Y@tDD, Cos@j@tD - Y@tDD,
Cos@Y@tDD, Cos@j@tD + Y@tDD, Cos@2 j@tD + Y@tDD, Cos@j@tD + 2 Y@tDD,
Sin@j@tDD, Sin@2 j@tDD, Sin@3 j@tDD, Sin@j@tD - 2 Y@tDD, Sin@j@tD - Y@tDD,
Sin@Y@tDD, Sin@j@tD + Y@tDD, Sin@2 j@tD + Y@tDD, Sin@j@tD + 2 Y@tDD<<
```

Using the principle of superposition, we seek a particular solution corresponding to the fast-varying terms as a linear combination of these possible forms:

sol1Form = MapIndexed@Hcoeffs1@#2@@@1DDD = Array@c, Length@#1DDL.#1 &, FVT1FormsD

```
8c@1D Cos@j@tDD + c@2D Cos@2 j@tDD + c@3D Cos@3 j@tDD +
c@4D Cos@j@tD - 2 Y@tDD + c@5D Cos@j@tD - Y@tDD + c@6D Cos@j@tD + Y@tDD +
c@7D Cos@2 j@tD + Y@tDD + c@8D Cos@j@tD + 2 Y@tDD + c@9D Sin@j@tDD + c@10D Sin@2 j@tDD +
c@11D Sin@3 j@tDD + c@12D Sin@j@tD - 2 Y@tDD + c@13D Sin@j@tD - Y@tDD +
c@14D Sin@j@tD + Y@tDD + c@15D Sin@2 j@tD + Y@tDD + c@16D Sin@j@tD + 2 Y@tDD,
c@1D Cos@j@tDD + c@2D Cos@2 j@tDD + c@3D Cos@3 j@tDD + c@4D Cos@j@tD - 2 Y@tDD +
c@5D Cos@j@tD - Y@tDD + c@6D Cos@Y@tDD + c@7D Cos@j@tD + Y@tDD + c@8D Cos@2 j@tD + Y@tDD +
c@9D Cos@j@tD + 2 Y@tDD + c@10D Sin@j@tDD + c@11D Sin@2 j@tDD + c@12D Sin@3 j@tDD +
c@13D Sin@j@tD - 2 Y@tDD + c@14D Sin@j@tD - Y@tDD + c@15D Sin@Y@tDD +
c@16D Sin@j@tD + Y@tDD + c@17D Sin@2 j@tD + Y@tDD + c@18D Sin@j@tD + 2 Y@tDD<
```

Substituting **sol1Form** into **eqEps[[1]]**, using **SVT1Rule**, collecting the coefficients of **FVT1Forms**, solving the resulting algebraic equations for the undetermined coefficients, and then substituting the result back into **sol1Form**, we obtain the solution

```

sol1rhs = Table@sol1Form@@iDD •.
Solve@Coefficient@Subtract žž eqEps@@1, iDD •. SVT1Rule •. solVar@1D@@iDD -> HEvaluate@
sol1Form@@iDD •. Thread@basicTerms -> 8#1, #2, #3<DD &L, FVT1Forms@@iDDD ==
0 •• Thread, coeffs1@iDD@@1DD •. D@psiRule, tD •• Expand, 8i, 2<D

:- 
$$\frac{d^2 L^2 \cos@i@tD}{2 w^2} - \frac{d^2 L^2 \cos@i@tD - 2 Y@tD}{4 w Hw - 2 WL} - \frac{d^2 L^2 \cos@i@tD + 2 Y@tD}{4 w Hw + 2 WL} -$$


$$\frac{L m W \sin@j@tD - Y@tD}{w Hw - WL} + \frac{L m W \sin@j@tD + Y@tD}{w Hw + WL} - \frac{d L \cos@2 j@tD + Y@tD h@tD}{2 w H2 w + WL} +$$


$$\frac{m \sin@2 j@tD h@tD}{2 w} - \frac{d \cos@i@tD h@tD^2}{4 w^2} - \frac{d \cos@3 j@tD h@tD^2}{12 w^2},$$


$$\frac{m \cos@2 j@tD}{2 w} + \frac{d L \sin@Y@tD}{w W} + \frac{d L \sin@2 j@tD + Y@tD}{2 w H2 w + WL} - \frac{L m W \cos@j@tD - Y@tD}{w Hw - WL h@tD} +$$


$$\frac{L m W \cos@j@tD + Y@tD}{w Hw + WL h@tD} + \frac{d L^2 \sin@j@tD}{2 w^2 h@tD} + \frac{d L^2 \sin@j@tD - 2 Y@tD}{4 w Hw - 2 WL h@tD} +$$


$$\frac{d L^2 \sin@j@tD + 2 Y@tD}{4 w Hw + 2 WL h@tD} + \frac{3 d \sin@j@tD h@tD}{4 w^2} + \frac{d \sin@3 j@tD h@tD}{12 w^2} >$$


```

which can be expressed in pure function form as

```

sol1Rule = Table@solVar@1D@@iDD ->
HEvaluate@sol1rhs@@iDD •. Thread@basicTerms -> 8#1, #2, #3<DD &L, 8i, 2<D

:a1 ® 
$$\left\{ \begin{aligned} &\frac{d^2 L^2 \cos@#2D}{2 w^2} - \frac{d^2 L^2 \cos@#2 - 2 #3D}{4 w Hw - 2 WL} - \\ &\frac{d^2 L^2 \cos@#2 + 2 #3D}{4 w Hw + 2 WL} - \frac{L m W \sin@#2 - #3D}{w Hw - WL} + \frac{L m W \sin@#2 + #3D}{w Hw + WL} - \\ &\frac{d L \cos@2 #2 + #3D #1}{2 w H2 w + WL} + \frac{m \sin@2 #2D #1}{2 w} - \frac{d \cos@#2D #1^2}{4 w^2} - \frac{d \cos@3 #2D #1^2}{12 w^2} \end{aligned} \right\},$$

f1 ® 
$$\left\{ \begin{aligned} &\frac{L m W \cos@2 #2D}{2 w} + \frac{d L \sin@#3D}{w W} + \frac{d L \sin@2 #2 + #3D}{2 w H2 w + WL} - \frac{L m W \cos@#2 - #3D}{w Hw - WL #1} + \frac{L m W \cos@#2 + #3D}{w Hw + WL #1} + \\ &\frac{d L^2 \sin@#2D}{2 w^2 #1} + \frac{d L^2 \sin@#2 - 2 #3D}{4 w Hw - 2 WL #1} + \frac{d L^2 \sin@#2 + 2 #3D}{4 w Hw + 2 WL #1} + \frac{3 d \sin@#2D #1}{4 w^2} + \frac{d \sin@3 #2D #1}{12 w^2} \end{aligned} \right\}$$


```

Substituting the slow- and fast-varying components of the first-order solution, **SVT1Rule** and **sol1Rule**, into the second-order equations, we have

```

Horder2Expr = Table@TrigReduce •ž
HSubtract žž eqEps@@2, iDD •. SVT1Rule •. sol1Rule •• ExpandL, 8i, 2<D;L •• Timing

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```

Since we are seeking an expansion valid up to $O(\epsilon^2)$, we do not need to solve for a_2 and f_2 . All we need to do is to investigate the above expressions to determine the slowly varying parts and determine A_2 and F_2 . The result is

SVT2Rule =

```
Table@Solve@order2Expr@@iDD == 0 . Thread@solVar@2D -> H0 &LD . expRule1 . f_@_ t + _ . D ->
0 . expRule2 . H0mgRule . e -> 0L, SVT2D@@iDDD, 8i, 2<D . Flatten . ExpandAll
```

$$:A_2@h@tD, j@tD, Y@tDD @ - \frac{7 d L m \cos@2 j@tD - Y@tDD h@tD}{6 w^2}, F_2@h@tD, j@tD, Y@tDD @$$

$$- \frac{9 d^2 L^2}{16 w^3} + \frac{3 a L^2}{4 w} - \frac{m^2}{2 w} + \frac{7 d L m \sin@2 j@tD - Y@tDD}{6 w^2} - \frac{5 d^2 h@tD^2}{12 w^3} + \frac{3 a h@tD^2}{8 w} >$$

Hence, to the second approximation, we find that

u@tD ==

```
HTrigReduce . % HNormal@Series@u . transformRule . solRule, 8e, 0, 1<D . sol1Rule .
ExpandL . Collect@#, eD &L
```

u@tD ==

$$L \cos@Y@tDD + \cos@j@tDD h@tD + e \left\{ - \frac{d L^2}{2 w^2} - \frac{d L^2 \cos@2 Y@tDD}{4 w H w - 2 W L} - \frac{d L^2 \cos@2 Y@tDD}{4 w H w + 2 W L} + \frac{L m W \sin@Y@tDD}{w H w - W L} \right.$$

$$\left. - \frac{L m W \sin@Y@tDD}{w H w + W L} - \frac{d L \cos@j@tD - Y@tDD h@tD}{2 w W} + \frac{d L \cos@j@tD + Y@tDD h@tD}{2 w W} - \frac{d L \cos@j@tD + Y@tDD h@tD}{2 w H^2 w + W L} + \frac{m \sin@j@tDD h@tD}{2 w} - \frac{d h@tD^2}{2 w^2} + \frac{d \cos@2 j@tDD h@tD^2}{6 w^2} \right\}$$

where the equations governing the amplitude and phase are

```
8ampEqs, phaseEqs<= HbasicDRule . SVT1Rule . SVT2Rule . ExpandAllL .
```

```
f_@a_D := f@Collect@a . psiRule, tDD . Rule -> Equal
```

$$:h^e@tD == -e m h@tD - \frac{7 d e^2 L m \cos@t H^2 w - W L + 2 b@tDD h@tD}{6 w^2} + \frac{d e L \sin@t H^2 w - W L + 2 b@tDD h@tD}{2 w},$$

$$j^e@tD == - \frac{9 d^2 e^2 L^2}{16 w^3} + \frac{3 a e^2 L^2}{4 w} - \frac{e^2 m^2}{2 w} + w + \frac{d e L \cos@t H^2 w - W L + 2 b@tDD}{2 w} +$$

$$\frac{7 d e^2 L m \sin@t H^2 w - W L + 2 b@tDD}{6 w^2} - \frac{5 d^2 e^2 h@tD^2}{12 w^3} + \frac{3 a e^2 h@tD^2}{8 w} >$$

à 4.4 The Method of Normal Forms

In this section, we use the method of normal forms to determine a second-order uniform expansion of the solution of **FDuffingEq**. To this end, we start with the corresponding first-order complex-valued equation, which was derived in Section 4.2.6 and a summary of the derivation is given below:

$$\text{transformRule} = 9u@tD \rightarrow I z@tD + \dot{z}@tD M, u^e@tD \rightarrow I w I z@tD - \dot{z}@tD M, f \cos@W tD \rightarrow \frac{1}{2} H z@tD + \dot{z}@tD L =;$$

$$\text{gRule} = g \rightarrow -2 e m u^e@tD - e d u@tD^2 - e^2 a u@tD^3 + f \cos@W tD;$$

```
eq44a = 8z^e@tD == I w z@tD - I g • H2 wL • . gRule • . transformRule •• ExpandAll, z^e@tD == I W z@tD<
: z^e@tD ==
-  $\frac{I z@tD}{4 w}$  - e m z@tD + I w z@tD +  $\frac{I d e z@tD^2}{2 w}$  +  $\frac{I a e^2 z@tD^3}{2 w}$  -  $\frac{I \dot{z}@tD}{4 w}$  + e m  $\dot{z}@tD$  +  $\frac{I d e z@tD \dot{z}@tD}{w}$  +
 $\frac{3 I a e^2 z@tD^2 \dot{z}@tD}{2 w}$  +  $\frac{I d e \dot{z}@tD^2}{2 w}$  +  $\frac{3 I a e^2 z@tD \dot{z}@tD^2}{2 w}$  +  $\frac{I a e^2 \dot{z}@tD^3}{2 w}$ , z^e@tD == I W z@tD>
```

According to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = 8h@tD, ḣ@tD, z@tD, ż@tD<;

zetaRule = 9z -> FunctionAt, h@tD + SumAe^j h_j@Sequence žž basicTermsD, 8j, 0, 2<E •• EvaluateE,
ż -> FunctionAt, ḣ@tD + SumAe^j ḣ_j@Sequence žž basicTermsD, 8j, 0, 2<E •• EvaluateE=

8z @ Function@t, h@tD + h_0@h@tD, ḣ@tD, z@tD, ż@tDD +
e h_1@h@tD, ḣ@tD, z@tD, ż@tDD + e^2 h_2@h@tD, ḣ@tD, z@tD, ż@tDDD,
ż @ Function@t, ḣ@tD + h_0@ḣ@tD, ḣ@tD, z@tD, ż@tDD +
e h_1@ḣ@tD, ḣ@tD, z@tD, ż@tDD + e^2 h_2@ḣ@tD, ḣ@tD, z@tD, ż@tDDD<
```

that results in the simplest possible equation

```
etaRule = 9h^e@tD -> I w h@tD + SumAe^j g_j@tD, 8j, 2<E, z^e@tD -> I W z@tD=
8h^e@tD @ I w h@tD + e g_1@tD + e^2 g_2@tD, z^e@tD @ I W z@tD<
```

where the overbar denotes the complexconjugate

```
conjugateRule = 8h -> h̄, h̄ -> h, g -> ḡ, ḡ -> g, z -> z̄, z̄ -> z, Complex@0, n_D -> Complex@0, -nD<;
```

Substituting the expansion for Z, the **zetaRule**, into **eq44a**, using the **etaRule**, expanding the result for small e, and discarding terms of order higher than e², we have

```
eq44b = Heq44a@@1DD • . zetaRule • . etaRule • . HetaRule • . conjugateRuleL •• ExpandAllL • .
e^n_•; n>2 -> 0;
```

Equating coefficients of like powers of e in **eq44b** yields

```
eqEps = CoefficientList@Subtract žž eq44b, eD == 0 •• Thread;
```

Ÿ First-Order Solution

Using the method of undetermined coefficients, we assume

```
coeffsL = Table@D_i, 8i, Length@basicTermsD<D
8D_1, D_2, D_3, D_4<
```



```

hFormL = 9h0 -> HEvaluate@coeffsL.basicTerms • Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
  h0 -> HEvaluate@coeffsL.basicTerms • conjugateRule •
  Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=;

```

Substituting the **hFormL** into **eqEps[[1]]**, collecting the coefficients of the **basicTerms**, and solving for the **coeffsL**, we obtain

```

coeffsLRule =
  Solve@Coefficient@eqEps@@1, 1DD • hFormL, basicTermsD == 0 • Thread, coeffsLD@@1DD

Solve::svars: Equations may not give solutions for all "solve" variables.

```

$$:D_2 \otimes 0, D_3 \otimes \frac{1}{4wHw - WL}, D_4 \otimes \frac{1}{4wHw + WL} >$$

Substituting the **coeffsLRule** into the **hFormL** yields

$$hSolL = h_0 \text{ \texttt{\textcircled{z}} } basicTerms \cdot hFormL \cdot coeffsLRule \cdot D_i \rightarrow 0$$

$$\frac{z \otimes tD}{4wHw - WL} + \frac{\dot{z} \otimes tD}{4wHw + WL}$$

Then, we write the first-order solution in pure function form as

```

hRuleL = 9h0 -> HEvaluate@hSolL • Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
  h0 -> HEvaluate@hSolL • conjugateRule • Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=

:h0 \otimes \int \frac{\#3}{k 4wHw - WL} + \frac{\#4}{4wHw + WL} \&\forall, \dot{h}_0 \otimes \int \frac{\#3}{k 4wHw + WL} + \frac{\#4}{4wHw - WL} \&\forall >

```

¶ Second-Order Solution

Substituting the **hRuleL** into the left-hand side of the second-order equation, **eqEps[[2]]**, yields

```

order2expr = eqEps@@2, 1DD • hRuleL • Expand

```

$$\begin{aligned}
& \frac{mz \otimes tD}{4wHw - WL} - \frac{mz \otimes tD}{4wHw + WL} - \frac{Idz \otimes tD^2}{32w^3Hw - WL^2} - \frac{Idz \otimes tD^2}{32w^3Hw + WL^2} - \frac{Idz \otimes tD^2}{16w^3Hw - WLHw + WL} + m h \otimes tD - \\
& \frac{Idz \otimes tD h \otimes tD}{4w^2Hw - WL} - \frac{Idz \otimes tD h \otimes tD}{4w^2Hw + WL} - \frac{Idh \otimes tD^2}{2w} - \frac{m \dot{z} \otimes tD}{4wHw - WL} + \frac{m \dot{z} \otimes tD}{4wHw + WL} - \frac{Idz \otimes tD \dot{z} \otimes tD}{16w^3Hw - WL^2} - \\
& \frac{Idz \otimes tD \dot{z} \otimes tD}{16w^3Hw + WL^2} - \frac{Idz \otimes tD \dot{z} \otimes tD}{8w^3Hw - WLHw + WL} - \frac{Idh \otimes tD \dot{z} \otimes tD}{4w^2Hw - WL} - \frac{Idh \otimes tD \dot{z} \otimes tD}{4w^2Hw + WL} - \frac{Id \dot{z} \otimes tD^2}{32w^3Hw - WL^2} - \\
& \frac{Id \dot{z} \otimes tD^2}{32w^3Hw + WL^2} - \frac{Id \dot{z} \otimes tD^2}{16w^3Hw - WLHw + WL} - m \dot{h} \otimes tD - \frac{Idz \otimes tD \dot{h} \otimes tD}{4w^2Hw - WL} - \frac{Idz \otimes tD \dot{h} \otimes tD}{4w^2Hw + WL} - \frac{Idh \otimes tD \dot{h} \otimes tD}{w} - \\
& \frac{Id \dot{z} \otimes tD \dot{h} \otimes tD}{4w^2Hw - WL} - \frac{Id \dot{z} \otimes tD \dot{h} \otimes tD}{4w^2Hw + WL} - \frac{Id \dot{h} \otimes tD^2}{2w} + g_1 \otimes tD - I w h_1 \otimes h \otimes tD, \dot{h} \otimes tD, z \otimes tD, \dot{z} \otimes tDD - \\
& I w \dot{z} \otimes tD h_1^{H_0, 0, 0, 1L} \otimes h \otimes tD, \dot{h} \otimes tD, z \otimes tD, \dot{z} \otimes tDD + I w z \otimes tD h_1^{H_0, 0, 1, 0L} \otimes h \otimes tD, \dot{h} \otimes tD, z \otimes tD, \dot{z} \otimes tDD - \\
& I w \dot{h} \otimes tD h_1^{H_0, 1, 0, 0L} \otimes h \otimes tD, \dot{h} \otimes tD, z \otimes tD, \dot{z} \otimes tDD + I w h \otimes tD h_1^{H_1, 0, 0, 0L} \otimes h \otimes tD, \dot{h} \otimes tD, z \otimes tD, \dot{z} \otimes tDD
\end{aligned}$$

We choose h_1 to eliminate as many terms from **order2expr**, thereby reducing it to its simplest possible form. It turns out that we can eliminate all nonresonance terms. To determine the resonance terms in **order2expr**, we first determine all of the possible forms of its nonhomogeneous terms:

```
possibleQTerms =
  basicTerms ~ Join ~ HOuter@Times, basicTerms, basicTermsD • Flatten • UnionL
9h@tD, h@tD, z@tD, z@tD^2, z@tD h@tD, h@tD^2,
  z@tD z@tD, h@tD z@tD, z@tD^2, z@tD h@tD, h@tD h@tD, z@tD h@tD, h@tD^2=
```

It follows from the linear parts of the **etaRule** that

```
form = 9h@tD -> E^I w t, h@tD -> E^-I w t, z@tD -> E^I w t, z@tD -> E^-I w t=;
```

Hence, the possible resonance terms are given by

```
ResonantQTerm =
  I E^-I w t possibleQTerms • form • W -> 2 w • E^-t -> 0M possibleQTerms • Union • Rest
8h@tD, z@tD h@tD<
```

We choose g_1 to eliminate the resonance terms in **order2expr** according to

```
gRuleQ =
  g1@tD -> -HCoefficient@order2expr • h1 -> H0 &L, ResonantQTermD • Thread@basicTerms -> 0DL.
  ResonantQTerm
g1@tD ® -m h@tD - ∫_k - (I d / (4 w^2 Hw - WL) - (I d / (4 w^2 Hw + WL) ) z@tD h@tD
```

The nonresonance terms are the complement of the resonance terms; that is,

```
NRQT = Complement@possibleQTerms, ResonantQTermD;
```

We associate with each of them an undetermined coefficient according to

```
coeffsQ = Table@Gj, 8j, Length@NRQTD<D
8G1, G2, G3, G4, G5, G6, G7, G8, G9, G10, G11, G12<
```

Therefore, h_1 has the form

```
hFormQ = 8h1 -> HEvaluate@coeffsQ.NRQT • Thread@basicTerms -> 8#1, #2, #3, #4<DD &L<
8h1 ® H#3 G1 + #3^2 G2 + #1 #3 G3 + #1^2 G4 + #4 G5 +
  #3 #4 G6 + #1 #4 G7 + #4^2 G8 + #2 G9 + #1 #2 G10 + #2 #4 G11 + #2^2 G12 &L<
```

Substituting for h_1 in **order2expr**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting equations for the G_i , we obtain

```
coeffsQRule = Solve@Coefficient@order2expr . gRuleQ . hFormQ . Expand, NRQTD == 0 .
Thread@basicTerms -> 0D . Thread, coeffsQD@1DD
```

$$\begin{aligned}
9G_3 &\otimes - \frac{d}{2wWH-w+WLHw+WL}, G_4 \otimes \frac{d}{2w^2}, G_6 \otimes - \frac{d}{4w^2Hw-WL^2Hw+WL^2}, G_7 \otimes \frac{d}{2wWH-w+WLHw+WL}, \\
G_9 &\otimes \frac{Im}{2w}, G_{10} \otimes - \frac{d}{w^2}, G_{12} \otimes - \frac{d}{6w^2}, G_1 \otimes - \frac{ImW}{2wHw-WL^2Hw+WL}, G_2 \otimes - \frac{d}{8wHw-2WLHw-WL^2Hw+WL^2}, \\
G_5 &\otimes \frac{ImW}{2wHw-WLHw+WL^2}, G_{11} \otimes - \frac{d}{2wHw-WLHw+WLH_2w+WL}, G_8 \otimes - \frac{d}{8wHw-WL^2Hw+WL^2Hw+2WL} =
\end{aligned}$$

Substituting the **coeffsQRule** into the **hFormQ** yields

```
hSolQ = h1 . Thread@basicTerms . hFormQ . coeffsQRule
```

$$\begin{aligned}
& - \frac{ImWz@tD}{2wHw-WL^2Hw+WL} - \frac{dz@tD^2}{8wHw-2WLHw-WL^2Hw+WL^2} - \frac{dz@tDh@tD}{2wWH-w+WLHw+WL} + \\
& \frac{dh@tD^2}{2w^2} + \frac{ImWz@tD}{2wHw-WLHw+WL^2} - \frac{dz@tDz@tD}{4w^2Hw-WL^2Hw+WL^2} + \frac{dh@tDz@tD}{2wWH-w+WLHw+WL} - \\
& \frac{dz@tD^2}{8wHw-WL^2Hw+WL^2Hw+2WL} + \frac{Imh@tD}{2w} - \frac{dh@tDh@tD}{w^2} - \frac{dz@tDh@tD}{2wHw-WLHw+WLH_2w+WL} - \frac{dh@tD^2}{6w^2}
\end{aligned}$$

Hence, we can write the second-order solution in pure function form as

```
hRuleQ = 9h1 -> HEvaluate@hSolQ . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
h1 -> HEvaluate@hSolQ . conjugateRule . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=;
```

Third-Order Equations

Substituting the **hRuleL**, **hRuleQ**, and **gRuleQ** into the left-hand side of the third-order equation, **eqEps[[3]]**, yields

```
order3expr =
eqEps@@@3, 1DD . hRuleL . hRuleQ . gRuleQ . HgRuleQ . conjugateRuleL . Expand;
```

The nonhomogeneous terms in **order3expr** are proportional to

```
possibleCTerms = Outer@Times, possibleQTerms, basicTermsD . Flatten . Union
```

$$\begin{aligned}
& 9z@tD^2, z@tD^3, z@tDh@tD, z@tD^2h@tD, h@tD^2, z@tDh@tD^2, h@tD^3, z@tDz@tD, z@tD^2z@tD, \\
& h@tDz@tD, z@tDh@tDz@tD, h@tD^2z@tD, z@tDz@tD^2, h@tDz@tD^2, z@tD^3, z@tDh@tD, \\
& z@tD^2h@tD, h@tDh@tD, z@tDh@tDh@tD, h@tD^2h@tD, z@tDz@tDh@tD, \\
& h@tDz@tDh@tD, z@tD^2h@tD, h@tD^2, z@tDh@tD^2, h@tDh@tD^2, z@tDh@tD^2, h@tD^3=
\end{aligned}$$

```
possibleTerms = Join@possibleQTerms, possibleCTermsD •• Union
```

```
9z@tD, z@tD^2, z@tD^3, h@tD, z@tD h@tD, z@tD^2 h@tD, h@tD^2, z@tD h@tD^2, h@tD^3, z@tD z@tD,
z@tD^2 z@tD, h@tD z@tD, z@tD h@tD z@tD, h@tD^2 z@tD, z@tD^2, z@tD z@tD^2, h@tD z@tD^2,
z@tD^3, h@tD, z@tD h@tD, z@tD^2 h@tD, h@tD h@tD, z@tD h@tD h@tD, h@tD^2 h@tD, z@tD h@tD,
z@tD z@tD h@tD, h@tD z@tD h@tD, z@tD^2 h@tD, h@tD^2, z@tD h@tD^2, h@tD h@tD^2, z@tD h@tD^2, h@tD^3=
```

Next, we determine the resonance terms according to

```
ResonantCTerm =
```

```
I E-Iwt possibleTerms •. form •. W -> 2 w •. E-t -> 0M possibleTerms •• Union •• Rest
```

```
8h@tD, z@tD h@tD z@tD, z@tD h@tD, h@tD^2 h@tD<
```

Choosing g_2 to eliminate the resonance terms from **order3expr** yields

```
gRuleC = g2@tD ->
```

```
- Factor@Coefficient@order3expr •. h2 -> H0 &L, ResonantCTermD •. Thread@basicTerms -> 0DD.
ResonantCTerm
```

$$g_2@tD \otimes - \frac{I m^2 h@tD}{2 w} + \frac{I H - 5 d^2 w + 6 a w^3 - 2 d^2 W + 3 a w^2 W L z@tD h@tD z@tD}{4 w^3 H w - W L^2 H w + W L^2 H^2 w + W L} -$$

$$\frac{d m H^2 w^3 + w^2 W - 6 w W^2 - W^3 L z@tD h@tD}{4 w^2 H w - W L^2 W H w + W L^2} - \frac{I H 10 d^2 - 9 a w^2 L h@tD^2 h@tD}{6 w^3}$$

The nonresonance terms are the complement of the resonance terms; that is,

```
NRCT = Complement@possibleTerms, ResonantCTermD;
```

Again, we associate an undetermined coefficient with each of these terms as follows:

```
coeffsC = Table@Lj, 8j, Length@NRCTD<D
```

```
8L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, L11, L12, L13, L14, L15,
L16, L17, L18, L19, L20, L21, L22, L23, L24, L25, L26, L27, L28, L29, L30<
```

Consequently, h_2 has the form

```
hFormC = 8h2 -> HEvaluate@coeffsC.NRCT •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L<;
```

Substituting for h_2 into **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the L_i , we obtain

```
coeffsCRule = Solve@Coefficient@order3expr •. gRuleC •. hFormC •• Expand, NRCTD == 0 •.
Thread@basicTerms -> 0D •• Thread, coeffsCD@@1DD •• ExpandAll;
```

Substituting the **coeffsCRule** into the **hFormC** yields

```
hSolC = h2 žž basicTerms •. hFormC •. coeffsCRule;
```

Combining the **etaRule**, **gRuleQ**, and **gRuleC** and letting

$$\text{solForm} = 9h \rightarrow IA\#D E^{Iw\#} \&M, \dot{h} \rightarrow I\dot{A}\#D E^{-Iw\#} \&M, z \rightarrow If E^{Iw\#} \&M, \dot{z} \rightarrow If E^{-Iw\#} \&M;$$

we obtain the modulation equation

$$\text{moduEq} = I2 I w E^{-Iwt} Hh^c @tD - Hh^c @tD \cdot \text{etaRule} \cdot \text{gRuleQ} \cdot \text{gRuleCLL} \cdot \text{solForm} \cdot \text{Expand} \cdot \text{Collect} @\#, eD \&M == 0$$

$$\begin{aligned} e \int_k \left\{ 2 I m w A @tD + \frac{E^{-2Itw+ItW} f d \dot{A} @tD}{2 w Hw - WL} + \frac{E^{-2Itw+ItW} f d \dot{A} @tD y}{2 w Hw + WL} \right\} + \\ e^2 \int_k \left\{ -m^2 A @tD - \frac{5 f^2 d^2 A @tD}{2 w Hw - WL^2 Hw + WL^2 H2 w + WL} + \frac{3 f^2 a w A @tD}{Hw - WL^2 Hw + WL^2 H2 w + WL} + \right. \\ \frac{3 f^2 a w A @tD}{2 Hw - WL^2 Hw + WL^2 H2 w + WL} - \frac{f^2 d^2 w A @tD}{w^2 Hw - WL^2 Hw + WL^2 H2 w + WL} + \\ \frac{I E^{-2Itw+ItW} f d m w \dot{A} @tD}{2 Hw - WL^2 Hw + WL^2} + \frac{I E^{-2Itw+ItW} f d m w^2 \dot{A} @tD}{Hw - WL^2 W Hw + WL^2} - \frac{3 I E^{-2Itw+ItW} f d m W \dot{A} @tD}{Hw - WL^2 Hw + WL^2} - \\ \left. \frac{I E^{-2Itw+ItW} f d m W^2 \dot{A} @tD}{2 w Hw - WL^2 Hw + WL^2} + 3 a A @tD^2 \dot{A} @tD - \frac{10 d^2 A @tD^2 \dot{A} @tD y}{3 w^2} \right\} + 2 I w A^c @tD == 0 \end{aligned}$$

or

$$\text{moduEq} \cdot f \rightarrow 2 L I W^2 - w^2 M \cdot W \rightarrow 2 w$$

$$\begin{aligned} e H2 I m w A @tD - 2 d L \dot{A} @tDL + \\ e^2 \int_k \left\{ 6 a L^2 A @tD - m^2 A @tD^2 - \frac{9 d^2 L^2 A @tD}{2 w^2} - \frac{14 I d L m \dot{A} @tD}{3 w} + 3 a A @tD^2 \dot{A} @tD - \frac{10 d^2 A @tD^2 \dot{A} @tD y}{3 w^2} \right\} + \\ 2 I w A^c @tD == 0 \end{aligned}$$

which is in agreement with those obtained by using the method of multiple scales and the generalized method of averaging.

Chapter 5

Higher-Order Approximations for Systems with Internal Resonances

Off@General::spell1D

à 5.1 Euler-Lagrange Equations

In this chapter, we use different methods to determine approximate solutions of nonlinear systems possessing internal resonances to orders higher than the order at which the influence of the internal resonance first appears. To describe the methods with minimum algebra, we consider the free oscillations of a two-degree-of-freedom conservative system possessing a two-to-one internal resonance and having simple quadratic nonlinearities. In particular, we consider a system governed by the **Lagrangian**

$$\text{Lagrangian} = \frac{1}{2} \dot{u}_1^2 + \frac{1}{2} \dot{u}_2^2 - \frac{1}{2} \omega_1^2 u_1^2 - \frac{1}{2} \omega_2^2 u_2^2 + d u_1 \dot{u}_2^2 u_2;$$

where $\omega_2 \gg 2\omega_1$. Writing down the Euler-Lagrange equations corresponding to the **Lagrangian**, we obtain the following second-order equations of motion:

$$\text{EOM1} = \text{Table}[\text{D}[\text{Lagrangian}, u_1] == 0, \text{D}[\text{Lagrangian}, u_2] == 0, \{u_1, u_2\}]$$

$$8\omega_1^2 u_1 - 2d u_1 \dot{u}_2^2 + u_1^3 = 0, -d u_1 \dot{u}_2^2 + \omega_2^2 u_2 + u_2^3 = 0$$

Approximate solutions of general two-degree-of-freedom system with quadratic and cubic nonlinearities having a two-to-one internal resonance can be obtained by replacing the **Lagrangian** with the Lagrangian corresponding to the general system.

As discussed in the next section, treating this second-order form, **EOM1**, of the governing equations by using the method of multiple scales may lead to results that violate the conservative nature of the system under consideration. To determine an approximate solution that preserves the conservative nature of the system, we treat a first-order form of the governing equations. To transform **EOM1** into a system of first-order equations, we form a modified **Lagrangian** by introducing two more states v_1 and v_2 such that

$$\text{velRule} = \{u_1 \dot{u}_1 \rightarrow v_1, u_2 \dot{u}_2 \rightarrow v_2\};$$

Then, the **Lagrangian** becomes

$$\text{Lag1} = \text{Lagrangian} \cdot \text{velRule}$$

$$-\frac{1}{2} \omega_1^2 u_1^2 + d u_1 v_2^2 - \frac{1}{2} \omega_2^2 u_2^2 + \frac{1}{2} v_1^2 + \frac{1}{2} v_2^2$$

Substituting **Lag1** into Euler-Lagrange equations, we obtain the following first-order equations

$$\begin{aligned} \text{eq51a} &= \text{Table}[\text{D}[\text{Lag1}, \text{v}_i[t]], \text{tD} - \text{D}[\text{Lag1}, u_i[t]] == 0, \{i, 2\}] \\ 8w_1^2 u_1[t] - 2 \frac{d}{dt} u_1[t] u_2[t] + v_1^2[t] &= 0, \quad -\frac{d}{dt} u_1[t]^2 + w_2^2 u_2[t] + v_2^2[t] = 0 \end{aligned}$$

Using the **velRule** and **eq51a**, we obtain the following first-order equations of motion:

$$\begin{aligned} \text{EOM2} &= \text{8velRule} \cdot \text{Rule} \rightarrow \text{Equal}, \text{eq51a} \ll \text{Transpose} \ll \text{Flatten} \\ 8u_1^2[t] &= v_1[t], \quad w_1^2 u_1[t] - 2 \frac{d}{dt} u_1[t] u_2[t] + v_1^2[t] = 0, \\ u_2^2[t] &= v_2[t], \quad -\frac{d}{dt} u_1[t]^2 + w_2^2 u_2[t] + v_2^2[t] = 0 \end{aligned}$$

The corresponding linear system of **EOM2** can be defined as $\dot{x} = Ax$, where x is the state vector; that is,

$$\begin{aligned} \text{states} &= \text{Table}[u_i[t], v_i[t], \{i, 2\}] \ll \text{Flatten} \\ 8u_1[t], v_1[t], u_2[t], v_2[t] \end{aligned}$$

Using these states, we find that the coefficient matrix A of the linear part of **EOM2** is given by

$$\begin{aligned} \text{matrixA} &= \text{Outer}[\text{D}, \text{states} \ll \text{Solve}[\text{EOM2}, \text{states}, \text{tD}]] \cdot \text{D} \rightarrow 0, \text{statesD} \\ 8w_1^2, 1, 0, 0, 8-w_1^2, 0, 0, 0, 8, 0, 0, 1, 8, 0, -w_2^2, 0 \end{aligned}$$

whose adjoint is defined by

$$\begin{aligned} \text{conjugateRule} &= 9A \rightarrow \dot{A}, \dot{A} \rightarrow A, \text{Complex}[0, n_D] \rightarrow \text{Complex}[0, -n_D]; \\ \text{hermitian}[\text{mat}_? \text{MatrixQ}] &:= \text{mat} \cdot \text{conjugateRule} \ll \text{Transpose} \end{aligned}$$

The eigenvalues and eigenvectors of **matrixA** can be obtained as

$$\begin{aligned} \text{matrixA} \ll \text{Eigensystem} \\ \{8 - w_1^2, w_1^2, -w_2^2, w_2^2\}, \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \} \end{aligned}$$

Hence, the right eigenvectors of **matrixA** corresponding to the eigenvalues w_1^2 and w_2^2 , respectively, are

$$\begin{aligned} \text{rightVec} &= \text{Eigenvectors}[\text{matrixA}], \{2, 4\} \\ \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} \end{aligned}$$

Moreover, the left eigenvectors of **matrixA** corresponding to the eigenvalues w_1^2 and w_2^2 , respectively, are

$$\begin{aligned} \text{leftVec} &= \text{Eigenvectors}[\text{hermitian}[\text{matrixA}]], \{3, 4\} \\ \{8 - w_1^2, 1, 0, 0, 8, 0, -w_2^2, 1\} \end{aligned}$$

whose complex conjugate is

```
ccleftVec = leftVec . conjugateRule
```

```
88I w1, 1, 0, 0<, 80, 0, I w2, 1<<
```

à 5.2 Method of Multiple Scales

We use the method of multiple scales to obtain a second-order uniform expansion of the solution of **EOM1** in Section 5.2.1, **EOM2** in Section 5.2.2, and the corresponding first-order complex-valued equations of **EOM1** in Section 5.2.3. Again, we need the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, which we symbolize by

```
Needs@"Utilities`Notation`"D
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = 8T0, T1, T2<;
```

In terms of the new time scales, we can express the time derivatives as

```
dt@1D@expr_D := SumAei D@expr, timeScales@@i+1DDD, 8i, 0, 2<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD • ExpandL • ei-•;i>3 -> 0;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
  9Derivative@a__DAui,j E@__D := SequenceFormATimes žž MapIndexedAD#1#2@@1DD-1 &, 8a<E, ui,jE,
  Derivative@a__DAAi E@__D := SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AiE,
  ui,j@__D -> ui,j, Ai@__D -> Ai;
```

Ÿ 5.2.1 Second-Order Real-Valued System

Using method of multiple scales, we assume that the solution of **EOM1** can be expressed in the form

```
solRule = ui -> I SumAej ui,j@#1, #2, #3D, 8j, 3<E &M;
```

```
multiScales = 8ui@tD -> ui@T0, T1, T2D, Derivative@n_D@ui@tD := dt@nD@ui@T0, T1, T2DD<;
```

Substituting **multiScales** and **solRule** into **EOM1**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain


```
eq521a = HEOM1 •. multiScales •. solRule •• ExpandAllL •. en_*; n>3 -> 0;
eq521a •. displayRule

8e HD02u1,1L + e2 HD02u1,2L + e3 HD02u1,3L + 2 e2 HD0 D1u1,1L + 2 e3 HD0 D1u1,2L + e3 HD12u1,1L + 2 e3 HD0 D2u1,1L +
e w12 u1,1 + e2 w12 u1,2 + e3 w12 u1,3 - 2 d e2 u1,1 u2,1 - 2 d e3 u1,2 u2,1 - 2 d e3 u1,1 u2,2 == 0,
e HD02u2,1L + e2 HD02u2,2L + e3 HD02u2,3L + 2 e2 HD0 D1u2,1L + 2 e3 HD0 D1u2,2L + e3 HD12u2,1L +
2 e3 HD0 D2u2,1L - d e2 u1,12 - 2 d e3 u1,1 u1,2 + e w22 u2,1 + e2 w22 u2,2 + e3 w22 u2,3 == 0<
```

Equating coefficients of like powers of e , we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq521a •• Transpose;
```

To place the linear operators on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := I#@@1DD & •ž eqEps@@1DD •. u-k,1 -> uk,iM ==
I#@@1DD & •ž eqEps@@1DD •. u-k,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using `eqOrder[i]` and `displayRule`, we rewrite `eqEps` in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm

D02u1,1 + w12 u1,1 == 0
D02u2,1 + w22 u2,1 == 0

D02u1,2 + w12 u1,2 == -2 HD0 D1u1,1L + 2 d u1,1 u2,1
D02u2,2 + w22 u2,2 == -2 HD0 D1u2,1L + d u1,12

D02u1,3 + w12 u1,3 == -2 HD0 D1u1,2L - D12u1,1 - 2 HD0 D2u1,1L + 2 d u1,2 u2,1 + 2 d u1,1 u2,2
D02u2,3 + w22 u2,3 == -2 HD0 D1u2,2L - D12u2,1 - 2 HD0 D2u2,1L + 2 d u1,1 u1,2
```

§ First-Order Equations: Linear System

To obtain the solution of `eqOrder[1]` by using `DSolve`, we transform `eqOrder[1]` into a set of ordinary-differential equations as

```
order1Eq = eqOrder@1D •. ui,j -> Hui,j@#1D &L

8w12 u1,1@T0D + u1,12@T0D == 0, w22 u2,1@T0D + u2,12@T0D == 0<
```

The particular solution of `order1Eq` can be expressed as

```
sol1p = DSolve@order1Eq, 8u1,1@T0D, u2,1@T0D<, T0D@@1DD •. C@_D -> 0

8u1,1@T0D ® 0, u2,1@T0D ® 0<
```

Hence, we write the first-order solution in function form as

```
sol1 = TableAui,1 -> FunctionA8T0, T1, T2<,
  Ai@T1, T2D Exp@I wi T0D + Ai@T1, T2D Exp@-I wi T0D + ui,1@T0D •. sol1p •• EvaluateE, 8i, 2<E
8u1,1 @ Function@8T0, T1, T2<, EI T0 w1 A1@T1, T2D + E-I T0 w1 A1@T1, T2DD,
u2,1 @ Function@8T0, T1, T2<, EI T0 w2 A2@T1, T2D + E-I T0 w2 A2@T1, T2DD<
```

Second-Order Equations

Substituting **sol1** into **eqOrder[2]** yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

$$\begin{aligned} D_0^2 u_{1,2} + w_1^2 u_{1,2} &= 2 E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2 - 2 I E^{I T_0 w_1} H D_1 A_1 L w_1 + \\ & 2 I E^{-I T_0 w_1} H D_1 A_1 L w_1 + 2 E^{-I T_0 w_1 + I T_0 w_2} d A_2 A_1 + 2 E^{I T_0 w_1 - I T_0 w_2} d A_1 A_2 + 2 E^{-I T_0 w_1 - I T_0 w_2} d A_1 A_2, \\ D_0^2 u_{2,2} + w_2^2 u_{2,2} &= E^{2 I T_0 w_1} d A_1^2 - 2 I E^{I T_0 w_2} H D_1 A_2 L w_2 + 2 I E^{-I T_0 w_2} H D_1 A_2 L w_2 + 2 d A_1 A_1 + E^{-2 I T_0 w_1} d A_1^2 \end{aligned}$$

To describe quantitatively the nearness of the **two-to-one internal resonance** $w_2 \gg 2w_1$, we introduce the detuning parameter s defined by

```
ResonanceCond = 8w2 == 2 w1 + e s<;
```

In eliminating secular terms, we need to express w_2 in terms of w_1 in some of the equations and w_1 in terms of w_2 in the other equations. To accomplish this, we let

```
omgList = Table@wi, 8i, 2<D;
omgRule = Solve@ResonanceCond, Drop@omgList, 8#<DD@@1DD & •ž 81, 2<
98w2 @ e s + 2 w1<, 9w1 @  $\frac{1}{2}$  H - e s + w2L==
```

To convert small-divisor terms into secular terms, we define the rule

```
expRule1@j_D := Exp@arg_D :> Exp@Expand@arg •. omgRule@@jDDD •. e T0 -> T1D
```

Substituting **expRule1[i]** into the right-hand sides of **order2Eq** and collecting the coefficients of $E^{I w_i T_0}$, we obtain the solvability conditions as

```
SCond1 = Coefficient@order2Eq@@#, 2DD •. expRule1@#D, Exp@I w# T0DD == 0 & •ž 81, 2<
92 EI T1 s d A2@T1, T2D A1@T1, T2D - 2 I w1 A1H1, 0L@T1, T2D == 0,
E-I T1 s d A1@T1, T2D2 - 2 I w2 A2H1, 0L@T1, T2D == 0=
```

Because the system is conservative, the modulation equations given by **SCond1** must be derivable from a Lagrangian. These equations are the Euler-Lagrange equations corresponding to the Lagrangian

$$\begin{aligned} \text{Lag} = & E^{I T_1 s} d A_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D^2 + E^{-I T_1 s} d A_1 @ T_1, T_2 D^2 \dot{A}_2 @ T_1, T_2 D - \\ & I w_1 \dot{A}_1 @ T_1, T_2 D A_1^{H1, OL} @ T_1, T_2 D - I w_2 \dot{A}_2 @ T_1, T_2 D A_2^{H1, OL} @ T_1, T_2 D + \\ & I w_1 A_1 @ T_1, T_2 D \dot{A}_1^{H1, OL} @ T_1, T_2 D + I w_2 A_2 @ T_1, T_2 D \dot{A}_2^{H1, OL} @ T_1, T_2 D; \end{aligned}$$

because

$$\begin{aligned} \text{a1SCond1} = & \text{TableA} - \text{DADALag}, \dot{A}_k^{H1, OL} @ T_1, T_2 D E, T_1 E + \text{DALag}, \dot{A}_k @ T_1, T_2 D E == 0, 8k, 2 < E \\ 92 E^{I T_1 s} d A_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D - 2 I w_1 A_1^{H1, OL} @ T_1, T_2 D & == 0, \\ E^{-I T_1 s} d A_1 @ T_1, T_2 D^2 - 2 I w_2 A_2^{H1, OL} @ T_1, T_2 D & == 0 = \end{aligned}$$

Equations **a1SCond1** are identical to equations **SCond1** as shown below

$$\text{a1SCond1} === \text{SCond1}$$

True

Alternatively, we can determine the modulation equations by using the time-averaged Lagrangian. To this end, we first define the rule

$$\text{expRule2} = \text{Exp@a_D} :> \text{Exp@Expand@a} \cdot \text{HResonanceCond} \cdot \text{Equal} \rightarrow \text{RuleLD} \cdot e^{T_0} \rightarrow T_1 D;$$

Substituting **multiScales**, **solRule**, **sol1**, and **expRule2** into the **Lagrangian** and selecting the slow-varying terms, we obtain the following time-averaged Lagrangian:

$$\begin{aligned} \text{TAL1} = & \text{SelectAIHLagrangian} \cdot \text{multiScales} \cdot \text{solRule} \cdot \text{ExpandL} \cdot e^{n \cdot \tau} \rightarrow 0 \cdot \text{sol1} \cdot \text{ExpandM} \cdot \\ & \text{expRule2}, \text{FreeQ}[\#, T_0 D] \& E \cdot e \rightarrow 1; \\ \text{TAL1} \cdot \text{displayRule} & \\ I HD_1 \dot{A}_1 L A_1 w_1 + I HD_1 \dot{A}_2 L A_2 w_2 - I HD_1 A_1 L w_1 \dot{A}_1 + E^{I T_1 s} d A_2 \dot{A}_1^2 + E^{-I T_1 s} d A_1^2 \dot{A}_2 - I HD_1 A_2 L w_2 \dot{A}_2 & \end{aligned}$$

One can easily verify that **TAL1** is identical to **Lag**. In fact,

$$\text{TAL1} == \text{Lag}$$

True

Next, we rewrite **SCond1** as

$$\begin{aligned} \text{SCond1Rule1} = & \text{SolveASCond1}, 9 A_1^{H1, OL} @ T_1, T_2 D, A_2^{H1, OL} @ T_1, T_2 D = E @ 1 D D \\ : A_1^{H1, OL} @ T_1, T_2 D @ - \frac{I E^{I T_1 s} d A_2 @ T_1, T_2 D A_1 @ T_1, T_2 D}{w_1}, A_2^{H1, OL} @ T_1, T_2 D @ - \frac{I E^{-I T_1 s} d A_1 @ T_1, T_2 D^2}{2 w_2} > \end{aligned}$$

whose complex conjugate is

$$\text{ccSCond1Rule1} = \text{SCond1Rule1} \cdot \text{conjugateRule};$$

To solve for the particular solution of **order2Eq**, we express the detuning parameter S in terms of the w_i ; that is,

```
sigRule = Solve@ResonanceCond, sD@@1DD
```

$$9s \approx -\frac{2w_1 - w_2}{e} =$$

Moreover, we express the scale T_1 in terms of the scale T_0 by using the rule

```
expRule3 = Exp@a_D := Exp@a . T_1 -> e T_0 . sigRule . Expand;
```

With the solvability conditions and the **expRule3**, **order2Eq** becomes

```
order2Eqm = order2Eq . SCond1Rule1 . ccSCond1Rule1 . expRule3;
order2Eqm . displayRule
```

$$8D_0^2 u_{1,2} + w_1^2 u_{1,2} == 2 E^{IT_0 w_1 + IT_0 w_2} d A_1 A_2 + 2 E^{-IT_0 w_1 - IT_0 w_2} d \dot{A}_1 \dot{A}_2, D_0^2 u_{2,2} + w_2^2 u_{2,2} == 2 d A_1 \dot{A}_1 <$$

Again, to use **DSolve**, we first transform **order2Eqm** into a set of ordinary-differential equations and then obtain the particular solution of **order2Eqm** as

```
sol2p = DSolve@order2Eqm . u_{i,j} -> Hu_{i,j}@#1D &L, 8u_{1,2}@T_0D, u_{2,2}@T_0D<, T_0D@@1DD . C@_D -> 0 . .
Simplify
```

$$\begin{aligned} &: u_{1,2}@T_0D \approx \\ &- \frac{1}{w_2 H_2 w_1 + w_2 L} H_2 E^{-IT_0 H w_1 + w_2 L} d H E^{2IT_0 H w_1 + w_2 L} A_1@T_1, T_2 D A_2@T_1, T_2 D + \dot{A}_1@T_1, T_2 D \dot{A}_2@T_1, T_2 D L L, \\ &u_{2,2}@T_0D \approx \frac{2 d A_1@T_1, T_2 D \dot{A}_1@T_1, T_2 D}{w_2^2} > \end{aligned}$$

The second-order solution can be expressed in pure function form as

```
sol2 = TableAu_{i,2} -> FunctionA8T_0, T_1, T_2<, B_i@T_1, T_2D Exp@I w_i T_0D +
      B_i@T_1, T_2D Exp@-I w_i T_0D + sol2p@i, 2DD . . Expand . . EvaluateE, 8i, 2<E;
sol2 . displayRule
```

$$\begin{aligned} &: u_{1,2} \approx \text{FunctionB8} T_0, T_1, T_2 <, - \frac{2 E^{IT_0 H w_1 + w_2 L} d A_1 A_2}{w_2 H_2 w_1 + w_2 L} + E^{IT_0 w_1} B_1 - \frac{2 E^{-IT_0 H w_1 + w_2 L} d \dot{A}_1 \dot{A}_2}{w_2 H_2 w_1 + w_2 L} + E^{-IT_0 w_1} \dot{B}_1 F, \\ &u_{2,2} \approx \text{FunctionB8} T_0, T_1, T_2 <, E^{IT_0 w_2} B_2 + \frac{2 d A_1 \dot{A}_1}{w_2^2} + E^{-IT_0 w_2} \dot{B}_2 F > \end{aligned}$$

As shown below, the obtained modulation equations will violate the conservative nature of system unless we include an appropriate part of the homogeneous solutions of the second-order problem. This is the reason why we included the homogeneous solutions with the undetermined functions $B_i(t)$. They will be determined so as to preserve the conservative nature of the system.

Third-Order Equations

Substituting **sol1** and **sol2** into **eqOrder[3]** yields

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Substituting **expRule1[i]** into the right-hand sides of **order3Eq** and collecting the coefficients of $E^{I w_i T_0}$, we obtain the solvability conditions as

```
SCond2 = Coefficient@order3Eq@@#, 2DD •. expRule1@#D, Exp@I w# T0DD == 0 & •ž 81, 2<;
```

```
SCond2 •. displayRule
```

$$\begin{aligned} &: -HD_1^2 A_1 L - 2 I HD_1 B_1 L w_1 - 2 I HD_2 A_1 L w_1 + 2 E^{I T_1 s} d B_2 \dot{A}_1 + \frac{4 d^2 A_1^2 \dot{A}_1}{w_2^2} - \frac{4 d^2 A_1 A_2 \dot{A}_2}{2 w_1 w_2 + w_2^2} + 2 E^{I T_1 s} d A_2 \dot{B}_1 == 0, \\ &- HD_1^2 A_2 L + 2 E^{-I T_1 s} d A_1 B_1 - 2 I HD_1 B_2 L w_2 - 2 I HD_2 A_2 L w_2 - \frac{4 d^2 A_1 A_2 \dot{A}_1}{2 w_1 w_2 + w_2^2} == 0 > \end{aligned}$$

Using the solvability conditions at second order, **SCond1Rule1**, we obtain

```
SCond1Rule2 = D@SCond1Rule1, T1D •. SCond1Rule1 •. ccSCond1Rule1;
```

```
SCond1Rule2 •. displayRule
```

$$\begin{aligned} &: D_1^2 A_1 @ \frac{E^{I T_1 s} d s A_2 \dot{A}_1}{w_1} - \frac{d^2 A_1^2 \dot{A}_1}{2 w_1 w_2} + \frac{d^2 A_1 A_2 \dot{A}_2}{w_1^2}, D_1^2 A_2 @ - \frac{E^{-I T_1 s} d s A_1^2}{2 w_2} - \frac{d^2 A_1 A_2 \dot{A}_1}{w_1 w_2} > \end{aligned}$$

Hence, we rewrite the solvability conditions at third order as

```
SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD •. SCond1Rule2 •• ExpandAll;
```

```
SCond2Rule1 •. displayRule
```

$$\begin{aligned} &: D_2 A_1 @ - HD_1 B_1 L + \frac{I E^{I T_1 s} d s A_2 \dot{A}_1}{2 w_1^2} - \frac{I E^{I T_1 s} d B_2 \dot{A}_1}{w_1} - \\ &\frac{2 I d^2 A_1^2 \dot{A}_1}{w_1 w_2^2} - \frac{I d^2 A_1^2 \dot{A}_1}{4 w_1^2 w_2} + \frac{I d^2 A_1 A_2 \dot{A}_2}{2 w_1^3} + \frac{2 I d^2 A_1 A_2 \dot{A}_2}{w_1 H_2 w_1 w_2 + w_2^2 L} - \frac{I E^{I T_1 s} d A_2 \dot{B}_1}{w_1}, \\ &D_2 A_2 @ - HD_1 B_2 L - \frac{I E^{-I T_1 s} d s A_1^2}{4 w_2^2} - \frac{I E^{-I T_1 s} d A_1 B_1}{w_2} - \frac{I d^2 A_1 A_2 \dot{A}_1}{2 w_1 w_2^2} + \frac{2 I d^2 A_1 A_2 \dot{A}_1}{w_2 H_2 w_1 w_2 + w_2^2 L} > \end{aligned}$$

Reconstitution

Using **SCond1Rule1** and **SCond2Rule1**, we reconstitute the modulation equations as

```

moduEq = Table@2 I w_k A_k^e, 8k, 2<D ==
  HTable@2 I w_k dt@1D@ A_k@T_1, T_2DD, 8k, 2<D . SCond1Rule1 . SCond2Rule1 . Expand .
  Collect@#, eD &L . Thread;
moduEq . displayRule

```

$$\begin{aligned}
& 2 I w_1 A_1^e == \\
& 2 E^{IT_1 s} d e A_2 \dot{A}_1 + e^2 \Bigg\} - 2 I H D_1 B_1 L w_1 + 2 E^{IT_1 s} d B_2 \dot{A}_1 - \frac{E^{IT_1 s} d s A_2 \dot{A}_1}{w_1} + \frac{4 d^2 A_1^2 \dot{A}_1}{w_2^2} + \frac{d^2 A_1^2 \dot{A}_1}{2 w_1 w_2} - \\
& \frac{d^2 A_1 A_2 \dot{A}_2}{w_1^2} - \frac{4 d^2 A_1 A_2 \dot{A}_2}{2 w_1 w_2 + w_2^2} + 2 E^{IT_1 s} d A_2 \dot{B}_1 \Bigg\} \frac{y}{z}, 2 I w_2 A_2^e == \\
& E^{-IT_1 s} d e A_1^2 + e^2 \Bigg\} 2 E^{-IT_1 s} d A_1 B_1 + \frac{E^{-IT_1 s} d s A_2^2}{2 w_2} - 2 I H D_1 B_2 L w_2 + \frac{d^2 A_1 A_2 \dot{A}_1}{w_1 w_2} - \frac{4 d^2 A_1 A_2 \dot{A}_1 y}{2 w_1 w_2 + w_2^2} \Bigg\} \frac{y}{z}
\end{aligned}$$

Without including the B_i in the second-order solution, the following two coefficients:

```

CoefficientAmoduEq@@1, 2DD, A_1@T_1, T_2D A_2@T_1, T_2D \dot{A}_2@T_1, T_2DE . w_2 -> 2 w_1 . Simplify

```

$$- \frac{3 d^2 e^2}{2 w_1^2}$$

```

CoefficientAmoduEq@@2, 2DD, A_1@T_1, T_2D A_2@T_1, T_2D \dot{A}_1@T_1, T_2DE . w_2 -> 2 w_1 . Simplify

```

0

are not the same, which violates the conservative nature of the system in that the modulation equations must be derivable from a Lagrangian. Using the flexibility given by the homogeneous parts of the solutions of the second-order problem, we choose the functions $B_i(t)$ so that these coefficients are the same. To this end, we assume that

```

BForm = 9B_1 -> FunctionA8T_1, T_2<, c_1 E^{IT_1 s} \dot{A}_1@T_1, T_2D A_2@T_1, T_2DE,
  \dot{B}_1 -> FunctionA8T_1, T_2<, c_1 E^{-IT_1 s} A_1@T_1, T_2D \dot{A}_2@T_1, T_2DE,
  B_2 -> FunctionA8T_1, T_2<, c_2 E^{-IT_1 s} A_1@T_1, T_2D^2 E;

```

where c_1 and c_2 are undetermined constants. Substituting **BForm** and the solvability conditions into the right-hand sides of **moduEq** yields

```

moduEqMod = HExpand@# . BForm . SCond1Rule1 . ccSCond1Rule1D & . \dot{z} moduEqL . ExpandAll;
moduEqMod . displayRule

```

$$\begin{aligned}
& 2 I w_1 A_1^e == 2 E^{IT_1 s} d e A_2 \dot{A}_1 + 2 d e^2 A_1^2 c_2 \dot{A}_1 - \frac{E^{IT_1 s} d e^2 s A_2 \dot{A}_1}{w_1} + 2 E^{IT_1 s} e^2 s A_2 c_1 w_1 \dot{A}_1 + \frac{4 d^2 e^2 A_1^2 \dot{A}_1}{w_2^2} + \\
& \frac{d^2 e^2 A_1^2 \dot{A}_1}{2 w_1 w_2} - \frac{d e^2 A_1^2 c_1 w_1 \dot{A}_1}{w_2} + 4 d e^2 A_1 A_2 c_1 \dot{A}_2 - \frac{d^2 e^2 A_1 A_2 \dot{A}_2}{w_1^2} - \frac{4 d^2 e^2 A_1 A_2 \dot{A}_2}{2 w_1 w_2 + w_2^2}, \\
& 2 I w_2 A_2^e == E^{-IT_1 s} d e A_1^2 + \frac{E^{-IT_1 s} d e^2 s A_1^2}{2 w_2} - 2 E^{-IT_1 s} e^2 s A_1^2 c_2 w_2 + 2 d e^2 A_1 A_2 c_1 \dot{A}_1 + \\
& \frac{d^2 e^2 A_1 A_2 \dot{A}_1}{w_1 w_2} - \frac{4 d e^2 A_1 A_2 c_2 w_2 \dot{A}_1}{w_1} - \frac{4 d^2 e^2 A_1 A_2 \dot{A}_1}{2 w_1 w_2 + w_2^2} >
\end{aligned}$$

In order that the `moduEqMod` be derivable from a Lagrangian, the following condition must be satisfied:

$$\begin{aligned} & \text{CoefficientAmoduEqMod}@@1, 2DD, A_1@T_1, T_2D A_2@T_1, T_2D \dot{A}_2@T_1, T_2DE - \\ & \text{CoefficientAmoduEqMod}@@2, 2DD, A_1@T_1, T_2D A_2@T_1, T_2D \dot{A}_1@T_1, T_2DE == 0 \cdot e \rightarrow 1 \cdot w_2 \rightarrow 2 w_1 \\ & 2 d c_1 + 8 d c_2 - \frac{3 d^2}{2 w_1^2} == 0 \end{aligned}$$

This provides a compatibility condition. However, one extra constant is floating !!!

To obtain a consistent second-order uniform expansion, we apply the method of multiple scales to the system of four first-order real-valued equations `EOM2` in Section 5.2.2 and to a system of two first-order complex-valued equations in Section 5.2.3.

5.2.2 First-Order Real-Valued System

As shown in the preceding section, treating the second-order form of the equations governing a system may lead to inconsistent results unless appropriate parts of the homogeneous solutions of the perturbation equations are included. Moreover, the consistent expansions might contain arbitrary constants that need to be chosen judiciously. In this section, we show that treatment of an equivalent set of first-order real-valued equations, namely `EOM2`, leads to consistent results without floating constants.

We seek a second-order uniform expansion of the solution of `EOM2` in the form

$$\begin{aligned} \text{solRule} = \\ 9u_i \rightarrow \text{ISumAe}^j u_{i,j}@@\#1, \#2, \#3D, 8j, 3<E \&M, v_i \rightarrow \text{ISumAe}^j v_{i,j}@@\#1, \#2, \#3D, 8j, 3<E \&M=; \end{aligned}$$

Expressing the time derivative in `EOM2` in terms of the time scales T_0 , T_1 , and T_2 , we have

$$\text{multiScales} = 9u_{-i} @tD \rightarrow u_i@T_0, T_1, T_2D, \text{Derivative}@n_DAu_{-i} E@tD \rightarrow dt@nD@u_i@T_0, T_1, T_2DD=;$$

Substituting `multiScales` and `solRule` into `EOM2`, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

$$\text{eq522a} = \text{HEOM2} \cdot \text{multiScales} \cdot \text{solRule} \cdot \text{ExpandAll} \cdot e^{n \cdot \epsilon; n > 3} \rightarrow 0;$$

Equating coefficients of like powers of ϵ , we obtain

$$\text{eqEps} = \text{Rest}@Thread@CoefficientList@Subtract \# \& \epsilon D == 0DD \& \cdot \# \text{eq522a} \cdot \text{Transpose};$$

To place the linear operators on one side and the nonhomogeneous terms on the other side, we define

$$\begin{aligned} \text{eqOrder}@i_D := \text{I}@@1DD \& \cdot \# \text{eqEps}@@1DD \cdot u_{-k,1} \rightarrow u_{k,i}M == \\ \text{I}@@1DD \& \cdot \# \text{eqEps}@@1DD \cdot u_{-k,1} \rightarrow u_{k,i}M - \text{H}@@1DD \& \cdot \# \text{eqEps}@@iDDL \cdot \text{Thread} \end{aligned}$$

Using `eqOrder[i]` and `displayRule`, we rewrite `eqEps` in a concise way as

eqOrder@1D •. displayRule •• TableForm

eqOrder@2D •. displayRule •• TableForm

eqOrder@3D •. displayRule •• TableForm

$$D_0 u_{1,1} - v_{1,1} == 0$$

$$D_0 v_{1,1} + w_1^2 u_{1,1} == 0$$

$$D_0 u_{2,1} - v_{2,1} == 0$$

$$D_0 v_{2,1} + w_2^2 u_{2,1} == 0$$

$$D_0 u_{1,2} - v_{1,2} == -HD_1 u_{1,1} L$$

$$D_0 v_{1,2} + w_1^2 u_{1,2} == -HD_1 v_{1,1} L + 2 d u_{1,1} u_{2,1}$$

$$D_0 u_{2,2} - v_{2,2} == -HD_1 u_{2,1} L$$

$$D_0 v_{2,2} + w_2^2 u_{2,2} == -HD_1 v_{2,1} L + d u_{1,1}^2$$

$$D_0 u_{1,3} - v_{1,3} == -HD_1 u_{1,2} L - D_2 u_{1,1}$$

$$D_0 v_{1,3} + w_1^2 u_{1,3} == -HD_1 v_{1,2} L - D_2 v_{1,1} + 2 d u_{1,2} u_{2,1} + 2 d u_{1,1} u_{2,2}$$

$$D_0 u_{2,3} - v_{2,3} == -HD_1 u_{2,2} L - D_2 u_{2,1}$$

$$D_0 v_{2,3} + w_2^2 u_{2,3} == -HD_1 v_{2,2} L - D_2 v_{2,1} + 2 d u_{1,1} u_{1,2}$$

• First-Order Equations

The homogeneous solution of **eqOrder[1]** can be obtained by using

eigenForm = #@@2DD Exp@#@@1DD T₀D & •ž Transpose@Eigensystem@matrixADD

$$\begin{aligned} &: \left(\frac{I E^{-I T_0 w_1}}{w_1}, E^{-I T_0 w_1}, 0, 0 \right), \left(-\frac{I E^{I T_0 w_1}}{w_1}, E^{I T_0 w_1}, 0, 0 \right), \\ &: 0, 0, \left(\frac{I E^{-I T_0 w_2}}{w_2}, E^{-I T_0 w_2} \right), \left(-\frac{I E^{I T_0 w_2}}{w_2}, E^{I T_0 w_2} \right) \end{aligned}$$

sol1hForm = FlattenATableAI w_i 9 - A_i@T₁, T₂D, A_i@T₁, T₂D=, 8i, 2<EE.eigenForm

$$\begin{aligned} &8 E^{I T_0 w_1} A_1 @ T_1, T_2 D + E^{-I T_0 w_1} \dot{A}_1 @ T_1, T_2 D, I E^{I T_0 w_1} w_1 A_1 @ T_1, T_2 D - I E^{-I T_0 w_1} w_1 \dot{A}_1 @ T_1, T_2 D, \\ &E^{I T_0 w_2} A_2 @ T_1, T_2 D + E^{-I T_0 w_2} \dot{A}_2 @ T_1, T_2 D, I E^{I T_0 w_2} w_2 A_2 @ T_1, T_2 D - I E^{-I T_0 w_2} w_2 \dot{A}_2 @ T_1, T_2 D < \end{aligned}$$

Since the first-order equations are homogeneous, the solution of **eqOrder[1]** can be expressed as

states1 = 8u_{1,1}, v_{1,1}, u_{2,1}, v_{2,1}<;

sol1 = Table@states1@@iDD -> Function@8T₀, T₁, T₂<, sol1hForm@@iDD •• Evaluated, 8i, 4<D

$$\begin{aligned} &8 u_{1,1} @ \text{Function}@8 T_0, T_1, T_2 <, E^{I T_0 w_1} A_1 @ T_1, T_2 D + E^{-I T_0 w_1} \dot{A}_1 @ T_1, T_2 D, \\ &v_{1,1} @ \text{Function}@8 T_0, T_1, T_2 <, I E^{I T_0 w_1} w_1 A_1 @ T_1, T_2 D - I E^{-I T_0 w_1} w_1 \dot{A}_1 @ T_1, T_2 D, \\ &u_{2,1} @ \text{Function}@8 T_0, T_1, T_2 <, E^{I T_0 w_2} A_2 @ T_1, T_2 D + E^{-I T_0 w_2} \dot{A}_2 @ T_1, T_2 D, \\ &v_{2,1} @ \text{Function}@8 T_0, T_1, T_2 <, I E^{I T_0 w_2} w_2 A_2 @ T_1, T_2 D - I E^{-I T_0 w_2} w_2 \dot{A}_2 @ T_1, T_2 D < \end{aligned}$$

Second-Order Equations

Substituting the first-order solution into the second-order equations, **eqOrder[2]**, yields

```
order2Eq = eqOrder@2D •. sol1 •. ExpandAll;
order2Eq •. displayRule
```

$$\begin{aligned} D_0 u_{1,2} - v_{1,2} &= -E^{I T_0 w_1} HD_1 A_1 L - E^{-I T_0 w_1} HD_1 \dot{A}_1 L, \\ D_0 v_{1,2} + w_1^2 u_{1,2} &= 2 E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2 - I E^{I T_0 w_1} HD_1 A_1 L w_1 + I E^{-I T_0 w_1} HD_1 \dot{A}_1 L w_1 + 2 E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1 + \\ &\quad 2 E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2 + 2 E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2, \quad D_0 u_{2,2} - v_{2,2} = -E^{I T_0 w_2} HD_1 A_2 L - E^{-I T_0 w_2} HD_1 \dot{A}_2 L, \\ D_0 v_{2,2} + w_2^2 u_{2,2} &= E^{2 I T_0 w_1} d A_1^2 - I E^{I T_0 w_2} HD_1 A_2 L w_2 + I E^{-I T_0 w_2} HD_1 \dot{A}_2 L w_2 + 2 d A_1 \dot{A}_1 + E^{-2 I T_0 w_1} d \dot{A}_1^2 \end{aligned}$$

Next, we substitute the **expRule1[i]** into the right-hand sides of **order2Eq** to transform the small-divisor terms into secular terms, collect the terms responsible for secular terms, and obtain

```
ST11 = Coefficient@@@2DD •. expRule1@1D, Exp@I w1 T0DD & •. order2Eq;
ST11 •. displayRule
```

$$8 - HD_1 A_1 L, -I HD_1 A_1 L w_1 + 2 E^{I T_1 s} d A_2 \dot{A}_1, 0, 0 <$$

```
ST12 = Coefficient@@@2DD •. expRule1@2D, Exp@I w2 T0DD & •. order2Eq;
ST12 •. displayRule
```

$$80, 0, -HD_1 A_2 L, E^{-I T_1 s} d A_1^2 - I HD_1 A_2 L w_2 <$$

The solvability conditions of the second-order equations demands that **ST11** and **ST12** be orthogonal to every solution of the adjoint homogeneous problems, namely the components of **ccleftVec**. Imposing these conditions, we obtain

```
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0 <;
SCond1 •. displayRule
```

$$8 - 2 I HD_1 A_1 L w_1 + 2 E^{I T_1 s} d A_2 \dot{A}_1 == 0, E^{-I T_1 s} d A_1^2 - 2 I HD_1 A_2 L w_2 == 0 <$$

or

```
SCond1Rule1 = SolveASCond1, 9A1^H1, Q @T1, T2D, A2^H1, Q @T1, T2D=E@@1DD;
SCond1Rule1 •. displayRule
```

$$:D_1 A_1 \otimes - \frac{I E^{I T_1 s} d A_2 \dot{A}_1}{w_1}, D_1 A_2 \otimes - \frac{I E^{-I T_1 s} d A_1^2}{2 w_2} >$$

whose complex conjugate is

```
ccSCond1Rule1 = SCond1Rule1 •. conjugateRule;
```

Using these solvability conditions and **expRule3**, we rewrite **order2Eq** as

```
order2Eqm = order2Eq . SCond1Rule1 . ccSCond1Rule1 . expRule3;
order2Eqm . displayRule
```

$$\begin{aligned}
9D_0 u_{1,2} - v_{1,2} &= \frac{I E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{w_1} - \frac{I E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_1}, D_0 v_{1,2} + w_1^2 u_{1,2} = \\
& 2 E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2 + E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1 + E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2 + 2 E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2, \\
D_0 u_{2,2} - v_{2,2} &= \frac{I E^{2 I T_0 w_1} d A_1^2}{2 w_2} - \frac{I E^{-2 I T_0 w_1} d \dot{A}_1^2}{2 w_2}, \\
D_0 v_{2,2} + w_2^2 u_{2,2} &= \frac{1}{2} E^{2 I T_0 w_1} d A_1^2 + 2 d A_1 \dot{A}_1 + \frac{1}{2} E^{-2 I T_0 w_1} d \dot{A}_1^2 =
\end{aligned}$$

Transforming **order2Eqm** into a set of ordinary-differential equations and using **DSolve**, we obtain the particular solutions as

```
Isol2p = DSolveAorder2Eqm . u_{i,2} -> Hu_{i,2}@#1D &L,
      8u_{1,2}@T_0D, v_{1,2}@T_0D, u_{2,2}@T_0D, v_{2,2}@T_0D<, T_0E@@1DD . C@_D -> 0 . Simplify;M . Timing
813.139 Second, Null<
```

Hence, the second-order solutions of **order2Eqm** can be expressed as

```
states2 = states1 . u_{i,1} -> u_{i,2}
```

```
8u_{1,2}, v_{1,2}, u_{2,2}, v_{2,2}<
```

```
sol2 = Table@states2@@iDD -> Function@8T_0, T_1, T_2<,
      Expand@sol2p@@i, 2DDD . Exp@a_D -> Exp@a . ExpandD . EvaluatedD, 8i, 4<D;
sol2 . displayRule
```

$$\begin{aligned}
9u_{1,2} \otimes \text{FunctionB8}T_0, T_1, T_2<, & - \frac{2 E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_2 H^2 w_1 + w_2 L} + \frac{E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{w_1 H^2 w_1 + w_2 L} + \\
& \frac{2 E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{w_2 H^2 w_1 + w_2 L} + \frac{E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_1 H^2 w_1 + w_2 L} + \frac{2 E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_2 H^2 w_1 + w_2 L} - \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{w_2 H^2 w_1 + w_2 L} F, \\
v_{1,2} \otimes \text{FunctionB8}T_0, T_1, T_2<, & - \frac{2 I E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{2 w_1 + w_2} - \frac{2 I E^{I T_0 w_1 + I T_0 w_2} d w_1 A_1 A_2}{w_2 H^2 w_1 + w_2 L} - \\
& \frac{I E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{2 w_1 + w_2} - \frac{2 I E^{-I T_0 w_1 + I T_0 w_2} d w_1 A_2 \dot{A}_1}{w_2 H^2 w_1 + w_2 L} + \frac{I E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{2 w_1 + w_2} + \\
& \frac{2 I E^{I T_0 w_1 - I T_0 w_2} d w_1 A_1 \dot{A}_2}{w_2 H^2 w_1 + w_2 L} + \frac{2 I E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{2 w_1 + w_2} + \frac{2 I E^{-I T_0 w_1 - I T_0 w_2} d w_1 \dot{A}_1 \dot{A}_2}{w_2 H^2 w_1 + w_2 L} F, \\
u_{2,2} \otimes \text{FunctionA8}T_0, T_1, T_2<, & \frac{E^{2 I T_0 w_1} d A_1^2}{2 w_2 H^2 w_1 + w_2 L} + \frac{4 d w_1 A_1 \dot{A}_1}{w_2 H^2 w_1 + w_2 L} + \frac{2 d A_1 \dot{A}_1}{w_2 H^2 w_1 + w_2 L} + \frac{E^{-2 I T_0 w_1} d \dot{A}_1^2}{2 w_2 H^2 w_1 + w_2 L} E, \\
v_{2,2} \otimes \text{FunctionA8}T_0, T_1, T_2<, & - \frac{I E^{2 I T_0 w_1} d A_1^2}{2 H^2 w_1 + w_2 L} + \frac{I E^{-2 I T_0 w_1} d \dot{A}_1^2}{2 H^2 w_1 + w_2 L} E =
\end{aligned}$$

Third-Order Equations

Substituting the first- and second-order solutions into the right-hand sides of the third-order equations, `eqOrder[3]`, yields

```
order3Eqrhs = Expand@@@2DD •. sol1 •. sol2D & •. eqOrder@3D;
```

To eliminate the terms that lead to secular terms from `order3Eqrhs`, we use `expRule1[i]` to transform the terms that convert small-divisor terms into secular terms, calculate the vectors proportional to the $E^{I w_i T_0}$, and obtain

```
ST21 = Coefficient@order3Eqrhs •. expRule1@1D, Exp@I w1 T0DD;
```

```
ST21 •. displayRule
```

$$\begin{aligned} &: -HD_2 A_1 L - \frac{E^{IT_1} s d HD_1 A_1 L A_2}{w_1 H_2 w_1 + w_2 L} - \frac{2 E^{IT_1} s d HD_1 A_1 L A_2}{w_2 H_2 w_1 + w_2 L} - \frac{E^{IT_1} s d HD_1 A_2 L A_1}{w_1 H_2 w_1 + w_2 L} - \frac{2 E^{IT_1} s d HD_1 A_2 L A_1}{w_2 H_2 w_1 + w_2 L}, \\ &- I HD_2 A_1 L w_1 + \frac{I E^{IT_1} s d HD_1 A_1 L A_2}{2 w_1 + w_2} + \frac{2 I E^{IT_1} s d HD_1 A_1 L A_2 w_1}{w_2 H_2 w_1 + w_2 L} + \frac{I E^{IT_1} s d HD_1 A_2 L A_1}{2 w_1 + w_2} + \\ &\frac{8 d^2 A_2^2 w_1 A_1}{w_2^2 H_2 w_1 + w_2 L} + \frac{5 d^2 A_2^2 A_1}{w_2 H_2 w_1 + w_2 L} + \frac{2 I E^{IT_1} s d HD_1 A_2 L w_1 A_1}{w_2 H_2 w_1 + w_2 L} + \frac{2 d^2 A_1 A_2 A_2}{w_1 H_2 w_1 + w_2 L}, 0, 0 > \end{aligned}$$

```
ST22 = Coefficient@order3Eqrhs •. expRule1@2D, Exp@I w2 T0DD;
```

```
ST22 •. displayRule
```

$$: 0, 0, -HD_2 A_2 L - \frac{E^{-IT_1} s d HD_1 A_1 L A_1}{w_2 H_2 w_1 + w_2 L}, -I HD_2 A_2 L w_2 + \frac{I E^{-IT_1} s d HD_1 A_1 L A_1}{2 w_1 + w_2} + \frac{2 d^2 A_1 A_2 A_1}{w_1 H_2 w_1 + w_2 L} >$$

Hence, demanding that `ST21` and `ST22` be orthogonal to the components of `ccleftVec` yields the solvability conditions

```
SCond2 = 8Expand@ccleftVec@@1DD.ST21D == 0, Expand@ccleftVec@@2DD.ST22D == 0<;
```

```
SCond2 •. displayRule
```

$$\begin{aligned} &: -2 I HD_2 A_1 L w_1 + \frac{8 d^2 A_1^2 w_1 A_1}{w_2^2 H_2 w_1 + w_2 L} + \frac{5 d^2 A_1^2 A_1}{w_2 H_2 w_1 + w_2 L} + \frac{2 d^2 A_1 A_2 A_2}{w_1 H_2 w_1 + w_2 L} == 0, \\ &-2 I HD_2 A_2 L w_2 + \frac{2 d^2 A_1 A_2 A_1}{w_1 H_2 w_1 + w_2 L} == 0 > \end{aligned}$$

or

```
SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD;
```

```
SCond2Rule1 •. displayRule
```

$$: D_2 A_1 \otimes - \frac{I H_8 d^2 A_2^2 w_1^2 A_1 + 5 d^2 A_2^2 w_1 w_2 A_1 + 2 d^2 A_1 A_2 w_2^2 A_2 L}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L}, D_2 A_2 \otimes - \frac{I d^2 A_1 A_2 A_1}{w_1 w_2 H_2 w_1 + w_2 L} >$$

Reconstitution

We reconstitute `SCond1Rule1` and `SCond2Rule1` and obtain

```

moduEq = Table@2 I wk Akε, 8k, 2<D ==
  HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D . SCond1Rule1 . SCond2Rule1 . .
  ExpandL . . Thread;
moduEq . . displayRule

: 2 I w1 A1ε == 2 EI T1 s d e A2 A1 +  $\frac{8 d^2 e^2 A_1^2 w_1 \dot{A}_1}{w_2^2 H^2 w_1 + w_2 L}$  +  $\frac{5 d^2 e^2 A_1^2 \dot{A}_1}{w_2 H^2 w_1 + w_2 L}$  +  $\frac{2 d^2 e^2 A_1 A_2 \dot{A}_2}{w_1 H^2 w_1 + w_2 L}$ ,

2 I w2 A2ε == E-I T1 s d e A12 +  $\frac{2 d^2 e^2 A_1 A_2 \dot{A}_1}{w_1 H^2 w_1 + w_2 L}$  >

```

These equations are derivable from a Lagrangian because

```

CoefficientAmoduEq@@1, 2DD, A1@T1, T2D A2@T1, T2D A2@T1, T2DE -
CoefficientAmoduEq@@2, 2DD, A1@T1, T2D A2@T1, T2D A1@T1, T2DE == 0 . . e -> 1 . . w2 -> 2 w1

True

```

Therefore, treatment of the first-order version of the governing equations yields consistent results without floating constants.

5.2.3 First-Order Complex-Valued System

In this section, we show that treatment of a first-order complex-valued version of the governing equations yields consistent results without floating constants. We start by transforming the second-order form **EOM1** of the governing equations into a system of first-order complex-valued equations by introducing the transformation

```

transfRule = 9uk@tD -> zk@tD +  $\dot{z}_k$ @tD, ukε@tD -> I wk I zk@tD -  $\dot{z}_k$ @tD M =;

```

Then, we define the dependent variables

```

depVar = 8u1@tD, u1ε@tD, u2@tD, u2ε@tD<;

nmodes = 9z1@tD,  $\dot{z}_1$ @tD, z2@tD,  $\dot{z}_2$ @tD=;

```

Substituting **transfRule** into **depVar** and solving for the **nmodes**, we obtain

```

zetaRule = Solve@depVar == HdepVar . . transfRuleL . . Thread, nmodesD@@1DD

: z1@tD @ -  $\frac{-w_1 u_1@tD + I u_1^\epsilon@tD}{2 w_1}$ ,  $\dot{z}_1$ @tD @ -  $\frac{-w_1 u_1@tD - I u_1^\epsilon@tD}{2 w_1}$ ,

z2@tD @ -  $\frac{-w_2 u_2@tD + I u_2^\epsilon@tD}{2 w_2}$ ,  $\dot{z}_2$ @tD @ -  $\frac{-w_2 u_2@tD - I u_2^\epsilon@tD}{2 w_2}$  >

```

It follows from **EOM1** that the acceleration is given by

```

acceleration = Solve@EOM1, 8u12@tD, u22@tD<D@@1DD

8u12@tD @ - w12 u1@tD + 2 d u1@tD u2@tD, u22@tD @ d u1@tD2 - w22 u2@tD<

```

Then, differentiating the dependent variables in **zetaRule** and using **transfRule** and **acceleration**, we transform the system of two second-order equations **EOM1** into the following two first-order complex-valued equations:

$$\begin{aligned} \mathbf{EOM3} = \mathbf{z}^{\epsilon} @ \mathbf{tD} == & \mathbf{HD} @ \mathbf{z}_{\#} @ \mathbf{tD} \cdot \mathbf{zetaRule}, \mathbf{tD} \cdot \mathbf{acceleration} \cdot \mathbf{transfRule} \cdot \mathbf{ExpandL} \& \cdot \mathbf{\check{z}} \mathbf{81}, \mathbf{2} < \\ \mathbf{z}_1^{\epsilon} @ \mathbf{tD} == & \mathbf{I} \mathbf{w}_1 \mathbf{z}_1 @ \mathbf{tD} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1}, \\ \mathbf{z}_2^{\epsilon} @ \mathbf{tD} == & - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD}^2}{2 \mathbf{w}_2} + \mathbf{I} \mathbf{w}_2 \mathbf{z}_2 @ \mathbf{tD} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD}^2}{2 \mathbf{w}_2} > \end{aligned}$$

To determine a second-order uniform expansion of the solution of **EOM3** using the method of multiple scales, we expand $\mathbf{z}$ and $\mathbf{z}^{\epsilon}$ in the form

$$\begin{aligned} \mathbf{solRule} = \\ \mathbf{9z_{i_} -> ISumAe^j z_{i,j}@ \#1, \#2, \#3D, 8j, 3<E \&M, \dot{z}_{i_} -> ISumAe^j \dot{z}_{i,j}@ \#1, \#2, \#3D, 8j, 3<E \&M}; \\ \mathbf{eq523a} = \mathbf{HEOM3} \cdot \mathbf{multiScales} \cdot \mathbf{solRule} \cdot \mathbf{ExpandAllL} \cdot \mathbf{e^{n_}; n>3} -> \mathbf{0}; \end{aligned}$$

Equating coefficients of like powers of ϵ , we obtain

$$\mathbf{eqEps} = \mathbf{Rest@Thread@CoefficientList@Subtract \check{z} \check{z} \#, \mathbf{eD} == \mathbf{0DD} \& \cdot \mathbf{\check{z}} \mathbf{eq523a} \cdot \mathbf{Transpose};$$

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

$$\begin{aligned} \mathbf{eqOrder@i_D} := & \mathbf{I\#@@1DD} \& \cdot \mathbf{\check{z}} \mathbf{eqEps@@1DD} \cdot \mathbf{u_{-k,1}} -> \mathbf{u_{k,i}M} == \\ & \mathbf{I\#@@1DD} \& \cdot \mathbf{\check{z}} \mathbf{eqEps@@1DD} \cdot \mathbf{u_{-k,1}} -> \mathbf{u_{k,i}M} - \mathbf{H\#@@1DD} \& \cdot \mathbf{\check{z}} \mathbf{eqEps@@iDDL} \cdot \mathbf{Thread} \end{aligned}$$

Using **eqOrder[i]** and the **displayRule**, we rewrite **eqEps** in a concise way as

$$\begin{aligned} & \mathbf{eqOrder@1D} \cdot \mathbf{displayRule} \cdot \mathbf{TableForm} \\ & \mathbf{eqOrder@2D} \cdot \mathbf{displayRule} \cdot \mathbf{TableForm} \\ & \mathbf{eqOrder@3D} \cdot \mathbf{displayRule} \cdot \mathbf{TableForm} \\ D_0 z_{1,1} - \mathbf{I} w_1 z_{1,1} & == 0 \\ D_0 z_{2,1} - \mathbf{I} w_2 z_{2,1} & == 0 \\ D_0 z_{1,2} - \mathbf{I} w_1 z_{1,2} & == -\mathbf{HD}_1 z_{1,1} \mathbf{L} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} \\ D_0 z_{2,2} - \mathbf{I} w_2 z_{2,2} & == -\mathbf{HD}_1 z_{2,1} \mathbf{L} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1^2 @ \mathbf{tD}}{2 \mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1^2 @ \mathbf{tD}}{2 \mathbf{w}_2} \\ D_0 z_{1,3} - \mathbf{I} w_1 z_{1,3} & == -\mathbf{HD}_1 z_{1,2} \mathbf{L} - \mathbf{D}_2 z_{1,1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_1} \\ D_0 z_{2,3} - \mathbf{I} w_2 z_{2,3} & == -\mathbf{HD}_1 z_{2,2} \mathbf{L} - \mathbf{D}_2 z_{2,1} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_1 @ \mathbf{tD} \mathbf{z}_2 @ \mathbf{tD}}{\mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_2} - \frac{\mathbf{I} \mathbf{d} \mathbf{z}_2 @ \mathbf{tD} \mathbf{z}_1 @ \mathbf{tD}}{\mathbf{w}_2} \end{aligned}$$

First-Order Equations

Since **eqOrder[1]** are homogeneous, we can write their solutions as

```

sol1 = Table[A9zi,1 -> Function@8T0, T1, T2<, Ai@T1, T2D Exp@I wi T0D •• EvaluateD,
  {zi,1 -> Function@8T0, T1, T2<, Ai@T1, T2D Exp@-I wi T0D •• EvaluateE=, 8i, 2<E •• Flatten

8z1,1 ® Function@8T0, T1, T2<, EI T0 w1 A1@T1, T2DD,
z1,1 ® Function@8T0, T1, T2<, E-I T0 w1 A1@T1, T2DD,
z2,1 ® Function@8T0, T1, T2<, EI T0 w2 A2@T1, T2DD,
z2,1 ® Function@8T0, T1, T2<, E-I T0 w2 A2@T1, T2DD<

```

Second-Order Equations

Substituting the first-order solution into the second-order equations, [eqOrder\[2\]](#), we have

```

order2Eq = eqOrder@2D •. sol1;
order2Eq •. displayRule

```

$$\begin{aligned}
 9D_0 z_{1,2} - I w_1 z_{1,2} &= -E^{I T_0 w_1} H D_1 A_1 L - \frac{I E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_1} - \\
 &\frac{I E^{-I T_0 w_1 + I T_0 w_2} d A_2 A_1}{w_1} - \frac{I E^{I T_0 w_1 - I T_0 w_2} d A_1 A_2}{w_1} - \frac{I E^{-I T_0 w_1 - I T_0 w_2} d A_1 A_2}{w_1}, \\
 D_0 z_{2,2} - I w_2 z_{2,2} &= -E^{I T_0 w_2} H D_1 A_2 L - \frac{I E^{2 I T_0 w_1} d A_2^2}{2 w_2} - \frac{I d A_1 A_1}{w_2} - \frac{I E^{-2 I T_0 w_1} d A_1^2}{2 w_2} =
 \end{aligned}$$

Converting the terms that produce small-divisor terms by using the [expRule1](#) and then eliminating the terms that lead to secular terms from [order2Eq](#) yields

```

SCond1 = Coefficient@order2Eq@@#, 2DD •. expRule1@#D, Exp@I w# T0DD == 0 & •ž 81, 2<;
SCond1 •. displayRule

```

$$: -H D_1 A_1 L - \frac{I E^{I T_1 s} d A_2 A_1}{w_1} == 0, -H D_1 A_2 L - \frac{I E^{-I T_1 s} d A_1^2}{2 w_2} == 0 >$$

or

```

SCond1Rule1 = SolveASCond1, 9A1H1,0L@T1, T2D, A2H1,0L@T1, T2D=E@@1DD;
SCond1Rule1 •. displayRule

```

$$: D_1 A_1 \text{ ® } - \frac{I E^{I T_1 s} d A_2 A_1}{w_1}, D_1 A_2 \text{ ® } - \frac{I E^{-I T_1 s} d A_1^2}{2 w_2} >$$

Substituting [SCond1Rule1](#) and [expRule3](#) into [order2Eq](#) yields

```
order2Eqm = order2Eq . SCond1Rule1 . expRule3;
order2Eqm . displayRule
```

$$9D_0 z_{1,2} - I w_1 z_{1,2} == - \frac{I E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_1} - \frac{I E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_1} - \frac{I E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{w_1},$$

$$D_0 z_{2,2} - I w_2 z_{2,2} == - \frac{I d A_1 \dot{A}_1}{w_2} - \frac{I E^{-2 I T_0 w_1} d \dot{A}_1^2}{2 w_2} =$$

We transform **order2Eqm** into a set of ordinary-differential equations and obtain their particular solutions as

```
sol2p = DSolve@order2Eqm . z_{i,2} -> H z_{i,2}@#1D &L, 8 z_{1,2}@T_0D, z_{2,2}@T_0D<, T_0D@@1DD . C@_D -> 0;
expRule4 = Exp@a_D :> Exp@Expand@aDD;
```

We write the second-order solution in function form as

```
sol2 = TableA9 z_{i,2} -> Function@8 T_0, T_1, T_2<, H sol2p@@i, 2DD . ExpandL . expRule4 . EvaluateD,
      z_{i,2} -> Function@8 T_0, T_1, T_2<, H sol2p@@i, 2DD . ExpandL . expRule4 . conjugateRule .
      EvaluateD=, 8 i, 2<E . Flatten;
sol2 . displayRule
```

$$9 z_{1,2} \text{® FunctionB8 } T_0, T_1, T_2<, - \frac{E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_1 H_2 w_1 + w_2 L} - \frac{2 E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_2 H_2 w_1 + w_2 L} +$$

$$\frac{E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_1 H_2 w_1 + w_2 L} + \frac{2 E^{I T_0 w_1 - I T_0 w_2} d A_1 \dot{A}_2}{w_2 H_2 w_1 + w_2 L} + \frac{E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{w_1 H_2 w_1 + w_2 L} F,$$

$$z_{1,2} \text{® FunctionB8 } T_0, T_1, T_2<, \frac{E^{I T_0 w_1 + I T_0 w_2} d A_1 A_2}{w_1 H_2 w_1 + w_2 L} + \frac{E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{w_1 H_2 w_1 + w_2 L} +$$

$$\frac{2 E^{-I T_0 w_1 + I T_0 w_2} d A_2 \dot{A}_1}{w_2 H_2 w_1 + w_2 L} - \frac{E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{w_1 H_2 w_1 + w_2 L} - \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d \dot{A}_1 \dot{A}_2}{w_2 H_2 w_1 + w_2 L} F,$$

$$z_{2,2} \text{® FunctionA8 } T_0, T_1, T_2<, \frac{2 d w_1 A_1 \dot{A}_1}{w_2^2 H_2 w_1 + w_2 L} + \frac{d A_1 \dot{A}_1}{w_2 H_2 w_1 + w_2 L} + \frac{E^{-2 I T_0 w_1} d \dot{A}_1^2}{2 w_2 H_2 w_1 + w_2 L} E,$$

$$z_{2,2} \text{® FunctionB8 } T_0, T_1, T_2<, \frac{E^{2 I T_0 w_1} d A_1^2}{2 w_2 H_2 w_1 + w_2 L} + \frac{2 d w_1 A_1 \dot{A}_1}{w_2^2 H_2 w_1 + w_2 L} + \frac{d A_1 \dot{A}_1}{w_2 H_2 w_1 + w_2 L} F=$$

¶ Third-Order Equations

Substituting the first- and second-order solutions into the right-hand sides of the third-order equations, **eqOrder[3]**, we obtain

```
order3Eqrhs = Expand@@@2DD . sol1 . sol2D & . z eqOrder@3D;
```

Converting the terms that produce small-divisor terms and eliminating the terms that produce secular terms in $z_{i,3}$ from **order3Eqrhs** demands the following conditions:

```
SCond2 = Coefficient@order3Eqrhs@@#DD . expRule1@#D, Exp@I w# T0DD == 0 & . #1, 2<;
SCond2 . displayRule
```

$$: -HD_2A_1L - \frac{4I d^2 A_1^2 \dot{A}_1}{w_2^2 H_2 w_1 + w_2 L} - \frac{5I d^2 A_1^2 \dot{A}_1}{2 w_1 w_2 H_2 w_1 + w_2 L} - \frac{I d^2 A_1 A_2 \dot{A}_2}{w_1^2 H_2 w_1 + w_2 L} == 0, -HD_2A_2L - \frac{I d^2 A_1 A_2 \dot{A}_1}{w_1 w_2 H_2 w_1 + w_2 L} == 0>$$

or

```
SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD;
SCond2Rule1 . displayRule
```

$$: D_2A_1 \otimes - \frac{I H_8 d^2 A_1^2 w_1^2 \dot{A}_1 + 5 d^2 A_1^2 w_1 w_2 \dot{A}_1 + 2 d^2 A_1 A_2 w_2^2 \dot{A}_2 L}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L}, D_2A_2 \otimes - \frac{I d^2 A_1 A_2 \dot{A}_1}{w_1 w_2 H_2 w_1 + w_2 L} >$$

Reconstitution

We reconstitute **SCond1Rule1** and **SCond2Rule1** and obtain

```
moduEq = Table@2 I wk Ak^e, 8k, 2<D ==
  HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D . SCond1Rule1 . SCond2Rule1 .
  ExpandL . Thread;
moduEq . displayRule
```

$$: 2 I w_1 A_1^e == 2 E^{I T_1 s} d e A_2 \dot{A}_1 + \frac{8 d^2 e^2 A_1^2 w_1 \dot{A}_1}{w_2^2 H_2 w_1 + w_2 L} + \frac{5 d^2 e^2 A_1^2 \dot{A}_1}{w_2 H_2 w_1 + w_2 L} + \frac{2 d^2 e^2 A_1 A_2 \dot{A}_2}{w_1 H_2 w_1 + w_2 L},$$

$$2 I w_2 A_2^e == E^{-I T_1 s} d e A_1^2 + \frac{2 d^2 e^2 A_1 A_2 \dot{A}_1}{w_1 H_2 w_1 + w_2 L} >$$

à 5.3 Method of Normal Forms

To apply the method of normal forms, we find it convenient to start with the first-order complex-valued form **EOM3**

$$EOM3 = 9z_1^e @tD == I w_1 z_1 @tD - \frac{I d z_1 @tD z_2 @tD}{w_1} - \frac{I d z_2 @tD z_1 @tD}{w_1} - \frac{I d z_1 @tD z_2 @tD}{w_1} - \frac{I d z_1 @tD z_2 @tD}{w_1},$$

$$z_2^e @tD == - \frac{I d z_1 @tD^2}{2 w_2} + I w_2 z_2 @tD - \frac{I d z_1 @tD z_1 @tD}{w_2} - \frac{I d z_1 @tD^2}{2 w_2} =;$$

of the governing equations. Then, according to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = Table@8hk@tD, h_k@tD<, 8k, 2<D . Flatten
8h1@tD, h1@tD, h2@tD, h2@tD<
```



```

zetaRule =
9z_i_ -> FunctionAt, e h_i@tD + SumAej+1 h_i,j@Sequence žž basicTermsD, 8j, 2<E •• EvaluateE,
z_i_ -> FunctionAt, e ħ_i@tD + SumAej+1 ħ_i,j@Sequence žž basicTermsD, 8j, 2<E •• EvaluateE=

8z_i_ @ Function@t,
e h_i@tD + e2 h_i,1@h_1@tD, ħ_1@tD, h_2@tD, ħ_2@tDD + e3 h_i,2@h_1@tD, ħ_1@tD, h_2@tD, ħ_2@tDDD, z_i_ @
Function@t, e ħ_i@tD + e2 ħ_i,1@h_1@tD, ħ_1@tD, h_2@tD, ħ_2@tDD + e3 ħ_i,2@h_1@tD, ħ_1@tD, h_2@tD, ħ_2@tDDD<

```

that results in the simplest possible equations

```

etaRule = h_i_ε@tD -> I w_i h_i@tD + SumAej g_i,j@tD, 8j, 2<E

h_i_ε@tD @ I w_i h_i@tD + e g_i,1@tD + e2 g_i,2@tD

```

where the overbar denotes the complex conjugate

```

conjugateRule = 8h -> ħ, ħ -> h, g -> ġ, ġ -> g, Complex@0, n_D -> Complex@0, -nD<;

```

Substituting the expansion for the z_i , the **zetaRule**, into **EOMB**, using the **etaRule**, expanding the result for small e , and discarding terms of order higher than e^3 , we have

```

eq53a =
HEOM3 •• zetaRule •• etaRule •• HetaRule •• conjugateRuleL •• ExpandAllL •• en-•;n>3 -> 0;

```

Ÿ Second-Order Equations

Collecting the coefficients of e^2 in **eq53a** yields

```

order2expr = CoefficientASubtract žž #, e2E & •ž eq53a;

```

To determine the resonance terms in **order2expr**, we note that all of the possible forms of its nonhomogeneous terms are

```

possibleQTerms = Outer@Times, basicTerms, basicTermsD •• Flatten •• Union

9h_1@tD2, h_1@tD h_2@tD, h_2@tD2, h_1@tD ħ_1@tD,
h_2@tD ħ_1@tD, ħ_1@tD2, h_1@tD ħ_2@tD, h_2@tD ħ_2@tD, ħ_1@tD ħ_2@tD, ħ_2@tD2=

```

It follows from the linear parts of the **etaRule** that to the first approximation

```

form = 9h_i_@tD -> EI w_i t, ħ_i_@tD -> E-I w_i t;

```

Hence, the possible resonance terms are given by

```

ResonantQTerm@1D =
I E-I w_1 t possibleQTerms •• form •• w_2 -> 2 w_1 •• E-t -> 0M possibleQTerms •• Union •• Rest

8h_2@tD ħ_1@tD<

```

ResonantQTerm@2D =

$$\int_K E^{-I w_2 t} \text{possibleQTerms} \cdot \text{form} \cdot w_1 \rightarrow \frac{1}{2} w_2 \cdot E^{-t} \rightarrow 0 \quad \text{possibleQTerms} \cdot \text{Union} \cdot \text{Rest}$$

$$8h_1 @ t D^2 <$$

Then, the nonresonance terms and their associated coefficients can be defined as

```
Do@NRQT@iD = Complement@possibleQTerms, ResonantQTerm@iDD;
coeffsQ@iD = Table@Gi,j, 8j, Length@NRQT@iDD<D, 8i, 2<D
```

Thus, the $h_{i,1}$ and their complex conjugates have the form

```
hFormQ =
TableA9hi,1 -> HEvaluate@coeffsQ@iD.NRQT@iD . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
hi,1 -> HEvaluate@coeffsQ@iD.NRQT@iD . conjugateRule .
Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E . Flatten;
```

Substituting for the $h_{i,1}$ in **order2expr**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting algebraic equations for the $G_{i,j}$, we obtain

```
coeffsQRule = Table@
Solve@Coefficient@order2expr@@iDD . hFormQ, NRQT@iDD == 0 . Thread, coeffsQ@iDD@@1DD,
8i, 2<D . Flatten
```

$$: G_{1,1} \otimes 0, G_{1,2} \otimes -\frac{d}{w_1 w_2}, G_{1,4} \otimes 0, G_{1,5} \otimes 0, G_{1,6} \otimes \frac{d}{w_1 w_2}, G_{1,7} \otimes 0,$$

$$G_{1,3} \otimes 0, G_{1,8} \otimes \frac{d}{w_1 H^2 w_1 + w_2 L}, G_{1,9} \otimes 0, G_{2,1} \otimes 0, G_{2,2} \otimes 0, G_{2,3} \otimes \frac{d}{w_2^2},$$

$$G_{2,4} \otimes 0, G_{2,7} \otimes 0, G_{2,9} \otimes 0, G_{2,5} \otimes \frac{d}{2 w_2 H^2 w_1 + w_2 L}, G_{2,6} \otimes 0, G_{2,8} \otimes 0 >$$

Using these coefficients, we rewrite **hFormQ** as

```
hSolQ = Table@hi,1@Sequence žž basicTermsD ->
Hhi,1@Sequence žž basicTermsD . hFormQ . coeffsQRuleL, 8i, 2<D
```

$$: h_{1,1} @ h_1 @ t D, \dot{h}_1 @ t D, h_2 @ t D, \dot{h}_2 @ t D D \otimes -\frac{d h_1 @ t D h_2 @ t D}{w_1 w_2} + \frac{d h_1 @ t D \dot{h}_2 @ t D}{w_1 w_2} + \frac{d \dot{h}_1 @ t D h_2 @ t D}{w_1 H^2 w_1 + w_2 L},$$

$$h_{2,1} @ h_1 @ t D, \dot{h}_1 @ t D, h_2 @ t D, \dot{h}_2 @ t D D \otimes \frac{d h_1 @ t D \dot{h}_1 @ t D}{w_2^2} + \frac{d \dot{h}_1 @ t D^2}{2 w_2 H^2 w_1 + w_2 L} >$$

or in pure function form as

```
hRuleQ = TableA9hi,1 -> HEvaluate@hSolQ@@i, 2DD . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
hi,1 -> HEvaluate@hSolQ@@i, 2DD . conjugateRule .
Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E . Flatten;
```

We choose the $g_{i,1}$ to eliminate the resonance terms in **order2expr** according to

```
gRuleQ = Table@
  gi,1@tD -> - Coefficient@order2expr@@iDD, ResonantQTerm@iDD.ResonantQTerm@iD, 8i, 2<D
: g1,1@tD @ -  $\frac{\text{Id } h_2@tD \dot{h}_1@tD}{w_1}$ , g2,1@tD @ -  $\frac{\text{Id } h_1@tD^2}{2 w_2}$  >
```

Third-Order Equations

Collecting the coefficients of e^3 in **eq53a** and using **hRuleQ** and **gRuleQ**, we have

```
order3expr = MapACoefficientASubtract žž #, e3E &, eq53aE . hRuleQ . gRuleQ .
HgRuleQ . conjugateRuleL . Expand;
```

The nonhomogeneous terms in **order3expr** are proportional to

```
possibleCTerms = Outer@Times, possibleQTerms, basicTermsD . Flatten . Union
9h1@tD3, h1@tD2 h2@tD, h1@tD h2@tD2, h2@tD3, h1@tD2  $\dot{h}_1$ @tD,
h1@tD h2@tD  $\dot{h}_1$ @tD, h2@tD2  $\dot{h}_1$ @tD, h1@tD  $\dot{h}_1$ @tD2, h2@tD  $\dot{h}_1$ @tD2,  $\dot{h}_1$ @tD3,
h1@tD2  $\dot{h}_2$ @tD, h1@tD h2@tD  $\dot{h}_2$ @tD, h2@tD2  $\dot{h}_2$ @tD, h1@tD  $\dot{h}_1$ @tD  $\dot{h}_2$ @tD,
h2@tD  $\dot{h}_1$ @tD  $\dot{h}_2$ @tD,  $\dot{h}_1$ @tD2  $\dot{h}_2$ @tD, h1@tD  $\dot{h}_2$ @tD2, h2@tD  $\dot{h}_2$ @tD2,  $\dot{h}_1$ @tD  $\dot{h}_2$ @tD2,  $\dot{h}_2$ @tD3=
```

Next, we determine the resonance terms according to

```
ResonantCTerm@1D =
  I E-I w1 t possibleCTerms . form . w2 -> 2 w1 . E-t -> 0M possibleCTerms . Union . Rest
8h1@tD2  $\dot{h}_1$ @tD, h1@tD h2@tD  $\dot{h}_2$ @tD<
ResonantCTerm@2D =
   $\int_k E^{-I w_2 t}$  possibleCTerms . form . w1 ->  $\frac{1}{2} w_2$  . E-t -> 0  $\frac{1}{2}$  possibleCTerms . Union . Rest
8h1@tD h2@tD  $\dot{h}_1$ @tD, h2@tD2  $\dot{h}_2$ @tD<
```

Then, the nonresonance terms and their associated coefficients are given by

```
Do@NRCT@iD = Complement@possibleCTerms, ResonantCTerm@iDD;
coeffsC@iD = Table@Li,j, 8j, Length@NRCT@iDD<D, 8i, 2<D
```

Consequently, the $h_{i,2}$ and their complexconjugates have the forms

```
hFormC =
  TableA9hi,2 -> HEvaluate@coeffsC@iD.NRCT@iD . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
 $\dot{h}_{i,2}$  -> HEvaluate@coeffsC@iD.NRCT@iD . conjugateRule .
  Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E . Flatten;
```

Substituting for the $h_{i,2}$ in **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the $L_{i,j}$, we obtain

```
coeffsCRule = Table@
  Solve@Coefficient@order3expr@@iDD •. hFormC, NRCT@iDD == 0 •• Thread, coeffsC@iDD@@1DD,
  8i, 2<D •• Flatten

: L1,1 @ -  $\frac{d^2 w_1 + d^2 w_2}{2 w_1^2 w_2^2 H^2 w_1 + w_2 L}$ , L1,3 @  $\frac{d^2}{w_1 w_2^2 H^2 w_1 + w_2 L}$ , L1,7 @ -  $\frac{-5 d^2 w_1 - 3 d^2 w_2}{2 w_1^2 w_2^2 H^2 w_1 + w_2 L}$ ,
L1,9 @  $\frac{d^2}{4 w_1^2 w_2^2 H^2 w_1 + w_2 L}$ , L1,13 @ -  $\frac{d^2}{w_1^2 w_2^2 H^2 w_1 + w_2 L}$ , L1,15 @ -  $\frac{-d^2 w_1 - d^2 w_2}{w_1^2 w_2^2 H^2 w_1 + w_2 L}$ ,
L1,2 @ 0, L1,16 @ 0, L1,4 @ 0, L1,17 @ -  $\frac{d^2}{w_1 w_2 H w_1 + w_2 L H^2 w_1 + w_2 L}$ , L1,5 @ 0,
L1,18 @ 0, L1,10 @ 0, L1,8 @ 0, L1,6 @  $\frac{d^2}{w_1^2 H w_1 - w_2 L w_2}$ , L1,11 @ 0, L1,14 @ 0, L1,12 @ 0,
L2,2 @  $\frac{d^2}{w_1 w_2^2 H^2 w_1 + w_2 L}$ , L2,4 @ 0, L2,8 @ 0, L2,12 @  $\frac{d^2}{w_1 w_2^2 H^2 w_1 + w_2 L}$ , L2,16 @ 0, L2,18 @ 0,
L2,1 @ 0, L2,3 @ 0, L2,5 @ 0, L2,6 @ 0, L2,7 @ 0, L2,9 @ 0, L2,10 @ -  $\frac{d^2}{w_1 H w_1 - w_2 L w_2^2}$ ,
L2,11 @ 0, L2,17 @ 0, L2,13 @ 0, L2,14 @ -  $\frac{d^2}{w_2^2 H w_1 + w_2 L H^2 w_1 + w_2 L}$ , L2,15 @ 0 >
```

Using these coefficients, we rewrite **hFormC** as

```
hSolC = Table@h1,2@Sequence žž basicTermsD ->
  Hh1,2@Sequence žž basicTermsD •. hFormC •. coeffsCRuleL, 8i, 2<D

: h1,2@h1@tD, h1@tD, h2@tD, h2@tDD @ -  $\frac{H d^2 w_1 + d^2 w_2 L h_1 @ t D^3}{2 w_1^2 w_2^2 H^2 w_1 + w_2 L} + \frac{d^2 h_1 @ t D h_2 @ t D^2}{w_1 w_2^2 H^2 w_1 + w_2 L} + \frac{d^2 h_2 @ t D^2 h_1 @ t D}{w_1^2 H w_1 - w_2 L w_2} -$ 
 $\frac{H - 5 d^2 w_1 - 3 d^2 w_2 L h_1 @ t D h_1 @ t D^2}{2 w_1^2 w_2^2 H^2 w_1 + w_2 L} + \frac{d^2 h_1 @ t D^3}{4 w_1^2 w_2 H^2 w_1 + w_2 L} - \frac{d^2 h_2 @ t D h_1 @ t D h_2 @ t D}{w_1^2 w_2 H^2 w_1 + w_2 L} -$ 
 $\frac{H - d^2 w_1 - d^2 w_2 L h_1 @ t D h_2 @ t D^2}{w_1^2 w_2^2 H^2 w_1 + w_2 L} - \frac{d^2 h_1 @ t D h_2 @ t D^2}{w_1 w_2 H w_1 + w_2 L H^2 w_1 + w_2 L}, h2,2@h1@tD, h1@tD, h2@tD, h2@tDD @$ 
 $\frac{d^2 h_1 @ t D^2 h_2 @ t D}{w_1 w_2^2 H^2 w_1 + w_2 L} - \frac{d^2 h_1 @ t D^2 h_2 @ t D}{w_1 H w_1 - w_2 L w_2^2} + \frac{d^2 h_1 @ t D h_1 @ t D h_2 @ t D}{w_1 w_2^2 H^2 w_1 + w_2 L} - \frac{d^2 h_1 @ t D^2 h_2 @ t D}{w_2^2 H w_1 + w_2 L H^2 w_1 + w_2 L} >$ 
```

or in pure function form as

```
hRuleC = TableA9h1,2 -> HEvaluate@hSolC@@i, 2DD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
  h1,2 -> HEvaluate@hSolC@@i, 2DD •. conjugateRule •.
  Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E •• Flatten;
```

Choosing the $g_{i,2}$ to eliminate the resonance terms from **order3expr** yields

```

gRuleC = Table@
  gi,2@tD -> - Coefficient@order3expr@@iDD, ResonantCTerm@iDD.ResonantCTerm@iD, 8i, 2<D
: g1,2@tD ® - j  $\frac{2 I d^2}{k w_1 w_2^2} + \frac{I d^2}{2 w_1 w_2 H^2 w_1 + w_2 L} \sum h_1@tD^2 \dot{h}_1@tD - \frac{I d^2 h_1@tD h_2@tD \dot{h}_2@tD}{w_1^2 H^2 w_1 + w_2 L},$ 
  g2,2@tD ® -  $\frac{I d^2 h_1@tD h_2@tD \dot{h}_1@tD}{w_1 w_2 H^2 w_1 + w_2 L} >$ 

```

Combining **etaRule**, **gRuleQ**, and **gRuleC** and letting $h_i = A_i @tD E^{I w_i t}$, we obtain the modulation equation

```

moduEq = TableA2 I wk E-I wk t H hkε@tD - H hkε@tD • etaRule • gRuleQ • gRuleCLL == 0 • .
  9hi -> I Ai@#D EI wi # &M, hi -> I Ai@#D E-I wi # &M= • .
  Exp@a_D :> ExpAa • . w2 -> 2 w1 + e2 s • ExpandE • ExpandAll, 8k, 2<E
: -  $\frac{4 d^2 e^2 A_1@tD^2 \dot{A}_1@tD}{w_2^2} - \frac{2 d^2 e^2 w_1 A_1@tD^2 \dot{A}_1@tD}{4 w_1^2 w_2 + 2 w_1 w_2^2} -$ 
   $2 E^{I t e^2 s} d e A_2@tD \dot{A}_1@tD - \frac{2 d^2 e^2 w_1 A_1@tD A_2@tD \dot{A}_2@tD}{2 w_1^3 + w_1^2 w_2} + 2 I w_1 A_1^\epsilon@tD == 0,$ 
  - E-I t e2 s d e A1@tD2 -  $\frac{2 d^2 e^2 w_2 A_1@tD A_2@tD \dot{A}_1@tD}{2 w_1^2 w_2 + w_1 w_2^2} + 2 I w_2 A_2^\epsilon@tD == 0 >$ 

```

Using **zetaRule**, **hRuleQ**, and **hRuleC**, we obtain the following second-order uniform expansion of the solution of **FOM3**:

solution = TableAu_k@tD == CollectAz_k@tD + $\dot{z}_k$ @tD . . zetaRule . . hRuleQ . . hRuleC, eE, 8k, 2<E

$$\begin{aligned}
 :u_1@tD == & e Hh_1@tD + \dot{h}_1@tD L + e^2 \int_k \left\{ -\frac{d h_1@tD h_2@tD}{w_1 w_2} + \frac{d h_1@tD h_2@tD}{w_1 H_2 w_1 + w_2 L} + \right. \\
 & \frac{d h_2@tD \dot{h}_1@tD}{w_1 w_2} + \frac{d h_1@tD \dot{h}_2@tD}{w_1 w_2} - \frac{d \dot{h}_1@tD \dot{h}_2@tD}{w_1 w_2} + \frac{d \dot{h}_1@tD \dot{h}_2@tD}{w_1 H_2 w_1 + w_2 L} \Big\} \\
 & e^3 \int_k \left\{ \frac{d^2 h_1@tD^3}{4 w_1^2 w_2 H_2 w_1 + w_2 L} - \frac{H d^2 w_1 + d^2 w_2 L h_1@tD^3}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L} + \frac{d^2 h_1@tD h_2@tD^2}{w_1 w_2^2 H_2 w_1 + w_2 L} - \frac{d^2 h_1@tD h_2@tD^2}{w_1 w_2 H w_1 + w_2 L H_2 w_1 + w_2 L} - \right. \\
 & \frac{H-5 d^2 w_1 - 3 d^2 w_2 L h_1@tD^2 \dot{h}_1@tD}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L} + \frac{d^2 h_2@tD^2 \dot{h}_1@tD}{w_1^2 H w_1 - w_2 L w_2} - \frac{H-d^2 w_1 - d^2 w_2 L h_2@tD^2 \dot{h}_1@tD}{w_1^2 w_2^2 H_2 w_1 + w_2 L} - \\
 & \frac{H-5 d^2 w_1 - 3 d^2 w_2 L h_1@tD \dot{h}_1@tD^2}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L} + \frac{d^2 \dot{h}_1@tD^3}{4 w_1^2 w_2 H_2 w_1 + w_2 L} - \frac{H d^2 w_1 + d^2 w_2 L \dot{h}_1@tD^3}{2 w_1^2 w_2^2 H_2 w_1 + w_2 L} - \\
 & \frac{d^2 h_1@tD h_2@tD \dot{h}_2@tD}{w_1^2 w_2 H_2 w_1 + w_2 L} - \frac{d^2 h_2@tD \dot{h}_1@tD \dot{h}_2@tD}{w_1^2 w_2 H_2 w_1 + w_2 L} + \frac{d^2 h_1@tD \dot{h}_2@tD^2}{w_1^2 H w_1 - w_2 L w_2} - \\
 & \frac{H-d^2 w_1 - d^2 w_2 L h_1@tD \dot{h}_2@tD^2}{w_1^2 w_2^2 H_2 w_1 + w_2 L} + \frac{d^2 \dot{h}_1@tD \dot{h}_2@tD^2}{w_1 w_2^2 H_2 w_1 + w_2 L} - \frac{d^2 \dot{h}_1@tD \dot{h}_2@tD^2}{w_1 w_2 H w_1 + w_2 L H_2 w_1 + w_2 L} \Big\} \\
 u_2@tD == & e^2 \int_k \left\{ \frac{d h_1@tD^2}{2 w_2 H_2 w_1 + w_2 L} + \frac{2 d h_1@tD \dot{h}_1@tD}{w_2^2} + \frac{d \dot{h}_1@tD^2}{2 w_2 H_2 w_1 + w_2 L} \right\} + e Hh_2@tD + \dot{h}_2@tD L + \\
 & e^3 \int_k \left\{ \frac{d^2 h_1@tD^2 h_2@tD}{w_1 w_2^2 H_2 w_1 + w_2 L} - \frac{d^2 h_1@tD^2 h_2@tD}{w_2^2 H w_1 + w_2 L H_2 w_1 + w_2 L} + \frac{d^2 h_1@tD h_2@tD \dot{h}_1@tD}{w_1 w_2^2 H_2 w_1 + w_2 L} - \frac{d^2 h_2@tD \dot{h}_1@tD^2}{w_1 H w_1 - w_2 L w_2^2} - \right. \\
 & \frac{d^2 h_1@tD^2 \dot{h}_2@tD}{w_1 H w_1 - w_2 L w_2^2} + \frac{d^2 h_1@tD \dot{h}_1@tD \dot{h}_2@tD}{w_1 w_2^2 H_2 w_1 + w_2 L} + \frac{d^2 \dot{h}_1@tD^2 \dot{h}_2@tD}{w_1 w_2^2 H_2 w_1 + w_2 L} - \frac{d^2 \dot{h}_1@tD^2 \dot{h}_2@tD}{w_2^2 H w_1 + w_2 L H_2 w_1 + w_2 L} \Big\}
 \end{aligned}$$

à 5.4 Generalized Method of Averaging

We define the nonlinear part of **FOM1**, a transformation rule, and the states vector as

```

NLTerms = 9f1 -> 2 e d u1@tD u2@tD, f2 -> e d u1@tD2;
transformRule = ui -> Hai@#D Sin@fi@#DD &L;
states = 8a1@tD, a2@tD, f1@tD, f2@tD<;

```

Substituting the above rules into the four first-order equations obtained by using the method of variation of parameters, we have

```

eq54a = D@states, tD ==
  jg f1 Cos@f1@tDD, f2 Cos@f2@tDD, w1 - f1 Sin@f1@tDD, w2 - f2 Sin@f2@tDD == .
  k w1 w2 a1@tD w1 a2@tD w2

  NLTerms . transformRule . TrigReduce . Expand . Thread
  {
: a1@tD == d e Cos@2 f1@tD - f2@tDD a1@tD a2@tD - d e Cos@2 f1@tD + f2@tDD a1@tD a2@tD, a2@tD ==
  2 w1 2 w1
- d e Cos@2 f1@tD - f2@tDD a1@tD^2 + d e Cos@f2@tDD a1@tD^2 - d e Cos@2 f1@tD + f2@tDD a1@tD^2,
  4 w2 2 w2 4 w2
f1@tD == w1 - d e Sin@2 f1@tD - f2@tDD a2@tD - d e Sin@f2@tDD a2@tD +
  2 w1 w1
d e Sin@2 f1@tD + f2@tDD a2@tD, f2@tD ==
  2 w1
w2 - d e Sin@2 f1@tD - f2@tDD a1@tD^2 - d e Sin@f2@tDD a1@tD^2 + d e Sin@2 f1@tD + f2@tDD a1@tD^2 >
  4 w2 a2@tD 2 w2 a2@tD 4 w2 a2@tD

```

We seek a second-order approximate solution of **eq54a** in the form

```

basicTerms = 8h1@tD, h2@tD, j1@tD, j2@tD<;

solRule = 9 a1_ -> I EvaluateAh1@tD + SumAe^j a1,j@Sequence z z basicTermsD, 8j, 2<E . t -> #E &M,
  f1_ -> I EvaluateAj1@tD + SumAe^j f1,j@Sequence z z basicTermsD, 8j, 2<E . t -> #E &M=;

```

where the $h_i@tD$ and $j_i@tD$ are expanded in powerseries in e as

```

basicDRule =
  D@basicTerms, tD -> 9SumAe^i A1,i@h1@tD, h2@tDD, 8i, 2<E, SumAe^i A2,i@h1@tD, h2@tDD, 8i, 2<E,
  w1 + SumAe^i F1,i@h1@tD, h2@tDD, 8i, 2<E, w2 + SumAe^i F2,i@h1@tD, h2@tDD, 8i, 2<E . Thread

8h1@tD @ e A1,1@h1@tD, h2@tDD + e^2 A1,2@h1@tD, h2@tDD,
h2@tD @ e A2,1@h1@tD, h2@tDD + e^2 A2,2@h1@tD, h2@tDD,
j1@tD @ w1 + e F1,1@h1@tD, h2@tDD + e^2 F1,2@h1@tD, h2@tDD,
j2@tD @ w2 + e F2,1@h1@tD, h2@tDD + e^2 F2,2@h1@tD, h2@tDD<

```

The functions $a_{i,j}$ and $f_{i,j}$ are fast varying functions of the j_i , while it follows from **basicDRule** that the h_i , and hence the $A_{i,j}$ and $F_{i,j}$, are slowly varying functions of t .

To the second approximation, we differentiate the $a_i@tD$ and $f_i@tD$ once with respect to t , use **solRule** and **basicDRule**, expand the result for small e , discard terms of order higher than e^2 , and obtain

```

eq54bLHS =
  CoefficientListAExpand@#@1DD . solRule . basicDRuleD . e^n; n>2 -> 0, eE & . z eq54a;

```

Next, we substitute **solRule** into the right-hand sides of **eq54a**, expand the result for small e , keep terms up to $O(e^2)$, and rewrite their right-hand sides as

```

eq54bRHS = CoefficientList@Series@#@2DD . solRule, 8e, 0, 2<D . Normal, eD & . z eq54a;

```

Equating coefficients of like powers of ϵ in **eq54bLHS** and **eq54bRHS**, we obtain

```
eqEps = MapThread@Equal, 8eq54bLHS, eq54bRHS<, 2D •• Transpose •• Rest;
```

To express the nearness of the two-to-one internal resonance, we introduce the detuning parameter S defined according to

```
omgRule = 8w2 -> 2 w1 + e s<;
```

Hence, S can be related to the w_i by

```
sigRule = Solve@omgRule •. Rule -> Equal, sD@@1DD
```

$$9s \approx - \frac{2w_1 - w_2}{e} =$$

To use the method of averaging, we define the following rules:

```
psiRule = 8j1@tD -> w1 t + b1, j2@tD -> w2 t + b2<;
```

```
betaRule = Solve@psiRule •. Rule -> Equal, 8b1, b2<D@@1DD
```

```
8b1 @ -t w1 + j1@tD, b2 @ -t w2 + j2@tD<
```

```
expRule1 = f@a_D -> f@Expand@a •. psiRule •. omgRuleD •. e t -> t1D;
```

```
expRule2 = f@a_D -> f@Expand@a •. t1 -> e t •. sigRule •. betaRuleDD;
```

Next, we separate the fast and slowly varying terms in the first-order equations, **eqEps[[1]]**. To this end, we define a slow state vector and a fast state vector according to

```
SVT@j_D := 8A1,j@h1@tD, h2@tDD, A2,j@h1@tD, h2@tDD, F1,j@h1@tD, h2@tDD, F2,j@h1@tD, h2@tDD<
solVar@j_D := 8a1,j, a2,j, f1,j, f2,j<
```

Then, the slowly varying parts of **eqEps[[1]]** are given by

```
SVT1Rule =
```

```
Table@Solve@eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD •. expRule1 •. f@_ t + _.D -> 0 •.
expRule2 •. f@a_D -> f@Expand@aDD, SVT@1D@@iDDD @@1DD, 8i, 4<D •• Flatten
```

```
:A1,1@h1@tD, h2@tDD @  $\frac{d \cos^2 j_1@tD - j_2@tDD h_1@tD h_2@tD}{2 w_1},$ 
```

```
A2,1@h1@tD, h2@tDD @  $-\frac{d \cos^2 j_1@tD - j_2@tDD h_1@tD^2}{4 w_2},$ 
```

```
F1,1@h1@tD, h2@tDD @  $-\frac{d \sin^2 j_1@tD - j_2@tDD h_2@tD}{2 w_1},$ 
```

```
F2,1@h1@tD, h2@tDD @  $-\frac{d \sin^2 j_1@tD - j_2@tDD h_1@tD^2}{4 w_2 h_2@tD}>$ 
```

whereas the fast varying parts are given by


```
FVT1 = Table@Subtract žž eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD, 8i, 4<D •. SVT1Rule
```

$$: \frac{d \cos@2 j_1@tD + j_2@tDD h_1@tD h_2@tD}{2 w_1}, - \frac{d \cos@j_2@tDD h_1@tD^2}{2 w_2} + \frac{d \cos@2 j_1@tD + j_2@tDD h_1@tD^2}{4 w_2},$$

$$\frac{d \sin@j_2@tDD h_2@tD}{w_1} - \frac{d \sin@2 j_1@tD + j_2@tDD h_2@tD}{2 w_1},$$

$$\frac{d \sin@j_2@tDD h_1@tD^2}{2 w_2 h_2@tD} - \frac{d \sin@2 j_1@tD + j_2@tDD h_1@tD^2}{4 w_2 h_2@tD} >$$

To determine the solution corresponding to these fast varying terms, we use the method of undetermined coefficients. To accomplish this, we first identify the possible forms of the nonhomogeneous terms. The result is

```
FVT1Forms = Flatten@Cases@#, HCos Ė SinL@a_D -> 8Cos@aD, Sin@aD<, InfinityDD & •ž FVT1
```

```
88Cos@2 j_1@tD + j_2@tDD, Sin@2 j_1@tD + j_2@tDD<,
8Cos@j_2@tDD, Sin@j_2@tDD, Cos@2 j_1@tD + j_2@tDD, Sin@2 j_1@tD + j_2@tDD<,
8Cos@j_2@tDD, Sin@j_2@tDD, Cos@2 j_1@tD + j_2@tDD, Sin@2 j_1@tD + j_2@tDD<,
8Cos@j_2@tDD, Sin@j_2@tDD, Cos@2 j_1@tD + j_2@tDD, Sin@2 j_1@tD + j_2@tDD<<
```

The fast-varying component of the first-order solution can be expressed as a linear combination of these forms; that is,

```
sol1Form = MapIndexed@Hcoeffs1@#2@@1DDD = Array@c, Length@#1DDL.#1 &, FVT1FormsD
```

```
8c@1D Cos@2 j_1@tD + j_2@tDD + c@2D Sin@2 j_1@tD + j_2@tDD,
c@1D Cos@j_2@tDD + c@3D Cos@2 j_1@tD + j_2@tDD + c@2D Sin@j_2@tDD + c@4D Sin@2 j_1@tD + j_2@tDD,
c@1D Cos@j_2@tDD + c@3D Cos@2 j_1@tD + j_2@tDD + c@2D Sin@j_2@tDD + c@4D Sin@2 j_1@tD + j_2@tDD,
c@1D Cos@j_2@tDD + c@3D Cos@2 j_1@tD + j_2@tDD + c@2D Sin@j_2@tDD + c@4D Sin@2 j_1@tD + j_2@tDD<
```

Substituting **sol1Form** into **eqEps[[1]]**, using **SVT1Rule**, collecting the coefficients of **FVT1Forms**, solving the resulting algebraic equation for the undetermined coefficients, and then substituting the result back into **sol1Form**, we obtain

```
sol1rhs = Table@sol1Form@@iDD •.
```

```
Solve@Coefficient@Subtract žž eqEps@@1, iDD •. SVT1Rule •. solVar@1D@@iDD ->
```

```
HEvaluate@sol1Form@@iDD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
```

```
FVT1Forms@@iDDD == 0 •• Thread, coeffs1@iDD@@1DD, 8i, 4<D
```

$$: - \frac{d \sin@2 j_1@tD + j_2@tDD h_1@tD h_2@tD}{2 w_1 H_2 w_1 + w_2 L}, \frac{d \sin@j_2@tDD h_1@tD^2}{2 w_2^2} - \frac{d \sin@2 j_1@tD + j_2@tDD h_1@tD^2}{4 w_2 H_2 w_1 + w_2 L},$$

$$\frac{d \cos@j_2@tDD h_2@tD}{w_1 w_2} - \frac{d \cos@2 j_1@tD + j_2@tDD h_2@tD}{2 w_1 H_2 w_1 + w_2 L},$$

$$\frac{d \cos@j_2@tDD h_1@tD^2}{2 w_2^2 h_2@tD} - \frac{d \cos@2 j_1@tD + j_2@tDD h_1@tD^2}{4 w_2 H_2 w_1 + w_2 L h_2@tD} >$$

or in a pure function form as

```

sol1Rule = Table@solVar@1D@@iDD ->
  HEvaluate@sol1rhs@@iDD . Thread@basicTerms -> 8#1, #2, #3, #4<DD &L, 8i, 4<D
: a1,1 Ⓡ  $\int_k \frac{d \sin@2 \#3 + \#4D \#1 \#2}{2 w_1 H_2 w_1 + w_2 L} \&Z, a_{2,1} \textcircled{R} \int_k \frac{d \sin@4D \#1^2}{2 w_2^2} - \frac{d \sin@2 \#3 + \#4D \#1^2}{4 w_2 H_2 w_1 + w_2 L} \&Z,$ 
f1,1 Ⓡ  $\int_k \frac{d \cos@4D \#2}{w_1 w_2} - \frac{d \cos@2 \#3 + \#4D \#2}{2 w_1 H_2 w_1 + w_2 L} \&Z, f_{2,1} \textcircled{R} \int_k \frac{d \cos@4D \#1^2}{2 \#2 w_2^2} - \frac{d \cos@2 \#3 + \#4D \#1^2}{4 \#2 w_2 H_2 w_1 + w_2 L} \&Z>$ 

```

Substituting the slow- and fast-varying components of the first-order solution, **SVT1Rule** and **sol1Rule**, into the second-order equations, we have

```

Horder2Expr = Table@Subtract @@ eqEps@@2, iDD . SVT1Rule . sol1Rule . Expand . TrigReduce,
  8i, 4<D;L . Timing
85.748 Second, Null<

```

Since we are seeking an expansion valid up to $O(\epsilon^2 L)$, we do not need to solve for the $a_{i,2}$ and $f_{i,2}$. All we need to do is to investigate the above expressions to determine the slowly varying parts and hence determine the $A_{i,2}$ and $F_{i,2}$. The result is

```

SVT2Rule = Table@
  Solve@order2Expr@@iDD == 0 . Thread@solVar@2D -> H0 &LD . expRule1 . f@_ t + _ .D -> 0 .
  expRule2 . f@a_D :> f@Expand@aDD, SVT@2D@@iDDD, 8i, 4<D . Flatten
:A1,2@h1@tD, h2@tDD Ⓡ 0, A2,2@h1@tD, h2@tDD Ⓡ 0,
F1,2@h1@tD, h2@tDD Ⓡ  $-\frac{8 d^2 w_1^2 h_1 @tD^2 + 5 d^2 w_1 w_2 h_1 @tD^2 + 2 d^2 w_2^2 h_2 @tD^2}{8 w_1^2 w_2^2 H_2 w_1 + w_2 L},$ 
F2,2@h1@tD, h2@tDD Ⓡ  $-\frac{d^2 h_1 @tD^2}{4 w_1 w_2 H_2 w_1 + w_2 L}>$ 

```

Hence, we find that, to the second approximation, the solution is given by

```

eq54c =
8u1@tD, u2@tD< == HNormal@Series@8u1@tD, u2@tD< . transformRule . solRule, 8e, 0, 1<DD .
  sol1Rule . TrigReduce . ExpandL . Thread
: u1@tD ==  $\frac{2 \sin@j_1 @tDD w_1 h_1 @tD}{2 w_1 + w_2} + \frac{\sin@j_1 @tDD w_2 h_1 @tD}{2 w_1 + w_2} + \frac{d e \cos@j_1 @tD - j_2 @tDD h_1 @tD h_2 @tD}{2 w_1 H_2 w_1 + w_2 L} +$ 
 $\frac{d e \cos@j_1 @tD - j_2 @tDD h_1 @tD h_2 @tD}{w_2 H_2 w_1 + w_2 L} + \frac{d e \cos@j_1 @tD + j_2 @tDD h_1 @tD h_2 @tD}{w_2 H_2 w_1 + w_2 L},$ 
u2@tD ==  $\frac{d e w_1 h_1 @tD^2}{w_2^2 H_2 w_1 + w_2 L} + \frac{d e h_1 @tD^2}{2 w_2 H_2 w_1 + w_2 L} - \frac{d e \cos@2 j_1 @tDD h_1 @tD^2}{4 w_2 H_2 w_1 + w_2 L} +$ 
 $\frac{2 \sin@j_2 @tDD w_1 h_2 @tD}{2 w_1 + w_2} + \frac{\sin@j_2 @tDD w_2 h_2 @tD}{2 w_1 + w_2} >$ 

```

where the amplitudes and phases are given by

```
ampEqs = basicDRule@@81, 2<DD •. SVT1Rule •. SVT2Rule •. psiRule •.
```

```
  f_@a_D := f@Collect@a, tDD •. Rule -> Equal
```

```
: h1ε@tD == 
$$\frac{d e \cos 2 b_1 - b_2 - t H - 2 w_1 + w_2 L D h_1 @ t D h_2 @ t D}{2 w_1},$$

```

```
h2ε@tD == - 
$$\frac{d e \cos 2 b_1 - b_2 - t H - 2 w_1 + w_2 L D h_1 @ t D^2}{4 w_2} >$$

```

```
phaseEqs = basicDRule@@83, 4<DD •. SVT1Rule •. SVT2Rule •. psiRule •.
```

```
  f_@a_D := f@Collect@a, tDD •. Rule -> Equal
```

```
: j1ε@tD ==
```

```
w1 - 
$$\frac{d e \sin 2 b_1 - b_2 - t H - 2 w_1 + w_2 L D h_2 @ t D}{2 w_1} - \frac{e^2 H 8 d^2 w_1^2 h_1 @ t D^2 + 5 d^2 w_1 w_2 h_1 @ t D^2 + 2 d^2 w_2^2 h_2 @ t D^2 L}{8 w_1^2 w_2^2 H 2 w_1 + w_2 L},$$

```

```
j2ε@tD == w2 - 
$$\frac{d^2 e^2 h_1 @ t D^2}{4 w_1 w_2 H 2 w_1 + w_2 L} - \frac{d e \sin 2 b_1 - b_2 - t H - 2 w_1 + w_2 L D h_1 @ t D^2}{4 w_2 h_2 @ t D} >$$

```

Chapter 6

Forced Oscillators of Systems with Finite Degrees of Freedom

In this chapter, we discuss nonlinear systems having finite degrees of freedom. The discussion is limited to weakly nonlinear systems, and approximate solutions are obtained by using the method of multiple scales. In the case of strongly nonlinear systems, perturbation methods can be used in cases for which a basic exact nonlinear solution exists. For the other cases, recourse is often made to numerical methods and / or geometrical methods to obtain a qualitative description of the behavior of the system, including its stability.

In contrast with a single-degree-of-freedom system, which has only a single linear natural frequency and a single mode of oscillation, an n -degree-of-freedom system has n linear natural frequencies and n corresponding modes. Let us denote these frequencies by $w_1, w_2, \dots, w_n$ and assume that all of them are real and different from zero. An important case occurs whenever two or more of these frequencies are commensurate or nearly commensurate. Examples of near-commensurability are

$$8w_2 \gg 2w_1, w_3 \gg w_2 \pm w_1, w_2 \gg 3w_1, w_3 \gg 2w_2 \pm w_1, w_4 \gg w_3 \pm w_2 \pm w_1$$

Depending on the order of the nonlinearity in the system, these commensurate relationships of frequencies can cause the corresponding modes to be strongly coupled, and an [internal](#) or [autoparametric resonance](#) is said to exist. For example, if the system has quadratic nonlinearities, then to first order an internal resonance may exist if $w_m \gg 2w_k$ or $w_q \gg w_p \pm w_m$. For a system with cubic nonlinearities, to first order an internal resonance may exist if $w_m \gg 3w_k$ or $w_q \gg w_p \pm w_m \pm w_k$ or $w_m \gg w_k$ or $w_q \gg 2w_p \pm w_m$. When an internal resonance exists in a free undamped system, energy imparted initially to one of the modes involved in the internal resonance will be continuously exchanged among the modes involved in that internal resonance. If damping is present in the system, then the energy will decay with time as it is being continuously exchanged.

In a conservative nongyroscopic system, if the linear motion is oscillatory, then the nonlinear motion is bounded. For a conservative gyroscopic multidegree-of-freedom system, the nonlinear motion may be unbounded and hence unstable if an internal resonance exists.

If an external harmonic excitation of frequency W acts on a multidegree-of-freedom system, then in addition to all of the primary and secondary resonances ($pW \gg qw_m$, with p and q being integers) of a single-degree-of-freedom system, there might exist other [resonance combinations](#) of the frequencies in the form

$$pW = a_1 w_1 + a_2 w_2 + \dots + a_N w_N$$

where p and the a_n are positive or negative integers such that

$$|p| + \sum_{n=1}^N |a_n| = M$$

where M is the order of the nonlinearity plus one and N is the number of degrees of freedom. The type of combination resonance which might exist in a system depends on the order of its nonlinearity. For a system having quadratic nonlinearities, to first order the combination resonances that might exist involve two frequencies in addition to W ; that is, $W \gg w_m + w_k$

or $W \gg w_m - w_k$. The first of these is called a **summed combination resonance** or a **combination resonance of the additive type**, whereas the second is called a **difference combination resonance** or a **combination resonance of the difference type**. These types of combination resonances were predicted theoretically by Malkin (1956) and found experimentally by Yamamoto (1957, 1960). For a system having cubic nonlinearities, to first order the combination resonances that might exist involve either two or three of the natural frequencies in addition to W ; that is,

$$8W \gg w_p \pm w_m \pm w_k, \quad W \gg 2w_p \pm w_m, \quad W \gg w_p \pm 2w_m, \quad 2W \gg w_m \pm w_k$$

à Preliminaries

```
Off@General::spell1D
```

```
Needs@"Utilities`Notation`"D
```

To use the method of multiple scales, we introduce different time scales, symbolize them as

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

and form a list of them as follows:

```
timeScales = {T0, T1, T2};
```

In terms of the time scales T_n , the time derivatives become

```
dt@1D@expr_D := Sum[Ae^i D@expr, timeScales@@i+1DDD, {i, 0, maxOrder<E};
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD • ExpandL • e^i_ ; i>maxOrder -> 0;
```

In the course of the analysis, we need the complex conjugates of A and G . We define them using the following rule:

```
conjugateRule = {A -> A-bar, A-bar -> A, G -> G-bar, G-bar -> G, Complex@0, n_D -> Complex@0, -nD=};
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
9Derivative@a__DAu__i,j__E@__D := SequenceFormATimes [z, MapIndexedAD^#1
#2@@1DD-1 &, {8a<E, u_i,jE,
Derivative@a__DAA__i__E@__D := SequenceFormATimes [z, MapIndexedAD^#1
#2@@1DD &, {8a<E, A_iE,
u__i,j__@__D := u_i,j, A__i__@__D := A_i=;
```

à 6.1 Externally Excited Linearly Uncoupled Systems

ÿ System of Equations

We consider the response of the following two-degree-of-freedom system with quadratic and cubic geometric nonlinearities to an external (additive) excitation:

eq61a = 9

$$u_1^2 @tD + w_1^2 u_1 @tD + 2 m_1 u_1^c @tD + D @V @u_1 @tD, u_2 @tDD, u_1 @tDD == F_1 \cos @W t + t_1 D,$$

$$u_2^2 @tD + w_2^2 u_2 @tD + 2 m_2 u_2^c @tD + D @V @u_1 @tD, u_2 @tDD, u_2 @tDD == F_2 \cos @W t + t_2 D=$$

$$8w_1^2 u_1 @tD + 2 m_1 u_1^c @tD + u_1^2 @tD + V^{H1, 0L} @u_1 @tD, u_2 @tDD == \cos @t W + t_1 D F_1,$$

$$w_2^2 u_2 @tD + 2 m_2 u_2^c @tD + u_2^2 @tD + V^{H0, 1L} @u_1 @tD, u_2 @tDD == \cos @t W + t_2 D F_2<$$

where the potential function V possesses cubic and quartic nonlinearities. The most general such potential function can be expressed as follows:

basicModes = 8u₁@tD, u₂@tD<;

cubicTerms = Nest@Outer@Times, basicModes, #D &, basicModes, 2D •• Flatten •• Union

8u₁@tD³, u₁@tD² u₂@tD, u₁@tD u₂@tD², u₂@tD³<

quarticTerms = Outer@Times, basicModes, cubicTermsD •• Flatten •• Union

8u₁@tD⁴, u₁@tD³ u₂@tD, u₁@tD² u₂@tD², u₁@tD u₂@tD³, u₂@tD⁴<

potential =

V -> HEvaluate@Sum@d_i cubicTerms@@iDD, 8i, Length@cubicTermsD<D + Sum@a_i quarticTerms@@iDD,
8i, Length@quarticTermsD<D •. Thread@basicModes -> 8#1, #2<DD &L

$$V @ H \#1^4 a_1 + \#1^3 \#2 a_2 + \#1^2 \#2^2 a_3 + \#1 \#2^3 a_4 + \#2^4 a_5 + \#1^3 d_1 + \#1^2 \#2 d_2 + \#1 \#2^2 d_3 + \#2^3 d_4 \&L$$

Approximate solutions of such a system can be obtained using a number of techniques, including the method of normal forms, the method of multiple scales, the Krylov-Bogoliubov-Mitropolsky technique, and the method of averaging. With these methods, one determines a set of ordinary-differential equations (modulation equations) that govern the time evolution of the amplitudes and phases of the modes participating in the response. If these equations are time-dependent, one usually transforms them into a set of autonomous equations. In this chapter, we describe how the method of multiple scales can be used to determine uniform expansions of this system, including the modulation equations.

Secondary resonances of nonlinear systems include subharmonic, superharmonic, and combination resonances. One or more of these resonances might be activated in the presence or absence of internal resonances. When a combination resonance is activated in the presence of an internal resonance, fractional harmonics might be excited.

For a consistent expansion, we first transform **eq61a** into a system of four first-order equations. To this end, we introduce the two states $v_1@tD$ and $v_2@tD$ defined by

$$ve1 = 8u_1^c @tD -> v_1 @tD, u_2^c @tD -> v_2 @tD<;$$

Substituting the velocity and acceleration terms, using **ve1** and **potential**, into **eq61a** and combining the result with **ve1**, we transform **eq61a** into the following set of four first-order equations:

eq61b = 8vel . Rule -> Equal, eq61a . D@vel, tD . vel . potential< . Transpose . Flatten

$$\begin{aligned}
 8u_1^{\epsilon} @tD &= v_1 @tD, \quad w_1^2 u_1 @tD + 3 d_1 u_1 @tD^2 + 4 a_1 u_1 @tD^3 + 2 d_2 u_1 @tD u_2 @tD + 3 a_2 u_1 @tD^2 u_2 @tD + \\
 &\quad d_3 u_2 @tD^2 + 2 a_3 u_1 @tD u_2 @tD^2 + a_4 u_2 @tD^3 + 2 m_1 v_1 @tD + v_1^{\epsilon} @tD == \text{Cos}@t W + t_1 D F_1, \\
 u_2^{\epsilon} @tD &= v_2 @tD, \quad d_2 u_1 @tD^2 + a_2 u_1 @tD^3 + w_2^2 u_2 @tD + 2 d_3 u_1 @tD u_2 @tD + 2 a_3 u_1 @tD^2 u_2 @tD + \\
 &\quad 3 d_4 u_2 @tD^2 + 3 a_4 u_1 @tD u_2 @tD^2 + 4 a_5 u_2 @tD^3 + 2 m_2 v_2 @tD + v_2^{\epsilon} @tD == \text{Cos}@t W + t_2 D F_2
 \end{aligned}$$

We seek a uniform second-order expansion of the solution of **eq61b** in the form

solRule =

$$9u_{i_} \rightarrow \text{ISumAe}^j u_{i,j} @\#1, \#2, \#3D, 8j, 3<E \&M, v_{i_} \rightarrow \text{ISumAe}^j v_{i,j} @\#1, \#2, \#3D, 8j, 3<E \&M=;$$

and set

maxOrder = 2;

The damping and forcing terms in **eq61b** need to be scaled so that they balance the influence of the nonlinearity. The scaling of the forcing depends on the type of resonance. Next, we treat different resonance cases in the following sections.

¶ 6.1.1 $W \gg w_2$ and $w_2 \gg 2 w_1$

In this case, we have a combination of a primary resonance of the second mode, a subharmonic resonance of order one-half of the first mode, and a two-to-one internal resonance. In order to bring the effects of damping, forcing, and nonlinearity at the same order, we let

$$\text{scaling} = 9m_{n_} \rightarrow e m_n, F_1 \rightarrow e f_1, F_2 \rightarrow e^2 f_2=;$$

To describe quantitatively the nearness of the resonances, we introduce the two detunings S_1 and S_2 defined by

$$\text{ResonanceCond} = 8w_2 == 2 w_1 + e s_1, W == w_2 + e s_2<;$$

Then, we define the following lists:

$$\text{omgList} = 8w_1, w_2<;$$

$$\text{omgRule} = \text{Solve}@ResonanceCond, 8W, \#< . FlattenD@@1DD \& . \checkmark \text{omgList} . Reverse$$

$$98W @ e s_1 + e s_2 + 2 w_1, w_2 @ e s_1 + 2 w_1<, 9W @ e s_2 + w_2, w_1 @ \frac{1}{2} H - e s_1 + w_2 L ==$$

Using the time scales T_0, T_1 , and T_2 , we express the time derivative as

$$\begin{aligned}
 \text{multiScales} &= 8u_{i_} @tD \rightarrow u_i \checkmark \checkmark \text{timeScales}, v_{i_} @tD \rightarrow v_i \checkmark \checkmark \text{timeScales}, \\
 \text{Derivative}@1D@u_D @tD &\rightarrow dt@1D@u \checkmark \checkmark \text{timeScalesD}, t \rightarrow T_0<;
 \end{aligned}$$

Substituting the **scaling**, **multiScales**, and **solRule** into **eq61b**, expanding the result for small e , and discarding terms of order higher than e^3 , we obtain

```
eq611a = Heq61b . scaling . multiScales . solRule . TrigToExp . ExpandAllL . en_*; n>3 -> 0;
```

Equating coefficients of like powers of e , we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & . žž eq611a . Transpose;
```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```
eqOrder@i_D := I#@@1DD & . žž eqEps@@1DD . f_s_ -> 0 . u_{-k,1} -> u_{k,i}M ==  
I#@@1DD & . žž eqEps@@1DD . f_s_ -> 0 . u_{-k,1} -> u_{k,i}M - H#@@1DD & . žž eqEps@@iDDL . Thread
```

Using the **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D . displayRule
```

```
eqOrder@2D . displayRule
```

```
eqOrder@3D . displayRule
```

$$9D_0 u_{1,1} - v_{1,1} == 0, D_0 v_{1,1} + w_1^2 u_{1,1} == \frac{1}{2} E^{-I T_0 W - I t_1} f_1 + \frac{1}{2} E^{I T_0 W + I t_1} f_1,$$

$$D_0 u_{2,1} - v_{2,1} == 0, D_0 v_{2,1} + w_2^2 u_{2,1} == 0$$

$$9D_0 u_{1,2} - v_{1,2} == -HD_1 u_{1,1} L, D_0 v_{1,2} + w_1^2 u_{1,2} == -HD_1 v_{1,1} L - 3 d_1 u_{1,1}^2 - 2 d_2 u_{1,1} u_{2,1} - d_3 u_{2,1}^2 - 2 m_1 v_{1,1},$$

$$D_0 u_{2,2} - v_{2,2} == -HD_1 u_{2,1} L, D_0 v_{2,2} + w_2^2 u_{2,2} ==$$

$$-HD_1 v_{2,1} L + \frac{1}{2} E^{-I T_0 W - I t_2} f_2 + \frac{1}{2} E^{I T_0 W + I t_2} f_2 - d_2 u_{1,1}^2 - 2 d_3 u_{1,1} u_{2,1} - 3 d_4 u_{2,1}^2 - 2 m_2 v_{2,1} =$$

$$8D_0 u_{1,3} - v_{1,3} == -HD_1 u_{1,2} L - D_2 u_{1,1},$$

$$D_0 v_{1,3} + w_1^2 u_{1,3} == -HD_1 v_{1,2} L - D_2 v_{1,1} - 4 a_1 u_{1,1}^3 - 6 d_1 u_{1,1} u_{1,2} - 3 a_2 u_{1,1}^2 u_{2,1} -$$

$$2 d_2 u_{1,2} u_{2,1} - 2 a_3 u_{1,1} u_{2,1}^2 - a_4 u_{2,1}^3 - 2 d_2 u_{1,1} u_{2,2} - 2 d_3 u_{2,1} u_{2,2} - 2 m_1 v_{1,2},$$

$$D_0 u_{2,3} - v_{2,3} == -HD_1 u_{2,2} L - D_2 u_{2,1}, D_0 v_{2,3} + w_2^2 u_{2,3} == -HD_1 v_{2,2} L - D_2 v_{2,1} - a_2 u_{1,1}^3 - 2 d_2 u_{1,1} u_{1,2} -$$

$$2 a_3 u_{1,1}^2 u_{2,1} - 2 d_3 u_{1,2} u_{2,1} - 3 a_4 u_{1,1} u_{2,1}^2 - 4 a_5 u_{2,1}^3 - 2 d_3 u_{1,1} u_{2,2} - 6 d_4 u_{2,1} u_{2,2} - 2 m_2 v_{2,2} <$$

§ First-Order Problem: Linear System

The first-order problem, **eqOrder[1]**, consists of two sets of uncoupled linear nonhomogeneous differential equations. Hence, the general solution of each set can be obtained by using the principle of superposition as the sum of any particular solution and the solution of the homogeneous equations. To determine the general solution of the homogeneous sets, we rewrite the homogeneous part of **eqOrder[1]** as

```
linearSys = #@@1DD & . žž eqOrder@1D;
```

```
linearSys . displayRule
```

$$8D_0 u_{1,1} - v_{1,1}, D_0 v_{1,1} + w_1^2 u_{1,1}, D_0 u_{2,1} - v_{2,1}, D_0 v_{2,1} + w_2^2 u_{2,1} <$$

Next, we seek a solution of the **linearSys** in the form $u_{i,1} = P E^{I w_i T_0}$ and $v_{i,1} = Q E^{I w_i T_0}$ and obtain


```

expr1 =
linearSys •. 9ui,1 -> IP EI w1 #1 &M, vi,1 -> IQ EI w1 #1 &M= •. Exp@a_D -> 1 • Partition@#, 2D &
88-Q + IP w1, IQ w1 + P w12<, 8-Q + IP w2, IQ w2 + P w22<<

```

The coefficient matrices of **expr1** are

```

coefMat = Outer@D, #, 8P, Q<D & •ž expr1
88I w1, -1<, 8w12, I w1<<, 88I w2, -1<, 8w22, I w2<<<

```

and their adjoints are defined by

```

hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose

```

Hence, the right and left eigenvectors of **coefMat** are

```

rightVec = # • @@1DD & •ž HNullSpace@#D@@1DD & •ž coefMatL
881, I w1<, 81, I w2<<

leftVec = NullSpace@hermitian@#D@@1DD & •ž coefMat
88-I w1, 1<, 8-I w2, 1<<

```

whose complex conjugates are

```

ccleftVec = leftVec •. conjugateRule
88I w1, 1<, 8I w2, 1<<

```

To determine particular solutions of the partial-differential equations **eqOrder[1]** using the *Mathematica* function **DSolve**, we first transform them into a set of ordinary-differential equations as

```

order1Eq = eqOrder@1D •. ui,1 -> Hui,1@#1D &L
9- v1,1@T0D + u1,1€@T0D == 0, w12 u1,1@T0D + v1,1€@T0D ==  $\frac{1}{2} E^{-I T_0 W - I t_1} f_1 + \frac{1}{2} E^{I T_0 W + I t_1} f_1$ ,
- v2,1@T0D + u2,1€@T0D == 0, w22 u2,1@T0D + v2,1€@T0D == 0=

```

Then, the particular solutions of **order1Eq** are

```

sol1p =
DSolve@order1Eq, 8u1,1@T0D, v1,1@T0D, u2,1@T0D, v2,1@T0D<, T0D@@1DD •. C@_D -> 0 •• Simplify
: u1,1@T0D ® -  $\frac{E^{-I H T_0 W + t_1 L} H_1 + E^{2 I H T_0 W + t_1 L} f_1}{2 H W^2 - w_1^2 L}$ ,
v1,1@T0D ® -  $\frac{I E^{-I H T_0 W + t_1 L} H - 1 + E^{2 I H T_0 W + t_1 L} L W f_1}{2 H W^2 - w_1^2 L}$ , u2,1@T0D ® 0, v2,1@T0D ® 0>

```

To simplify some of the expressions, we let

$$\mathbf{fRule} = 9\mathbf{f}_i \rightarrow 2\mathbf{L}_i \mathbf{I} \mathbf{w}_i^2 - \mathbf{W}^2 \mathbf{M};$$

and express $u_{i,1}$ and $v_{i,1}$ in pure function form as

$$\begin{aligned} \mathbf{sol1u} &= \mathbf{TableAu}_{i,1} \rightarrow \mathbf{FunctionA8T_0, T_1, T_2}, \mathbf{A_i@T_1, T_2D Exp@I w_i T_0D} + \dot{\mathbf{A_i@T_1, T_2D Exp@-I w_i T_0D}} + \\ &\quad \mathbf{Hu}_{i,1@T_0D} \cdot \mathbf{sol1p} \cdot \mathbf{fRule} \cdot \mathbf{Simplify} \cdot \mathbf{ExpandL} \cdot \mathbf{EvaluateE}, \mathbf{8i, 2<E} \\ 8u_{1,1} &\otimes \mathbf{Function@8T_0, T_1, T_2}, \mathbf{E^{-I T_0 W-I t_1} L_1} + \mathbf{E^{I T_0 W+I t_1} L_1} + \mathbf{E^{I T_0 w_1} A_1@T_1, T_2D} + \mathbf{E^{-I T_0 w_1} \dot{A_1@T_1, T_2DD}}, \\ u_{2,1} &\otimes \mathbf{Function@8T_0, T_1, T_2}, \mathbf{E^{I T_0 w_2} A_2@T_1, T_2D} + \mathbf{E^{-I T_0 w_2} \dot{A_2@T_1, T_2DD}} \\ \mathbf{sol1v} &= \\ \mathbf{Table@v}_{i,1} &\rightarrow \mathbf{Function@8T_0, T_1, T_2}, \mathbf{D@u}_{i,1} \mathbf{z\check{z} timeScales, T_0D} \cdot \mathbf{sol1u} \cdot \mathbf{Evaluated}, \mathbf{8i, 2<D} \\ 8v_{1,1} &\otimes \mathbf{Function@8T_0, T_1, T_2}, \\ &\quad \mathbf{-I E^{-I T_0 W-I t_1} W L_1} + \mathbf{I E^{I T_0 W+I t_1} W L_1} + \mathbf{I E^{I T_0 w_1} w_1 A_1@T_1, T_2D} - \mathbf{I E^{-I T_0 w_1} w_1 \dot{A_1@T_1, T_2DD}}, \\ v_{2,1} &\otimes \mathbf{Function@8T_0, T_1, T_2}, \mathbf{I E^{I T_0 w_2} w_2 A_2@T_1, T_2D} - \mathbf{I E^{-I T_0 w_2} w_2 \dot{A_2@T_1, T_2DD}} \end{aligned}$$

Then, we combine them using the *Mathematica* function **Join** as

$$\mathbf{sol1} = \mathbf{Join@sol1u, sol1vD};$$

¶ Second-Order Problem

Substituting the first-order solution **sol1** into the second-order problem, **eqOrder[2]**, yields

$$\begin{aligned} \mathbf{order2Eq} &= \mathbf{eqOrder@2D} \cdot \mathbf{sol1} \cdot \mathbf{ExpandAll}; \\ \mathbf{order2Eq} &\cdot \mathbf{displayRule} \\ 9D_0 u_{1,2} - v_{1,2} &= -\mathbf{E^{I T_0 w_1} HD_1 A_1 L} - \mathbf{E^{-I T_0 w_1} HD_1 \dot{A_1 L}}, \\ D_0 v_{1,2} + w_1^2 u_{1,2} &= -3\mathbf{E^{2I T_0 w_1} A_1^2 d_1} - 2\mathbf{E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2} - \mathbf{E^{2I T_0 w_2} A_2^2 d_3} - \\ &\quad 6\mathbf{E^{-I T_0 W-I t_1 + I T_0 w_1} A_1 d_1 L_1} - 6\mathbf{E^{I T_0 W+I t_1 + I T_0 w_1} A_1 d_1 L_1} - 2\mathbf{E^{-I T_0 W-I t_1 + I T_0 w_2} A_2 d_2 L_1} - \\ &\quad 2\mathbf{E^{I T_0 W+I t_1 + I T_0 w_2} A_2 d_2 L_1} - 6\mathbf{d_1 L_1^2} - 3\mathbf{E^{-2I T_0 W-2I t_1} d_1 L_1^2} - 3\mathbf{E^{2I T_0 W+2I t_1} d_1 L_1^2} + \\ &\quad 2\mathbf{I E^{-I T_0 W-I t_1} W L_1 m_1} - 2\mathbf{I E^{I T_0 W+I t_1} W L_1 m_1} - \mathbf{I E^{I T_0 w_1} HD_1 A_1 L w_1} + \mathbf{I E^{-I T_0 w_1} HD_1 \dot{A_1 L} w_1} - \\ &\quad 2\mathbf{I E^{I T_0 w_1} A_1 m_1 w_1} - 6\mathbf{A_1 d_1 \dot{A_1}} - 2\mathbf{E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A_1}} - 6\mathbf{E^{-I T_0 W-I t_1 - I T_0 w_1} d_1 L_1 \dot{A_1}} - \\ &\quad 6\mathbf{E^{I T_0 W+I t_1 - I T_0 w_1} d_1 L_1 \dot{A_1}} + 2\mathbf{I E^{-I T_0 w_1} m_1 w_1 \dot{A_1}} - 3\mathbf{E^{-2I T_0 w_1} d_1 \dot{A_1}^2} - 2\mathbf{E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A_2}} - 2\mathbf{A_2 d_3 \dot{A_2}} - \\ &\quad 2\mathbf{E^{-I T_0 W-I t_1 - I T_0 w_2} d_2 L_1 \dot{A_2}} - 2\mathbf{E^{I T_0 W+I t_1 - I T_0 w_2} d_2 L_1 \dot{A_2}} - 2\mathbf{E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A_1} \dot{A_2}} - \mathbf{E^{-2I T_0 w_2} d_3 \dot{A_2}^2}, \\ D_0 u_{2,2} - v_{2,2} &= -\mathbf{E^{I T_0 w_2} HD_1 A_2 L} - \mathbf{E^{-I T_0 w_2} HD_1 \dot{A_2 L}}, \\ D_0 v_{2,2} + w_2^2 u_{2,2} &= \frac{1}{2}\mathbf{E^{-I T_0 W-I t_2} f_2} + \frac{1}{2}\mathbf{E^{I T_0 W+I t_2} f_2} - \mathbf{E^{2I T_0 w_1} A_1^2 d_2} - 2\mathbf{E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3} - \\ &\quad 3\mathbf{E^{2I T_0 w_2} A_2^2 d_4} - 2\mathbf{E^{-I T_0 W-I t_1 + I T_0 w_1} A_1 d_2 L_1} - 2\mathbf{E^{I T_0 W+I t_1 + I T_0 w_1} A_1 d_2 L_1} - \\ &\quad 2\mathbf{E^{-I T_0 W-I t_1 + I T_0 w_2} A_2 d_3 L_1} - 2\mathbf{E^{I T_0 W+I t_1 + I T_0 w_2} A_2 d_3 L_1} - 2\mathbf{d_2 L_1^2} - \mathbf{E^{-2I T_0 W-2I t_1} d_2 L_1^2} - \\ &\quad \mathbf{E^{2I T_0 W+2I t_1} d_2 L_1^2} - \mathbf{I E^{I T_0 w_2} HD_1 A_2 L w_2} + \mathbf{I E^{-I T_0 w_2} HD_1 \dot{A_2 L} w_2} - 2\mathbf{I E^{I T_0 w_2} A_2 m_2 w_2} - \\ &\quad 2\mathbf{A_1 d_2 \dot{A_1}} - 2\mathbf{E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A_1}} - 2\mathbf{E^{-I T_0 W-I t_1 - I T_0 w_1} d_2 L_1 \dot{A_1}} - 2\mathbf{E^{I T_0 W+I t_1 - I T_0 w_1} d_2 L_1 \dot{A_1}} - \\ &\quad \mathbf{E^{-2I T_0 w_1} d_2 \dot{A_1}^2} - 2\mathbf{E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A_2}} - 6\mathbf{A_2 d_4 \dot{A_2}} - 2\mathbf{E^{-I T_0 W-I t_1 - I T_0 w_2} d_3 L_1 \dot{A_2}} - \\ &\quad 2\mathbf{E^{I T_0 W+I t_1 - I T_0 w_2} d_3 L_1 \dot{A_2}} + 2\mathbf{I E^{-I T_0 w_2} m_2 w_2 \dot{A_2}} - 2\mathbf{E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A_1} \dot{A_2}} - 3\mathbf{E^{-2I T_0 w_2} d_4 \dot{A_2}^2} = \end{aligned}$$

Any particular solution of **order2Eq** contains secular terms and small-divisor terms, making the expansion nonuniform. For a uniform expansion, we choose $D_1 A_1$ and $D_1 A_2$ to eliminate these terms. To accomplish this, we first convert the small-divisor terms into secular terms using the rule

```
expRule1@i_D := Exp@a_D :=> Exp@Expand@a . omgRule@@iDDD . e T_0 -> T_1 D
```

To eliminate the terms that produce secular terms (i.e., determine the solvability conditions) from **order2Eq**, we collect the terms proportional to $E^{I w_1 T_0}$ and $E^{I w_2 T_0}$ and obtain

```
ST11 = Coefficient@order2Eq@@#, 2DD . expRule1@1D, Exp@I w_1 T_0DD & . ž 81, 2<;
```

```
ST11 . displayRule
```

$$8 - \text{HD}_1 A_1 L, -I \text{HD}_1 A_1 L w_1 - 2 I A_1 m_1 w_1 - 2 E^{I T_1 s_1} A_2 d_2 \dot{A}_1 - 6 E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1 <$$

```
ST12 = Coefficient@order2Eq@@#, 2DD . expRule1@2D, Exp@I w_2 T_0DD & . ž 83, 4<;
```

```
ST12 . displayRule
```

$$9 - \text{HD}_1 A_2 L, \frac{1}{2} E^{I T_1 s_2 + I t_2} f_2 - E^{-I T_1 s_1} A_1^2 d_2 - I \text{HD}_1 A_2 L w_2 - 2 I A_2 m_2 w_2 =$$

Then, the solvability conditions, conditions for the elimination of the terms that produce secular terms, demand that **ST11** and **ST12** be orthogonal to every solution of the corresponding adjoint problem, namely, the components of **cleftVec**. The result is

```
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
```

```
SCond1 . displayRule
```

$$9 - 2 I \text{HD}_1 A_1 L w_1 - 2 I A_1 m_1 w_1 - 2 E^{I T_1 s_1} A_2 d_2 \dot{A}_1 - 6 E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1 == 0,$$

$$\frac{1}{2} E^{I T_1 s_2 + I t_2} f_2 - E^{-I T_1 s_1} A_1^2 d_2 - 2 I \text{HD}_1 A_2 L w_2 - 2 I A_2 m_2 w_2 == 0 =$$

Solving **SCond1** for $D_1 A_1$ and $D_1 A_2$ yields

```
SCond1Rule1 = SolveASCond1, 9A_1^{H1, 0L}@T_1, T_2 D, A_2^{H1, 0L}@T_1, T_2 D = E@@1DD . ExpandAll;
```

```
SCond1Rule1 . displayRule
```

$$: D_1 A_1 \text{ @ } - A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1}{w_1},$$

$$D_1 A_2 \text{ @ } - A_2 m_2 - \frac{I E^{I T_1 s_2 + I t_2} f_2}{4 w_2} + \frac{I E^{-I T_1 s_1} A_1^2 d_2}{2 w_2} >$$

With these conditions, **order2Eq** are solvable. However, their solutions are not unique. To render them unique, we demand that they be orthogonal to solutions of the corresponding adjoint problems; that is, we demand that $u_{1,2}, v_{1,2}$ be orthogonal to **cleftVec[[1]]** and $u_{2,2}, v_{2,2}$ be orthogonal to **cleftVec[[2]]**.

Next, we use the solvability conditions to eliminate $D_1 A_1$ and $D_1 A_2$ from **order2Eq**. To simplify the resulting equations using *Mathematica*, we replace the scale T_1 in **SCond1Rule1** with $e T_0$. To this end, we express the detuning parameters S_i in terms of the w_i and W as

```
sigRule = Solve@ResonanceCond, 8s1, s2<D@@1DD
```

$$9s_1 \otimes - \frac{2W_1 - W_2}{e}, s_2 \otimes - \frac{-W + W_2}{e} =$$

and replace T_1 with eT_0 using the rule

```
expRule2 = Exp@a_D :> Exp@a •. T1 -> e T0 •. sigRule •. ExpandD;
```

Using the **expRule2**, we rewrite **SCond1Rule1** as

```
SCond1Rule2 = SCond1Rule1 •. expRule2;
```

```
SCond1Rule2 •. displayRule
```

$$\begin{aligned} :D_1 A_1 \otimes - A_1 m_1 + \frac{I E^{-2 I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{w_1} + \frac{3 I E^{I T_0 W + I t_1 - 2 I T_0 w_1} d_1 I_4 \dot{A}_1}{w_1}, \\ D_1 A_2 \otimes - A_2 m_2 - \frac{I E^{I T_0 W + I t_2 - I T_0 w_2} f_2}{4 w_2} + \frac{I E^{2 I T_0 w_1 - I T_0 w_2} A_1^2 d_2}{2 w_2} > \end{aligned}$$

whose complex conjugate is

```
ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
```

Substituting this form for the solvability conditions into **order2Eq** yields

```
order2Eqm = order2Eq . SCond1Rule2 . ccSCond1Rule2 . ExpandAll;
order2Eqm . displayRule
```

$$\begin{aligned}
9D_0 u_{1,2} - v_{1,2} &= E^{I T_0 w_1} A_1 m_1 + \frac{3 I E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1}{w_1} + E^{-I T_0 w_1} m_1 \dot{A}_1 - \\
&\quad \frac{I E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{w_1} - \frac{3 I E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{w_1} + \frac{I E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{w_1}, \\
D_0 v_{1,2} + w_1^2 u_{1,2} &= -3 E^{2 I T_0 w_1} A_1^2 d_1 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2 - E^{2 I T_0 w_2} A_2^2 d_3 - 3 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1 - \\
&\quad 6 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_1 L_1 - 2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_2 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_2 L_1 - 6 d_1 L_1^2 - \\
&\quad 3 E^{-2 I T_0 W - 2 I t_1} d_1 L_1^2 - 3 E^{2 I T_0 W + 2 I t_1} d_1 L_1^2 + 2 I E^{-I T_0 W - I t_1} W L_1 m_1 - 2 I E^{I T_0 W + I t_1} W L_1 m_1 - \\
&\quad I E^{I T_0 w_1} A_1 m_1 w_1 - 6 A_1 d_1 \dot{A}_1 - E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1 - 6 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 - \\
&\quad 3 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 + I E^{-I T_0 w_1} m_1 w_1 \dot{A}_1 - 3 E^{-2 I T_0 w_1} d_1 \dot{A}_1^2 - E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2 - 2 A_2 d_3 \dot{A}_2 - \\
&\quad 2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2 - E^{-2 I T_0 w_2} d_3 \dot{A}_2^2, \\
D_0 u_{2,2} - v_{2,2} &= E^{I T_0 w_2} A_2 m_2 - \frac{I E^{-I T_0 W - I t_2} f_2}{4 w_2} + \frac{I E^{I T_0 W + I t_2} f_2}{4 w_2} - \\
&\quad \frac{I E^{2 I T_0 w_1} A_2^2 d_2}{2 w_2} + \frac{I E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{2 w_2} + E^{-I T_0 w_2} m_2 \dot{A}_2, \\
D_0 v_{2,2} + w_2^2 u_{2,2} &= \frac{1}{4} E^{-I T_0 W - I t_2} f_2 + \frac{1}{4} E^{I T_0 W + I t_2} f_2 - \frac{1}{2} E^{2 I T_0 w_1} A_1^2 d_2 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3 - \\
&\quad 3 E^{2 I T_0 w_2} A_2^2 d_4 - 2 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_2 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_2 L_1 - 2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_3 L_1 - \\
&\quad 2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_3 L_1 - 2 d_2 L_1^2 - E^{-2 I T_0 W - 2 I t_1} d_2 L_1^2 - E^{2 I T_0 W + 2 I t_1} d_2 L_1^2 - I E^{I T_0 w_2} A_2 m_2 w_2 - \\
&\quad 2 A_1 d_2 \dot{A}_1 - 2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1 - 2 E^{-I T_0 W - I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 - 2 E^{I T_0 W + I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 - \\
&\quad \frac{1}{2} E^{-2 I T_0 w_1} d_2 \dot{A}_1^2 - 2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2 - 6 A_2 d_4 \dot{A}_2 - 2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 - \\
&\quad 2 E^{I T_0 W + I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 + I E^{-I T_0 w_2} m_2 w_2 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2 - 3 E^{-2 I T_0 w_2} d_4 \dot{A}_2^2 =
\end{aligned}$$

Next, we use the method of undetermined coefficients to determine the particular solutions of **order2Eqm**. To accomplish this, we first identify the form of the nonhomogeneous terms. To this end, we let

```
basicH = Table[A9A1@T1, T2D E^{I w1 T0}, A1@T1, T2D E^{-I w1 T0}, 8i, 2<E . Flatten
8E^{I T0 w1} A1@T1, T2D, E^{-I T0 w1} A1@T1, T2D, E^{I T0 w2} A2@T1, T2D, E^{-I T0 w2} A2@T1, T2D<
collectForm = JoinAbasicH, 9L1 E^{I W T0 + I t1}, L1 E^{-I W T0 - I t1}=E
8E^{I T0 w1} A1@T1, T2D, E^{-I T0 w1} A1@T1, T2D,
E^{I T0 w2} A2@T1, T2D, E^{-I T0 w2} A2@T1, T2D, E^{I T0 W + I t1} L1, E^{-I T0 W - I t1} L1<
```

Then, the possible form of the nonhomogeneous terms in **order2Eqm** is

```

possibleTerms = JoinAcollectForm, 9EIWT0+It2, E-IWT0-It2,
Outer@Times, collectForm, collectFormD •• Flatten •• UnionE
: EIT0W1 A1@T1, T2D, E-IT0W1  $\dot{A}_1$ @T1, T2D, EIT0W2 A2@T1, T2D, E-IT0W2  $\dot{A}_2$ @T1, T2D,
EIT0W+It1 L1, E-IT0W-It1 L1, EIT0W+It2, E-IT0W-It2 L12, E-2IT0W-2It1 L12, E2IT0W+2It1 L12,
E-IT0W-It1+IT0W1 L1 A1@T1, T2D, EIT0W+It1+IT0W1 L1 A1@T1, T2D, E2IT0W1 A1@T1, T2D2,
E-IT0W-It1+IT0W2 L1 A2@T1, T2D, EIT0W+It1+IT0W2 L1 A2@T1, T2D, EIT0W1+IT0W2 A1@T1, T2D A2@T1, T2D,
E2IT0W2 A2@T1, T2D2, E-IT0W-It1-IT0W1 L1  $\dot{A}_1$ @T1, T2D, EIT0W+It1-IT0W1 L1  $\dot{A}_1$ @T1, T2D,
A1@T1, T2D  $\dot{A}_1$ @T1, T2D, E-IT0W1+IT0W2 A2@T1, T2D  $\dot{A}_1$ @T1, T2D, E-2IT0W1  $\dot{A}_1$ @T1, T2D2,
E-IT0W-It1-IT0W2 L1  $\dot{A}_2$ @T1, T2D, EIT0W+It1-IT0W2 L1  $\dot{A}_2$ @T1, T2D, EIT0W1-IT0W2 A1@T1, T2D  $\dot{A}_2$ @T1, T2D,
A2@T1, T2D  $\dot{A}_2$ @T1, T2D, E-IT0W1-IT0W2  $\dot{A}_1$ @T1, T2D  $\dot{A}_2$ @T1, T2D, E-2IT0W2  $\dot{A}_2$ @T1, T2D2>

```

Using the method of undetermined coefficients, we assume that the $u_{i,2}$ and $v_{i,2}$ are linear combinations of these possible terms, substitute the result into the governing equations, **order2Eqm**, equate the coefficients of each of the possible terms on both sides of each equation, and obtain a system of nonhomogeneous algebraic equations governing the unknown coefficients. Associated with each possible term is a pair of two algebraic equations. The solutions of all pairs, except those associated with resonance terms, are unique. The solutions of the pairs corresponding to the possible resonance terms are rendered unique by requiring them to be orthogonal to the components of **clefVec**. The possible resonance terms are proportional to $E^{I W_1 T_0}$, $E^{I W_2 T_0}$, and their complex conjugates; that is,

```

ResonantTerms@i_D :=
I# •. 8a_ •; a != 0 -> 1< & •ž I E-IWi T0 possibleTerms •. expRule1@iD •. Exp@_ T0 + _ .D -> 0MM
possibleTerms •• Union •• Rest

RT = Array@ResonantTerms, 2D

88EIT0W1 A1@T1, T2D, EIT0W+It1-IT0W1 L1  $\dot{A}_1$ @T1, T2D, E-IT0W1+IT0W2 A2@T1, T2D  $\dot{A}_1$ @T1, T2D<,
8EIT0W+It2, EIT0W+It1 L1, E2IT0W1 A1@T1, T2D2, EIT0W2 A2@T1, T2D<<

```

Hence, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[1]]** are

```

r1Rule =
MapIndexed@r1, #2@@1DD -> Coefficient@order2Eqm@@@1, 2DD, #1D &, RT@@1DDD •. Exp@_ T0 + _ .D -> 0

: r1,1 ® m1, r1,2 ® -  $\frac{3 I d_1}{w_1}$ , r1,3 ® -  $\frac{I d_2}{w_1}$  >

```

and the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[2]]** are

```

r2Rule =
MapIndexed@r2, #2@@1DD -> Coefficient@order2Eqm@@@2, 2DD, #1D &, RT@@1DDD •. Exp@_ T0 + _ .D -> 0

8r2,1 ® -I m1 w1, r2,2 ® -3 d1, r2,3 ® -d2<

```

It follows from

```
Table@ccleftVec@@1DD.8r1,i, r2,i<, 8i, 3<D . r1Rule . r2Rule
80, 0, 0<
```

that the parts **r1Rule** and **r2Rule** are orthogonal to the solution of the adjoint, the first component of **ccleftVec**, as they should.

Similarly, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[3]]** are

```
r3Rule = MapIndexed@r3, #2@@1DD -> Coefficient@order2Eqm@@3, 2DD, #1D &, RT@@2DDD
: r3,1 @  $\frac{I f_2}{4 w_2}$ , r3,2 @ 0, r3,3 @  $-\frac{I d_2}{2 w_2}$ , r3,4 @ m2>
```

and the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[4]]** are

```
r4Rule =
MapIndexed@r4, #2@@1DD -> Coefficient@order2Eqm@@4, 2DD, #1D &, RT@@2DDD . Exp@_T0 + _.D -> 0
9r4,1 @  $\frac{f_2}{4}$ , r4,2 @ 0, r4,3 @  $-\frac{d_2}{2}$ , r4,4 @  $-I m_2 w_2$ =
```

Again, it follows from

```
Table@ccleftVec@@2DD.8r3,i, r4,i<, 8i, 4<D . r3Rule . r4Rule
80, 0, 0, 0<
```

that the vector $8r_{3,i}, r_{4,i}<$ is orthogonal to the solution of the adjoint, the second component of **ccleftVec**, as it should.

The complement of the resonance terms **RT** in **possibleTerms** yields the nonresonance terms; that is,

```
NRT = Complement@possibleTerms, Join@#, # . conjugateRuleDD & . RT
:: E-I T0 W - I t2, EI T0 W + I t2, E-I T0 W - I t1 L1, EI T0 W + I t1 L1, L12, E-2 I T0 W - 2 I t1 L12,
E2 I T0 W + 2 I t1 L12, EI T0 W + I t1 + I T0 w1 L1 A1@T1, T2D, E2 I T0 w1 A1@T1, T2D2, EI T0 w2 A2@T1, T2D,
E-I T0 W - I t1 + I T0 w2 L1 A2@T1, T2D, EI T0 W + I t1 + I T0 w2 L1 A2@T1, T2D, EI T0 w1 + I T0 w2 A1@T1, T2D A2@T1, T2D,
E2 I T0 w2 A2@T1, T2D2, E-I T0 W - I t1 - I T0 w1 L1 A1@T1, T2D, A1@T1, T2D A1@T1, T2D, E-2 I T0 w1 A1@T1, T2D2,
E-I T0 w2 A2@T1, T2D, E-I T0 W - I t1 - I T0 w2 L1 A2@T1, T2D, EI T0 W + I t1 - I T0 w2 L1 A2@T1, T2D,
A2@T1, T2D A2@T1, T2D, E-I T0 w1 - I T0 w2 A1@T1, T2D A2@T1, T2D, E-2 I T0 w2 A2@T1, T2D2>,
: L12, E-2 I T0 W - 2 I t1 L12, E2 I T0 W + 2 I t1 L12, EI T0 w1 A1@T1, T2D, E-I T0 W - I t1 + I T0 w1 L1 A1@T1, T2D,
EI T0 W + I t1 + I T0 w1 L1 A1@T1, T2D, E-I T0 W - I t1 + I T0 w2 L1 A2@T1, T2D, EI T0 W + I t1 + I T0 w2 L1 A2@T1, T2D,
EI T0 w1 + I T0 w2 A1@T1, T2D A2@T1, T2D, E2 I T0 w2 A2@T1, T2D2, E-I T0 w1 A1@T1, T2D,
E-I T0 W - I t1 - I T0 w1 L1 A1@T1, T2D, EI T0 W + I t1 - I T0 w1 L1 A1@T1, T2D, A1@T1, T2D A1@T1, T2D,
E-I T0 w1 + I T0 w2 A2@T1, T2D A1@T1, T2D, E-I T0 W - I t1 - I T0 w2 L1 A2@T1, T2D,
EI T0 W + I t1 - I T0 w2 L1 A2@T1, T2D, EI T0 w1 - I T0 w2 A1@T1, T2D A2@T1, T2D,
A2@T1, T2D A2@T1, T2D, E-I T0 w1 - I T0 w2 A1@T1, T2D A2@T1, T2D, E-2 I T0 w2 A2@T1, T2D2>>
```

To implement the method of undetermined coefficients, we associate with each possible resonance term an unknown coefficient using the rules

```
RTsymbolList1 = Table@G1,j, 8j, Length@RT@@1DDD<D
```

```
8G1,1, G1,2, G1,3<
```

```
RTsymbolList2 = Table@G3,j, 8j, Length@RT@@2DDD<D
```

```
8G3,1, G3,2, G3,3, G3,4<
```

And, we associate with each possible nonresonance term two unknown coefficients using the rules

```
NRTsymbolList1@i_D = Table@Li,j, 8j, Length@NRT@@1DDD<D
```

```
8Li,1, Li,2, Li,3, Li,4, Li,5, Li,6, Li,7, Li,8, Li,9, Li,10, Li,11,  
Li,12, Li,13, Li,14, Li,15, Li,16, Li,17, Li,18, Li,19, Li,20, Li,21, Li,22, Li,23<
```

```
NRTsymbolList2@i_D = Table@Li+2,j, 8j, Length@NRT@@2DDD<D
```

```
8L2+i,1, L2+i,2, L2+i,3, L2+i,4, L2+i,5, L2+i,6, L2+i,7, L2+i,8, L2+i,9, L2+i,10, L2+i,11,  
L2+i,12, L2+i,13, L2+i,14, L2+i,15, L2+i,16, L2+i,17, L2+i,18, L2+i,19, L2+i,20, L2+i,21<
```

Accounting for the orthogonality conditions, $u_{1,2}$, $v_{1,2}$ must be orthogonal to $\text{ccleftVec}[[1]]$ and $u_{2,2}$, $v_{2,2}$ must be orthogonal to $\text{ccleftVec}[[2]]$, for the resonance terms, **RT**, we express $u_{i,2}$ and $v_{i,2}$ in the following function form:

```
8ratio1, ratio2< = Table@-ccleftVec@@i, 1DD, 8i, 2<D
```

```
8-I w1, -I w2<
```

```
sol2Form = 8u1,2 -> Function@8T0, T1, T2<,
```

```
RTsymbolList1.RT@@1DD + HRTsymbolList1.RT@@1DD • conjugateRuleL +  
NRTsymbolList1@1D.NRT@@1DD • EvaluateD, v1,2 -> Function@8T0, T1, T2<,  
ratio1 * RTsymbolList1.RT@@1DD + Hratio1 * RTsymbolList1.RT@@1DD • conjugateRuleL +  
NRTsymbolList1@2D.NRT@@1DD • EvaluateD, u2,2 -> Function@8T0, T1, T2<,  
RTsymbolList2.RT@@2DD + HRTsymbolList2.RT@@2DD • conjugateRuleL +  
NRTsymbolList2@1D.NRT@@2DD • EvaluateD, v2,2 -> Function@8T0, T1, T2<,  
ratio2 * RTsymbolList2.RT@@2DD + Hratio2 * RTsymbolList2.RT@@2DD • conjugateRuleL +  
NRTsymbolList2@2D.NRT@@2DD • EvaluateD<;
```

Substituting **sol2Form** into **order2Eqn** and collecting the coefficients of like **NRT**, we obtain two sets of algebraic equations as

**eq611b = HCoefficient@Subtract žž # •. sol2Form, NRT@@1DDD •. Exp@_ T₀ + _ . D -> 0 & •ž
order2Eqm@@81, 2<DD •• FlattenL == 0 •• Thread**

$$\begin{aligned}
&8 - I W L_{1,1} - L_{2,1} == 0, I W L_{1,2} - L_{2,2} == 0, -I W L_{1,3} - L_{2,3} == 0, I W L_{1,4} - L_{2,4} == 0, -L_{2,5} == 0, \\
&-2 I W L_{1,6} - L_{2,6} == 0, 2 I W L_{1,7} - L_{2,7} == 0, I W L_{1,8} + I w_1 L_{1,8} - L_{2,8} == 0, 2 I w_1 L_{1,9} - L_{2,9} == 0, \\
&I w_2 L_{1,10} - L_{2,10} == 0, -I W L_{1,11} + I w_2 L_{1,11} - L_{2,11} == 0, I W L_{1,12} + I w_2 L_{1,12} - L_{2,12} == 0, \\
&I w_1 L_{1,13} + I w_2 L_{1,13} - L_{2,13} == 0, 2 I w_2 L_{1,14} - L_{2,14} == 0, -I W L_{1,15} - I w_1 L_{1,15} - L_{2,15} == 0, \\
&-L_{2,16} == 0, -2 I w_1 L_{1,17} - L_{2,17} == 0, -I w_2 L_{1,18} - L_{2,18} == 0, -I W L_{1,19} - I w_2 L_{1,19} - L_{2,19} == 0, \\
&I W L_{1,20} - I w_2 L_{1,20} - L_{2,20} == 0, -L_{2,21} == 0, -I w_1 L_{1,22} - I w_2 L_{1,22} - L_{2,22} == 0, \\
&-2 I w_2 L_{1,23} - L_{2,23} == 0, w_1^2 L_{1,1} - I W L_{2,1} == 0, w_1^2 L_{1,2} + I W L_{2,2} == 0, \\
&-2 I W m_1 + w_1^2 L_{1,3} - I W L_{2,3} == 0, 2 I W m_1 + w_1^2 L_{1,4} + I W L_{2,4} == 0, 6 d_1 + w_1^2 L_{1,5} == 0, \\
&3 d_1 + w_1^2 L_{1,6} - 2 I W L_{2,6} == 0, 3 d_1 + w_1^2 L_{1,7} + 2 I W L_{2,7} == 0, 6 d_1 + w_1^2 L_{1,8} + I W L_{2,8} + I w_1 L_{2,8} == 0, \\
&3 d_1 + w_1^2 L_{1,9} + 2 I w_1 L_{2,9} == 0, w_1^2 L_{1,10} + I w_2 L_{2,10} == 0, 2 d_2 + w_1^2 L_{1,11} - I W L_{2,11} + I w_2 L_{2,11} == 0, \\
&2 d_2 + w_1^2 L_{1,12} + I W L_{2,12} + I w_2 L_{2,12} == 0, 2 d_2 + w_1^2 L_{1,13} + I w_1 L_{2,13} + I w_2 L_{2,13} == 0, \\
&d_3 + w_1^2 L_{1,14} + 2 I w_2 L_{2,14} == 0, 6 d_1 + w_1^2 L_{1,15} - I W L_{2,15} - I w_1 L_{2,15} == 0, \\
&6 d_1 + w_1^2 L_{1,16} == 0, 3 d_1 + w_1^2 L_{1,17} - 2 I w_1 L_{2,17} == 0, w_1^2 L_{1,18} - I w_2 L_{2,18} == 0, \\
&2 d_2 + w_1^2 L_{1,19} - I W L_{2,19} - I w_2 L_{2,19} == 0, 2 d_2 + w_1^2 L_{1,20} + I W L_{2,20} - I w_2 L_{2,20} == 0, \\
&2 d_3 + w_1^2 L_{1,21} == 0, 2 d_2 + w_1^2 L_{1,22} - I w_1 L_{2,22} - I w_2 L_{2,22} == 0, d_3 + w_1^2 L_{1,23} - 2 I w_2 L_{2,23} == 0 <
\end{aligned}$$

**eq611c = HCoefficient@Subtract žž # •. sol2Form, NRT@@2DDD •. Exp@_ T₀ + _ . D -> 0 & •ž
order2Eqm@@83, 4<DD •• FlattenL == 0 •• Thread**

$$\begin{aligned}
&8 - L_{4,1} == 0, -2 I W L_{3,2} - L_{4,2} == 0, 2 I W L_{3,3} - L_{4,3} == 0, I w_1 L_{3,4} - L_{4,4} == 0, \\
&-I W L_{3,5} + I w_1 L_{3,5} - L_{4,5} == 0, I W L_{3,6} + I w_1 L_{3,6} - L_{4,6} == 0, -I W L_{3,7} + I w_2 L_{3,7} - L_{4,7} == 0, \\
&I W L_{3,8} + I w_2 L_{3,8} - L_{4,8} == 0, I w_1 L_{3,9} + I w_2 L_{3,9} - L_{4,9} == 0, 2 I w_2 L_{3,10} - L_{4,10} == 0, \\
&-I w_1 L_{3,11} - L_{4,11} == 0, -I W L_{3,12} - I w_1 L_{3,12} - L_{4,12} == 0, I W L_{3,13} - I w_1 L_{3,13} - L_{4,13} == 0, -L_{4,14} == 0, \\
&-I w_1 L_{3,15} + I w_2 L_{3,15} - L_{4,15} == 0, -I W L_{3,16} - I w_2 L_{3,16} - L_{4,16} == 0, I W L_{3,17} - I w_2 L_{3,17} - L_{4,17} == 0, \\
&I w_1 L_{3,18} - I w_2 L_{3,18} - L_{4,18} == 0, -L_{4,19} == 0, -I w_1 L_{3,20} - I w_2 L_{3,20} - L_{4,20} == 0, \\
&-2 I w_2 L_{3,21} - L_{4,21} == 0, 2 d_2 + w_2^2 L_{3,1} == 0, d_2 + w_2^2 L_{3,2} - 2 I W L_{4,2} == 0, d_2 + w_2^2 L_{3,3} + 2 I W L_{4,3} == 0, \\
&w_2^2 L_{3,4} + I w_1 L_{4,4} == 0, 2 d_2 + w_2^2 L_{3,5} - I W L_{4,5} + I w_1 L_{4,5} == 0, 2 d_2 + w_2^2 L_{3,6} + I W L_{4,6} + I w_1 L_{4,6} == 0, \\
&2 d_3 + w_2^2 L_{3,7} - I W L_{4,7} + I w_2 L_{4,7} == 0, 2 d_3 + w_2^2 L_{3,8} + I W L_{4,8} + I w_2 L_{4,8} == 0, \\
&2 d_3 + w_2^2 L_{3,9} + I w_1 L_{4,9} + I w_2 L_{4,9} == 0, 3 d_4 + w_2^2 L_{3,10} + 2 I w_2 L_{4,10} == 0, w_2^2 L_{3,11} - I w_1 L_{4,11} == 0, \\
&2 d_2 + w_2^2 L_{3,12} - I W L_{4,12} - I w_1 L_{4,12} == 0, 2 d_2 + w_2^2 L_{3,13} + I W L_{4,13} - I w_1 L_{4,13} == 0, 2 d_2 + w_2^2 L_{3,14} == 0, \\
&2 d_3 + w_2^2 L_{3,15} - I w_1 L_{4,15} + I w_2 L_{4,15} == 0, 2 d_3 + w_2^2 L_{3,16} - I W L_{4,16} - I w_2 L_{4,16} == 0, \\
&2 d_3 + w_2^2 L_{3,17} + I W L_{4,17} - I w_2 L_{4,17} == 0, 2 d_3 + w_2^2 L_{3,18} + I w_1 L_{4,18} - I w_2 L_{4,18} == 0, \\
&6 d_4 + w_2^2 L_{3,19} == 0, 2 d_3 + w_2^2 L_{3,20} - I w_1 L_{4,20} - I w_2 L_{4,20} == 0, 3 d_4 + w_2^2 L_{3,21} - 2 I w_2 L_{4,21} == 0 <
\end{aligned}$$

Solving **eq611b** for the **NRTsymbolList1**, we obtain

coef11 = Solve@eq611b, Array@NRTsymbolList1, 2D •• FlattenD@@1DD

$$\begin{aligned}
 &:L_{2,5} \otimes 0, L_{2,16} \otimes 0, L_{2,21} \otimes 0, L_{1,5} \otimes -\frac{6 d_1}{w_1^2}, L_{1,16} \otimes -\frac{6 d_1}{w_1^2}, L_{1,21} \otimes -\frac{2 d_3}{w_1^2}, L_{2,1} \otimes 0, L_{2,2} \otimes 0, \\
 &L_{2,3} \otimes -\frac{2 W^2 m_1}{W^2 - w_1^2}, L_{2,4} \otimes -\frac{2 W^2 m_1}{W^2 - w_1^2}, L_{2,6} \otimes -\frac{6 I W d_1}{4 W^2 - w_1^2}, L_{2,7} \otimes -\frac{6 I W d_1}{4 W^2 - w_1^2}, L_{2,9} \otimes \frac{2 I d_1}{w_1}, \\
 &L_{2,10} \otimes 0, L_{2,14} \otimes -\frac{2 I d_3 w_2}{w_1^2 - 4 w_2^2}, L_{2,17} \otimes -\frac{2 I d_1}{w_1}, L_{2,18} \otimes 0, L_{2,23} \otimes \frac{2 I d_3 w_2}{w_1^2 - 4 w_2^2}, L_{1,1} \otimes 0, L_{1,2} \otimes 0, \\
 &L_{1,3} \otimes -\frac{2 I W m_1}{W^2 - w_1^2}, L_{1,4} \otimes \frac{2 I W m_1}{W^2 - w_1^2}, L_{1,6} \otimes \frac{3 d_1}{4 W^2 - w_1^2}, L_{1,7} \otimes \frac{3 d_1}{4 W^2 - w_1^2}, L_{1,9} \otimes \frac{d_1}{w_1^2}, L_{1,10} \otimes 0, \\
 &L_{1,14} \otimes -\frac{d_3}{w_1^2 - 4 w_2^2}, L_{1,17} \otimes \frac{d_1}{w_1^2}, L_{1,18} \otimes 0, L_{1,23} \otimes -\frac{d_3}{w_1^2 - 4 w_2^2}, L_{2,8} \otimes \frac{6 I d_1 H W + w_1 L}{W H W + 2 w_1 L}, \\
 &L_{2,11} \otimes -\frac{2 I d_2 H W - w_2 L}{W^2 - w_1^2 - 2 W w_2 + w_2^2}, L_{2,12} \otimes \frac{2 I d_2 H W + w_2 L}{W^2 - w_1^2 + 2 W w_2 + w_2^2}, L_{2,13} \otimes \frac{2 I d_2 H w_1 + w_2 L}{w_2 H^2 w_1 + w_2 L}, \\
 &L_{2,15} \otimes -\frac{6 I d_1 H W + w_1 L}{W H W + 2 w_1 L}, L_{2,19} \otimes -\frac{2 I d_2 H W + w_2 L}{W^2 - w_1^2 + 2 W w_2 + w_2^2}, L_{2,20} \otimes \frac{2 I d_2 H W - w_2 L}{W^2 - w_1^2 - 2 W w_2 + w_2^2}, \\
 &L_{2,22} \otimes -\frac{2 I d_2 H w_1 + w_2 L}{w_2 H^2 w_1 + w_2 L}, L_{1,8} \otimes \frac{6 d_1}{W H W + 2 w_1 L}, L_{1,11} \otimes -\frac{2 d_2}{-W^2 + w_1^2 + 2 W w_2 - w_2^2}, \\
 &L_{1,12} \otimes -\frac{2 d_2}{-W^2 + w_1^2 - 2 W w_2 - w_2^2}, L_{1,13} \otimes \frac{2 d_2}{w_2 H^2 w_1 + w_2 L}, L_{1,15} \otimes \frac{6 d_1}{W H W + 2 w_1 L}, \\
 &L_{1,19} \otimes -\frac{2 d_2}{-W^2 + w_1^2 - 2 W w_2 - w_2^2}, L_{1,20} \otimes -\frac{2 d_2}{-W^2 + w_1^2 + 2 W w_2 - w_2^2}, L_{1,22} \otimes \frac{2 d_2}{w_2 H^2 w_1 + w_2 L} >
 \end{aligned}$$

Solving **eq611c** for the **NRTsymbolList2**, we obtain

coef12 = Solve@eq611c, Array@NRTsymbolList2, 2D •• FlattenD@@1DD

$$\begin{aligned}
 &:L_{4,1} \otimes 0, L_{4,14} \otimes 0, L_{4,19} \otimes 0, L_{3,1} \otimes -\frac{2 d_2}{w_2^2}, L_{3,14} \otimes -\frac{2 d_2}{w_2^2}, L_{3,19} \otimes -\frac{6 d_4}{w_2^2}, L_{4,2} \otimes -\frac{2 I W d_2}{4 W^2 - w_2^2}, \\
 &L_{4,3} \otimes \frac{2 I W d_2}{4 W^2 - w_2^2}, L_{4,4} \otimes 0, L_{4,10} \otimes \frac{2 I d_4}{w_2}, L_{4,11} \otimes 0, L_{4,21} \otimes -\frac{2 I d_4}{w_2}, L_{3,2} \otimes \frac{d_2}{4 W^2 - w_2^2}, \\
 &L_{3,3} \otimes \frac{d_2}{4 W^2 - w_2^2}, L_{3,4} \otimes 0, L_{3,10} \otimes \frac{d_4}{w_2}, L_{3,11} \otimes 0, L_{3,21} \otimes \frac{d_4}{w_2}, L_{4,5} \otimes -\frac{2 I d_2 H W - w_1 L}{W^2 - 2 W w_1 + w_1^2 - w_2^2}, \\
 &L_{4,6} \otimes \frac{2 I d_2 H W + w_1 L}{W^2 + 2 W w_1 + w_1^2 - w_2^2}, L_{4,7} \otimes -\frac{2 I d_3 H W - w_2 L}{W H W - 2 w_2 L}, L_{4,8} \otimes \frac{2 I d_3 H W + w_2 L}{W H W + 2 w_2 L}, L_{4,9} \otimes \frac{2 I d_3 H w_1 + w_2 L}{w_1 H w_1 + 2 w_2 L}, \\
 &L_{4,12} \otimes -\frac{2 I d_2 H W + w_1 L}{W^2 + 2 W w_1 + w_1^2 - w_2^2}, L_{4,13} \otimes \frac{2 I d_2 H W - w_1 L}{W^2 - 2 W w_1 + w_1^2 - w_2^2}, L_{4,15} \otimes -\frac{2 I d_3 H w_1 - w_2 L}{w_1 H w_1 - 2 w_2 L}, \\
 &L_{4,16} \otimes -\frac{2 I d_3 H W + w_2 L}{W H W + 2 w_2 L}, L_{4,17} \otimes \frac{2 I d_3 H W - w_2 L}{W H W - 2 w_2 L}, L_{4,18} \otimes \frac{2 I d_3 H w_1 - w_2 L}{w_1 H w_1 - 2 w_2 L}, L_{4,20} \otimes -\frac{2 I d_3 H w_1 + w_2 L}{w_1 H w_1 + 2 w_2 L}, \\
 &L_{3,5} \otimes \frac{2 d_2}{W^2 - 2 W w_1 + w_1^2 - w_2^2}, L_{3,6} \otimes \frac{2 d_2}{W^2 + 2 W w_1 + w_1^2 - w_2^2}, L_{3,7} \otimes \frac{2 d_3}{W H W - 2 w_2 L}, L_{3,8} \otimes \frac{2 d_3}{W H W + 2 w_2 L}, \\
 &L_{3,9} \otimes \frac{2 d_2}{w_1 H w_1 + 2 w_2 L}, L_{3,12} \otimes \frac{2 d_2}{W^2 + 2 W w_1 + w_1^2 - w_2^2}, L_{3,13} \otimes \frac{2 d_2}{W^2 - 2 W w_1 + w_1^2 - w_2^2}, L_{3,15} \otimes \frac{2 d_3}{w_1 H w_1 - 2 w_2 L}, \\
 &L_{3,16} \otimes \frac{2 d_3}{W H W + 2 w_2 L}, L_{3,17} \otimes \frac{2 d_3}{W H W - 2 w_2 L}, L_{3,18} \otimes \frac{2 d_3}{w_1 H w_1 - 2 w_2 L}, L_{3,20} \otimes \frac{2 d_3}{w_1 H w_1 + 2 w_2 L} >
 \end{aligned}$$

Substituting **sol2Form** into **order2Eqm** and collecting the coefficients of like **RT**, we obtain two sets of algebraic equations as

```
eq1 = MapIndexed@Coefficient@order2Eqm@@1, 1DD . sol2Form, #1D == r1, #2@@1DD &, RT@@1DDD .
Exp@_ T0 + _ . D -> 0
```

```
82 I w1 G1,1 == r1,1, I W G1,2 == r1,2, I w2 G1,3 == r1,3<
```

```
eq2 = MapIndexed@Coefficient@order2Eqm@@3, 1DD . sol2Form, #1D == r3, #2@@1DD &, RT@@2DDD .
Exp@_ T0 + _ . D -> 0
```

```
8 I W G3,1 + I w2 G3,1 == r3,1, I W G3,2 + I w2 G3,2 == r3,2, 2 I w1 G3,3 + I w2 G3,3 == r3,3, 2 I w2 G3,4 == r3,4<
```

Solving **eq1** and **eq2** for the **RTsymbolList1** and **RTsymbolList2**, we obtain

```
coef21 = Solve@eq1, RTsymbolList1D@@1DD . r1Rule
```

```
: G1,1 @ - \frac{I m_1}{2 w_1}, G1,2 @ - \frac{3 d_1}{W w_1}, G1,3 @ - \frac{d_2}{w_1 w_2} >
```

```
coef22 = Solve@eq2, RTsymbolList2D@@1DD . r3Rule
```

```
: G3,4 @ - \frac{I m_2}{2 w_2}, G3,1 @ \frac{f_2}{4 w_2 H W + w_2 L}, G3,2 @ 0, G3,3 @ - \frac{d_2}{2 w_2 H^2 w_1 + w_2 L}
```

Combining these coefficients using the function **Join** yields

```
coeffs = Join@coef11, coef12, coef21, coef22,
coef21 . conjugateRule, coef22 . conjugateRuleD . HomgRule@@1DD . e -> 0L;
```

Then, we express the solution of the second-order equations in pure function form as

```
sol2 = sol2Form . Function@8T0, T1, T2<, a_D :=
Function@8T0, T1, T2< . Evaluate, a . coeffs . Expand . EvaluateD;
sol2 . displayRule
```

```
9 u_{1,2} @ FunctionA8T0, T1, T2<,
- \frac{6 d_1 L^2}{w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1} d_1 L^2}{5 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1} d_1 L^2}{5 w_1^2} - \frac{4 I E^{-I T_0 W - I t_1} L_1 m_1}{3 w_1} + \frac{4 I E^{I T_0 W + I t_1} L_1 m_1}{3 w_1} -
\frac{3 E^{-I T_0 W - I t_1 + I T_0 w_1} d_1 L_1 A_1}{2 w_1^2} + \frac{3 E^{I T_0 W + I t_1 + I T_0 w_1} d_1 L_1 A_1}{4 w_1^2} - \frac{I E^{I T_0 w_1} m_1 A_1}{2 w_1} + \frac{E^{2 I T_0 w_1} d_1 A_1^2}{w_1^2} -
\frac{2 E^{-I T_0 W - I t_1 + I T_0 w_2} d_2 L_1 A_2}{w_1^2} + \frac{2 E^{I T_0 W + I t_1 + I T_0 w_2} d_2 L_1 A_2}{15 w_1^2} + \frac{E^{I T_0 w_1 + I T_0 w_2} d_2 A_1 A_2}{4 w_1^2} + \frac{E^{2 I T_0 w_2} d_3 A_2^2}{15 w_1^2} +
\frac{3 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{4 w_1^2} - \frac{3 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{2 w_1^2} + \frac{I E^{-I T_0 w_1} m_1 \dot{A}_1}{2 w_1} - \frac{6 d_1 A_1 \dot{A}_1}{w_1^2} -
\frac{E^{-I T_0 w_1 + I T_0 w_2} d_2 A_2 \dot{A}_1}{2 w_1^2} + \frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} + \frac{2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{15 w_1^2} - \frac{2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{w_1^2} -
```

$$\begin{aligned}
& \frac{E^{-IT_0 W_1 - IT_0 W_2} d_2 \dot{A}_1 \dot{A}_2}{2 w_1^2} - \frac{2 d_3 \dot{A}_2 \dot{A}_2}{w_1^2} + \frac{E^{-IT_0 W_1 - IT_0 W_2} d_2 \dot{A}_1 \dot{A}_2}{4 w_1^2} + \frac{E^{-2IT_0 W_2} d_3 \dot{A}_2^2}{15 w_1^2} E, \\
v_{1,2} \text{ ® FunctionA8T}_0, T_1, T_2 <, & -\frac{8}{3} E^{-IT_0 W - IT_1} L_1 m_1 - \frac{8}{3} E^{IT_0 W + IT_1} L_1 m_1 - \frac{4 I E^{-2IT_0 W - 2IT_1} d_1 L_1^2}{5 w_1} + \\
& \frac{4 I E^{2IT_0 W + 2IT_1} d_1 L_1^2}{5 w_1} - \frac{1}{2} E^{IT_0 W_1} m_1 A_1 - \frac{3 I E^{-IT_0 W - IT_1 + IT_0 W_1} d_1 L_1 A_1}{2 w_1} + \frac{9 I E^{IT_0 W + IT_1 + IT_0 W_1} d_1 L_1 A_1}{4 w_1} + \\
& \frac{2 I E^{2IT_0 W_1} d_1 A_1^2}{w_1} + \frac{8 I E^{IT_0 W + IT_1 + IT_0 W_2} d_2 L_1 A_2}{15 w_1} + \frac{3 I E^{IT_0 W_1 + IT_0 W_2} d_2 A_1 A_2}{4 w_1} + \frac{4 I E^{2IT_0 W_2} d_3 A_2^2}{15 w_1} - \\
& \frac{1}{2} E^{-IT_0 W_1} m_1 \dot{A}_1 - \frac{9 I E^{-IT_0 W - IT_1 - IT_0 W_1} d_1 L_1 \dot{A}_1}{4 w_1} + \frac{3 I E^{IT_0 W + IT_1 - IT_0 W_1} d_1 L_1 \dot{A}_1}{2 w_1} + \\
& \frac{I E^{-IT_0 W_1 + IT_0 W_2} d_2 A_2 \dot{A}_1}{2 w_1} - \frac{2 I E^{-2IT_0 W_1} d_1 \dot{A}_1^2}{w_1} - \frac{8 I E^{-IT_0 W - IT_1 - IT_0 W_2} d_2 L_1 \dot{A}_2}{15 w_1} - \\
& \frac{I E^{IT_0 W_1 - IT_0 W_2} d_2 A_1 \dot{A}_2}{2 w_1} - \frac{3 I E^{-IT_0 W_1 - IT_0 W_2} d_2 \dot{A}_1 \dot{A}_2}{4 w_1} - \frac{4 I E^{-2IT_0 W_2} d_3 \dot{A}_2^2}{15 w_1} E, \\
u_{2,2} \text{ ® FunctionA8T}_0, T_1, T_2 <, & \frac{E^{-IT_0 W - IT_2} f_2}{32 w_1^2} + \frac{E^{IT_0 W + IT_2} f_2}{32 w_1^2} - \frac{d_2 L_1^2}{2 w_1^2} + \frac{E^{-2IT_0 W - 2IT_1} d_2 L_1^2}{12 w_1^2} + \\
& \frac{E^{2IT_0 W + 2IT_1} d_2 L_1^2}{12 w_1^2} - \frac{2 E^{-IT_0 W - IT_1 + IT_0 W_1} d_2 L_1 A_1}{3 w_1^2} + \frac{2 E^{IT_0 W + IT_1 + IT_0 W_1} d_2 L_1 A_1}{5 w_1^2} - \frac{E^{2IT_0 W_1} d_2 A_2^2}{16 w_1^2} - \\
& \frac{E^{-IT_0 W - IT_1 + IT_0 W_2} d_3 L_1 A_2}{2 w_1^2} + \frac{E^{IT_0 W + IT_1 + IT_0 W_2} d_3 L_1 A_2}{6 w_1^2} - \frac{I E^{IT_0 W_2} m_2 A_2}{4 w_1} + \frac{2 E^{IT_0 W_1 + IT_0 W_2} d_3 A_1 A_2}{5 w_1^2} + \\
& \frac{E^{2IT_0 W_2} d_4 A_2^2}{4 w_1^2} + \frac{2 E^{-IT_0 W - IT_1 - IT_0 W_1} d_2 L_1 \dot{A}_1}{5 w_1^2} - \frac{2 E^{IT_0 W + IT_1 - IT_0 W_1} d_2 L_1 \dot{A}_1}{3 w_1^2} - \frac{d_2 A_1 \dot{A}_1}{2 w_1^2} - \\
& \frac{2 E^{-IT_0 W_1 + IT_0 W_2} d_3 A_2 \dot{A}_1}{3 w_1^2} - \frac{E^{-2IT_0 W_1} d_2 \dot{A}_1^2}{16 w_1^2} + \frac{E^{-IT_0 W - IT_1 - IT_0 W_2} d_3 L_1 \dot{A}_2}{6 w_1^2} - \frac{E^{IT_0 W + IT_1 - IT_0 W_2} d_3 L_1 \dot{A}_2}{2 w_1^2} + \\
& \frac{I E^{-IT_0 W_2} m_2 \dot{A}_2}{4 w_1} - \frac{2 E^{IT_0 W_1 - IT_0 W_2} d_3 A_1 \dot{A}_2}{3 w_1^2} - \frac{3 d_4 A_2 \dot{A}_2}{2 w_1^2} + \frac{2 E^{-IT_0 W_1 - IT_0 W_2} d_3 \dot{A}_1 \dot{A}_2}{5 w_1^2} + \frac{E^{-2IT_0 W_2} d_4 \dot{A}_2^2}{4 w_1^2} E, \\
v_{2,2} \text{ ® FunctionA8T}_0, T_1, T_2 <, & -\frac{I E^{-2IT_0 W - 2IT_1} d_2 L_1^2}{3 w_1} + \frac{I E^{2IT_0 W + 2IT_1} d_2 L_1^2}{3 w_1} + \frac{I E^{-IT_0 W - IT_2} f_2 w_2}{32 w_1^2} - \\
& \frac{I E^{IT_0 W + IT_2} f_2 w_2}{32 w_1^2} + \frac{2 I E^{-IT_0 W - IT_1 + IT_0 W_1} d_2 L_1 A_1}{3 w_1} + \frac{6 I E^{IT_0 W + IT_1 + IT_0 W_1} d_2 L_1 A_1}{5 w_1} + \\
& \frac{I E^{2IT_0 W_1} d_2 w_2 A_1^2}{16 w_1^2} + \frac{2 I E^{IT_0 W + IT_1 + IT_0 W_2} d_3 L_1 A_2}{3 w_1} - \frac{E^{IT_0 W_2} m_2 w_2 A_2}{4 w_1} + \frac{6 I E^{IT_0 W_1 + IT_0 W_2} d_3 A_1 A_2}{5 w_1} + \\
& \frac{I E^{2IT_0 W_2} d_4 A_2^2}{w_1} - \frac{6 I E^{-IT_0 W - IT_1 - IT_0 W_1} d_2 L_1 \dot{A}_1}{5 w_1} - \frac{2 I E^{IT_0 W + IT_1 - IT_0 W_1} d_2 L_1 \dot{A}_1}{3 w_1} - \\
& \frac{2 I E^{-IT_0 W_1 + IT_0 W_2} d_3 A_2 \dot{A}_1}{3 w_1} - \frac{I E^{-2IT_0 W_1} d_2 w_2 \dot{A}_1^2}{16 w_1^2} - \frac{2 I E^{-IT_0 W - IT_1 - IT_0 W_2} d_3 L_1 \dot{A}_2}{3 w_1} - \\
& \frac{E^{-IT_0 W_2} m_2 w_2 \dot{A}_2}{4 w_1} + \frac{2 I E^{IT_0 W_1 - IT_0 W_2} d_3 A_1 \dot{A}_2}{3 w_1} - \frac{6 I E^{-IT_0 W_1 - IT_0 W_2} d_3 \dot{A}_1 \dot{A}_2}{5 w_1} - \frac{I E^{-2IT_0 W_2} d_4 \dot{A}_2^2}{w_1} E=
\end{aligned}$$

Y Third-Order Problem

Substituting the first- and second-order solutions into the third-order problem, **eqOrder[3]**, yields

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Substituting the **expRule1** into the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, the terms proportional to $E^{I w_1 T_0}$ and $E^{I w_2 T_0}$, we have

```
ST21 = Coefficient@order3Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
ST21 •. displayRule
```

$$\begin{aligned}
 &: -HD_2A_1L + \frac{E^{I T_1 s_1} HD_1 \dot{A}_1 L A_2 d_2}{2 w_1^2} + \frac{3 E^{I T_1 s_1 + I T_1 s_2 + I t_1} HD_1 \dot{A}_1 L d_1 L_1}{2 w_1^2} + \frac{I HD_1 A_1 L m_1}{2 w_1} + \frac{E^{I T_1 s_1} HD_1 A_2 L d_2 \dot{A}_1}{2 w_1^2}, \\
 &- 6 E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1 - 24 A_1 a_1 L_1^2 + \frac{1}{2} HD_1 A_1 L m_1 + A_1 m_1^2 + \frac{27 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{2 w_1^2} + \\
 &\frac{23 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{15 w_1^2} + \frac{81 A_1 d_1^2 L_1^2}{2 w_1^2} + \frac{23 A_1 d_2^2 L_1^2}{15 w_1^2} - \frac{I E^{I T_1 s_1} HD_1 \dot{A}_1 L A_2 d_2}{2 w_1} - \\
 &\frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} HD_1 \dot{A}_1 L d_1 L_1}{2 w_1} - I HD_2 A_1 L w_1 - 12 A_1^2 a_1 \dot{A}_1 + \frac{30 A_1^2 d_1^2 \dot{A}_1}{w_1^2} - \frac{E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \dot{A}_1}{16 w_1^2} + \\
 &\frac{9 A_1^2 d_2^2 \dot{A}_1}{8 w_1^2} - \frac{I E^{I T_1 s_1} HD_1 A_2 L d_2 \dot{A}_1}{2 w_1} - \frac{2 I E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1} - \frac{14 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1} + \\
 &\frac{I E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2 + \frac{A_1 A_2 d_2^2 \dot{A}_2}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^2} + \\
 &\frac{8 A_1 A_2 d_2^2 \dot{A}_2}{15 w_1^2} + \frac{3 A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} >
 \end{aligned}$$

ST22 = Coefficient@order3Eq@@#, 2DD •. expRule1@2D, Exp@I w₂ T₀DD & •ž 83, 4<;
 ST22 •. displayRule

$$\begin{aligned}
 &:-\text{HD}_2\text{A}_2\text{L} + \frac{E^{-I T_1 s_1} \text{HD}_1\text{A}_1\text{L}\text{A}_1\text{d}_2}{8 w_1^2} + \frac{I \text{HD}_1\text{A}_2\text{L}\text{m}_2}{4 w_1}, \\
 &-3 E^{-I T_1 s_2 - I t_1} \text{A}_2^2 \text{a}_4 \text{L}_1 - 4 \text{A}_2 \text{a}_3 \text{L}_1^2 - 3 E^{I T_1 s_2 + I t_1} \text{a}_2 \text{L}_1^3 + \frac{58 E^{-I T_1 s_2 - I t_1} \text{A}_2^2 \text{d}_2 \text{d}_3 \text{L}_1}{15 w_1^2} + \\
 &\frac{5 E^{-I T_1 s_2 - I t_1} \text{A}_2^2 \text{d}_3 \text{d}_4 \text{L}_1}{2 w_1^2} + \frac{56 \text{A}_2 \text{d}_5^2 \text{L}_1^2}{15 w_1^2} + \frac{12 \text{A}_2 \text{d}_1 \text{d}_3 \text{L}_1^2}{w_1^2} + \frac{2 \text{A}_2 \text{d}_5^2 \text{L}_1^2}{3 w_1^2} + \frac{3 \text{A}_2 \text{d}_2 \text{d}_4 \text{L}_1^2}{w_1^2} + \\
 &\frac{58 E^{I T_1 s_2 + I t_1} \text{d}_1 \text{d}_2 \text{L}_1^3}{5 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} \text{d}_2 \text{d}_3 \text{L}_1^3}{6 w_1^2} + \frac{I E^{-I T_1 s_1} \text{A}_2^2 \text{d}_2 \text{m}_1}{w_1} - I \text{HD}_2\text{A}_2\text{L} \text{w}_2 - \\
 &\frac{I E^{-I T_1 s_1} \text{HD}_1\text{A}_1\text{L}\text{A}_1\text{d}_2 \text{w}_2}{8 w_1^2} + \frac{I E^{I T_1 s_2 + I t_1} \text{f}_2 \text{m}_2 \text{w}_2}{16 w_1^2} - \frac{I E^{-I T_1 s_1} \text{A}_2^2 \text{d}_2 \text{m}_2 \text{w}_2}{8 w_1^2} + \frac{\text{HD}_1\text{A}_2\text{L}\text{m}_2 \text{w}_2}{4 w_1} + \\
 &\frac{\text{A}_2 \text{m}_5^2 \text{w}_2}{2 w_1} - 4 \text{A}_1 \text{A}_2 \text{a}_3 \dot{\text{A}}_1 - 6 E^{I T_1 s_2 + I t_1} \text{A}_1 \text{a}_2 \text{L}_1 \dot{\text{A}}_1 + \frac{\text{A}_1 \text{A}_2 \text{d}_5^2 \dot{\text{A}}_1}{2 w_1^2} + \frac{12 \text{A}_1 \text{A}_2 \text{d}_1 \text{d}_3 \dot{\text{A}}_1}{w_1^2} + \\
 &\frac{8 \text{A}_1 \text{A}_2 \text{d}_2^2 \dot{\text{A}}_1}{15 w_1^2} + \frac{3 \text{A}_1 \text{A}_2 \text{d}_2 \text{d}_4 \dot{\text{A}}_1}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} \text{A}_1 \text{d}_1 \text{d}_2 \text{L}_1 \dot{\text{A}}_1}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} \text{A}_1 \text{d}_2 \text{d}_3 \text{L}_1 \dot{\text{A}}_1}{15 w_1^2} - \\
 &12 \text{A}_2^2 \text{a}_5 \dot{\text{A}}_2 - 6 E^{I T_1 s_2 + I t_1} \text{A}_2 \text{a}_4 \text{L}_1 \dot{\text{A}}_2 - 2 E^{2 I T_1 s_2 + 2 I t_1} \text{a}_3 \text{L}_1^2 \dot{\text{A}}_2 + \frac{58 \text{A}_2^2 \text{d}_5^2 \dot{\text{A}}_2}{15 w_1^2} + \frac{15 \text{A}_2^2 \text{d}_4^2 \dot{\text{A}}_2}{2 w_1^2} + \\
 &\frac{116 E^{I T_1 s_2 + I t_1} \text{A}_2 \text{d}_2 \text{d}_3 \text{L}_1 \dot{\text{A}}_2}{15 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} \text{A}_2 \text{d}_3 \text{d}_4 \text{L}_1 \dot{\text{A}}_2}{w_1^2} + \frac{4 E^{2 I T_1 s_2 + 2 I t_1} \text{d}_5^2 \text{L}_1^2 \dot{\text{A}}_2}{w_1^2} - \\
 &\frac{2 E^{2 I T_1 s_2 + 2 I t_1} \text{d}_1 \text{d}_3 \text{L}_1^2 \dot{\text{A}}_2}{5 w_1^2} + \frac{E^{2 I T_1 s_2 + 2 I t_1} \text{d}_3^2 \text{L}_1^2 \dot{\text{A}}_2}{w_1^2} - \frac{E^{2 I T_1 s_2 + 2 I t_1} \text{d}_2 \text{d}_4 \text{L}_1^2 \dot{\text{A}}_2}{2 w_1^2} >
 \end{aligned}$$

Then, the solvability conditions demand that **ST21** and **ST22** be orthogonal to solutions of their corresponding adjoints, the components of **cleftVec**. The result is

SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0 < •• ExpandAll;

SCond2 •. displayRule

$$\begin{aligned}
 & : - 6 E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1 - 24 A_1 a_1 L_1^2 + A_1 m_1^2 + \frac{27 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{2 w_1^2} + \\
 & \frac{23 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{15 w_1^2} + \frac{81 A_1 d_1^2 L_1^2}{2 w_1^2} + \frac{23 A_1 d_2^2 L_1^2}{15 w_1^2} - 2 I H D_2 A_1 L w_1 - 12 A_1^2 a_1 \dot{A}_1 + \frac{30 A_1^2 d_1^2 \dot{A}_1}{w_1^2} - \\
 & \frac{E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \dot{A}_1}{16 w_1^2} + \frac{9 A_2^2 d_2^2 \dot{A}_1}{8 w_1^2} - \frac{2 I E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1} - \frac{14 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1} + \\
 & \frac{I E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2 + \frac{A_1 A_2 d_2^2 \dot{A}_2}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^2} + \\
 & \frac{8 A_1 A_2 d_2^2 \dot{A}_2}{15 w_1^2} + \frac{3 A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} == 0, \\
 & - 3 E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1 - 4 A_2 a_3 L_1^2 - 3 E^{I T_1 s_2 + I t_1} a_2 L_1^3 + \frac{58 E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2} + \\
 & \frac{5 E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{2 w_1^2} + \frac{56 A_2 d_2^2 L_1^2}{15 w_1^2} + \frac{12 A_2 d_1 d_3 L_1^2}{w_1^2} + \frac{2 A_2 d_3^2 L_1^2}{3 w_1^2} + \frac{3 A_2 d_2 d_4 L_1^2}{w_1^2} + \\
 & \frac{58 E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{6 w_1^2} + \frac{I E^{-I T_1 s_1} A_2^2 d_2 m_1}{w_1} - 2 I H D_2 A_2 L w_2 + \\
 & \frac{I E^{I T_1 s_2 + I t_2} f_2 m_2 w_2}{16 w_1^2} - \frac{I E^{-I T_1 s_1} A_2^2 d_2 m_2 w_2}{8 w_1^2} + \frac{A_2 m_2^2 w_2}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_1 - \\
 & 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1 + \frac{A_1 A_2 d_2^2 \dot{A}_1}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2} + \frac{8 A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2} + \\
 & \frac{3 A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{15 w_1^2} - \\
 & 12 A_2^2 a_5 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2 - 2 E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2 + \frac{58 A_2^2 d_2^2 \dot{A}_2}{15 w_1^2} + \frac{15 A_2^2 d_1^2 \dot{A}_2}{2 w_1^2} + \\
 & \frac{116 E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{w_1^2} + \frac{4 E^{2 I T_1 s_2 + 2 I t_1} d_3^2 L_1^2 \dot{A}_2}{w_1^2} - \\
 & \frac{2 E^{2 I T_1 s_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2} + \frac{E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2} - \frac{E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{2 w_1^2} == 0 >
 \end{aligned}$$

Solving **SCond2** for $D_2 A_1$ and $D_2 A_2$, we have

SCond2Rule1 = SolveASCond2, $9A_1^{H0,1L} @T_1, T_2D, A_2^{H0,1L} @T_1, T_2D=E@@1DD \bullet \bullet$ ExpandAll;

SCond2Rule1 . displayRule

$$\begin{aligned}
 :D_2A_1 \otimes & -\frac{27 I E^{-I T_1 S_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{4 w_1^3} - \frac{23 I E^{-I T_1 S_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{30 w_1^3} - \frac{81 I A_1 d_1^2 L_1^2}{4 w_1^3} - \frac{23 I A_1 d_2^2 L_1^2}{30 w_1^3} + \\
 & \frac{3 I E^{-I T_1 S_2 - I t_1} A_1 A_2 a_2 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} - \frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^3} + \frac{I E^{I T_1 S_1 + I T_1 S_2 + I t_2} f_2 d_2 \dot{A}_1}{32 w_1^3} - \\
 & \frac{9 I A_1^2 d_2^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 S_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2} - \frac{7 E^{I T_1 S_1 + I T_1 S_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^2} + \frac{E^{I T_1 S_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2} + \frac{6 I A_1^2 a_1 \dot{A}_1}{w_1} - \\
 & \frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{27 I E^{I T_1 S_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{4 w_1^3} - \\
 & \frac{23 I E^{I T_1 S_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{30 w_1^3} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 S_2 + I t_1} A_1 a_2 L_1 \dot{A}_2}{w_1}, \\
 D_2A_2 \otimes & \frac{E^{I T_1 S_2 + I t_2} f_2 m_2}{32 w_1^2} - \frac{E^{-I T_1 S_1} A_2^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{3 I E^{-I T_1 S_2 - I t_1} A_2^2 a_4 L_1}{2 w_2} + \frac{2 I A_2 a_2 L_1^2}{w_2} + \\
 & \frac{3 I E^{I T_1 S_2 + I t_1} a_2 L_1^3}{2 w_2} - \frac{29 I E^{-I T_1 S_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2 w_2} - \frac{5 I E^{-I T_1 S_2 - I t_1} A_2^2 d_3 d_4 L_1}{4 w_1^2 w_2} - \frac{28 I A_2 d_2^2 L_1^2}{15 w_1^2 w_2} - \\
 & \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \frac{I A_2 d_2^2 L_1^2}{3 w_1^2 w_2} - \frac{3 I A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2} - \frac{29 I E^{I T_1 S_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2 w_2} - \frac{5 I E^{I T_1 S_2 + I t_1} d_2 d_3 L_1^3}{12 w_1^2 w_2} + \\
 & \frac{E^{-I T_1 S_1} A_2^2 d_2 m_1}{2 w_1 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \frac{3 I E^{I T_1 S_2 + I t_1} A_1 a_2 L_1 \dot{A}_1}{w_2} - \frac{I A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \\
 & \frac{4 I A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2} - \frac{27 I E^{I T_1 S_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{4 w_1^2 w_2} - \frac{23 I E^{I T_1 S_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{30 w_1^2 w_2} + \\
 & \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} + \frac{3 I E^{I T_1 S_2 + I t_1} A_2 a_4 L_1 \dot{A}_2}{w_2} + \frac{I E^{2 I T_1 S_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2}{w_2} - \frac{29 I A_2^2 d_3^2 \dot{A}_2}{15 w_1^2 w_2} - \frac{15 I A_2^2 d_2^2 \dot{A}_2}{4 w_1^2 w_2} - \\
 & \frac{58 I E^{I T_1 S_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2 w_2} - \frac{5 I E^{I T_1 S_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{2 w_1^2 w_2} - \frac{2 I E^{2 I T_1 S_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2 w_2} + \\
 & \frac{I E^{2 I T_1 S_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2 w_2} - \frac{I E^{2 I T_1 S_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{2 w_1^2 w_2} + \frac{I E^{2 I T_1 S_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{4 w_1^2 w_2} >
 \end{aligned}$$

Reconstitution

Using the method of reconstitution, $A_k^\dagger = e D_1 A_k + e^2 D_2 A_k + \dots$, we combine the partial-differential equations **SCond1-Rule1** and **SCond2Rule1** into the following two ordinary-differential equations governing the modulation of the complex-valued functions A_k :

moduEq = Table@2 I w_k A_k^ε, 8k, 2<D ==

HTable@2 I w_k dt@1D@ A_k@T₁, T₂DD, 8k, 2<D •. SCond1Rule1 •. SCond2Rule1 •.

Collect@#, eD &L •. Thread;

moduEq •. displayRule

$$\begin{aligned}
 2 I w_1 A_1^\epsilon &= 2 I e w_1 \left\{ -A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1}{w_1} \frac{y}{z} \right\} \\
 2 I e^2 w_1 &\left\{ -\frac{27 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{4 w_1^3} - \frac{23 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{30 w_1^3} - \right. \\
 &\frac{81 I A_1 d_1^2 L_1^2}{4 w_1^3} - \frac{23 I A_1 d_2^2 L_1^2}{30 w_1^3} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} - \\
 &\frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^3} + \frac{I E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \dot{A}_1}{32 w_1^3} - \frac{9 I A_1^2 d_2^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2} - \\
 &\frac{7 E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^2} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2} + \frac{6 I A_1^2 a_1 \dot{A}_1}{w_1} - \frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \\
 &\frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{4 w_1^3} - \\
 &\frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{30 w_1^3} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2}{w_1} \frac{y}{z} \Big\} \\
 2 I w_2 A_2^\epsilon &= 2 I e \left\{ -A_2 m_2 - \frac{I E^{I T_1 s_2 + I t_2} f_2}{4 w_2} + \frac{I E^{-I T_1 s_1} A_2^2 d_2}{2 w_2} \frac{y}{z} \right\} w_2 + \\
 2 I e^2 w_2 &\left\{ \frac{E^{I T_1 s_2 + I t_2} f_2 m_2}{32 w_1^2} - \frac{E^{-I T_1 s_1} A_2^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1}{2 w_2} + \frac{2 I A_2 a_3 L_1^2}{w_2} + \right. \\
 &\frac{3 I E^{I T_1 s_2 + I t_1} a_2 L_1^3}{2 w_2} - \frac{29 I E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2 w_2} - \frac{5 I E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{4 w_1^2 w_2} - \frac{28 I A_2 d_2^2 L_1^2}{15 w_1^2 w_2} - \\
 &\frac{6 I A_2 d_1 d_2 L_1^2}{w_1^2 w_2} - \frac{I A_2 d_3^2 L_1^2}{3 w_1^2 w_2} - \frac{3 I A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2} - \frac{29 I E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2 w_2} - \frac{5 I E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{12 w_1^2 w_2} + \\
 &\frac{E^{-I T_1 s_1} A_2^2 d_2 m_1}{2 w_1 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1}{w_2} - \frac{I A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \\
 &\frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2} - \frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{4 w_1^2 w_2} - \frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{30 w_1^2 w_2} + \\
 &\frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} + \frac{3 I E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2}{w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2}{w_2} - \frac{29 I A_2^2 d_2^2 \dot{A}_2}{15 w_1^2 w_2} - \frac{15 I A_2^2 d_1^2 \dot{A}_2}{4 w_1^2 w_2} - \\
 &\frac{58 I E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2 w_2} - \frac{5 I E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{2 w_1^2 w_2} - \frac{2 I E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2 w_2} + \\
 &\frac{I E^{2 I T_1 s_2 + 2 I t_1} d_1 d_2 L_1^2 \dot{A}_2}{5 w_1^2 w_2} - \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{2 w_1^2 w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{4 w_1^2 w_2} \frac{y}{z} \Big\}
 \end{aligned}$$

¶ 6.1.2 The function **MMS**

According to the procedures described in the previous section, we can build a function named **MMS** (Method of Multiple Scales) specifically for **eq61b**. A more general function (a Package) can be created by considering as arguments the governing equations, symbols for dependent variables, independent variable, excitation amplitudes and frequencies, and all other related quantities which allow the program to identify their respective meanings. We will use **MMS** to solve for different resonance cases in the following sections.

```

MMS@scaling_List, ResonanceCond : 8__Equal<D :=
ModuleA8<,
  omgList = 8w1, w2<;
  omgRule = Solve@ResonanceCond, 8W, #< • FlattenD@@1DD & •ž omgList • Reverse;
  multiScales = 8ui@tD -> ui žž timeScales, vi@tD -> vi žž timeScales,
  Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
  eqa = Heq61b • scaling • multiScales • solRule • TrigToExp • ExpandAllL •
  en_*; n>3 -> 0;
  eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eqa • Transpose;
  eqOrder@i_D :=
  l#@@1DD & •ž eqEps@@1DD • fs -> 0 • u-k,1 -> uk,iM == l#@@1DD & •ž eqEps@@1DD • fs -> 0 •
  u-k,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL • Thread;

  H* First-Order Problem *L
  linearSys = #@@1DD & •ž eqOrder@1D;
  expr1 = linearSys • 9ui,1 -> IP EI wi #1 &M, vi,1 -> IQ EI wi #1 &M= • Exp@a_D -> 1 •
  Partition@#, 2D &;
  coefMat = Outer@D, #, 8P, Q<D & •ž expr1;
  hermitian@mat_?MatrixQD := mat • conjugateRule • Transpose;
  rightVec = # • #@@1DD & •ž HNullSpace@#D@@1DD & •ž coefMatL;
  leftVec = NullSpace@hermitian@#DD@@1DD & •ž coefMat;
  ccleftVec = leftVec • conjugateRule;
  order1Eq = eqOrder@1D • u-i,1 -> Hui,1@#1D &L;
  sol1p =
  DSolve@order1Eq, 8u1,1@T0D, v1,1@T0D, u2,1@T0D, v2,1@T0D<, T0D@@1DD • C@_D -> 0 • Simplify;
  fRule = 9fi -> 2 Li I wi2 - W2M=;
  sol1u =
  TableAui,1 -> FunctionA8T0, T1, T2<, Ai@T1, T2D Exp@I wi T0D + Āi@T1, T2D Exp@- I wi T0D +
  Hui,1@T0D • sol1p • fRule • Simplify • ExpandL • EvaluateE, 8i, 2<E;
  sol1v = Table@vi,1 -> Function@8T0, T1, T2<,
  D@ui,1 žž timeScales, T0D • sol1u • EvaluateD, 8i, 2<D;
  sol1 = Join@sol1u, sol1vD;

```

```

H* Second-Order Problem *L
order2Eq = eqOrder@2D •. sol1 •. ExpandAll;
expRule1@i_D :=
Exp@a_D :> Exp@Expand@a •. omgRule@@iDDD •. en• T0 :> timeScales@@n+1DDD;
ST11 = Coefficient@order2Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
ST12 = Coefficient@order2Eq@@#, 2DD •. expRule1@2D, Exp@I w2 T0DD & •ž 83, 4<;
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
SCond1Rule1 = SolveASCond1, 9A1H1,0L@T1, T2D, A2H1,0L@T1, T2D=E@@1DD •. ExpandAll;
sigRule = Solve@ResonanceCond, 8s1, s2<D@@1DD;
expRule2 = Exp@a_D :> ExpAa •. 9T1 -> e T0, T2 -> e2 T0= •. sigRule •. ExpandE;
SCond1Rule2 = SCond1Rule1 •. expRule2;
ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
order2Eqm = order2Eq •. SCond1Rule2 •. ccSCond1Rule2 •. ExpandAll;
IfAUnion@#@@2DD & •ž order2EqmD === 80<,
sol2 = 8u1,2 -> H0 &L, v1,2 -> H0 &L, u2,2 -> H0 &L, v2,2 -> H0 &L<,
basicH = TableA9Ai@T1, T2D EI wi T0, Ai@T1, T2D E-I wi T0=, 8i, 2<E •. Flatten;
collectForm =
Join@basicH, If@list1 = List žž Plus žž H#@@2DD & •ž eqOrder@1DL •. c_ fi Ea -> Li Ea;
Head@list1D === List, list1, 8<DD;
possibleTerms = JoinAcollectForm,
IfAlist1 = List žž Plus žž l#@@2DD & •ž eqOrder@2D •. ui,j -> H0 &LM •. c_ fi Ea -> Ea;
Head@list1D === List, list1, 8<E,
Outer@Times, collectForm, collectFormD •. Flatten •. UnionE;
ResonantTerms@i_D :=
l# •. 8a_ •; a != 0 -> 1< & •ž l E-I wi T0 possibleTerms •. expRule1@iD •. Exp@_ T0 + _D -> 0MM
possibleTerms •. Union •. Rest;
RT = Array@ResonantTerms, 2D;
r1Rule = MapIndexed@
r1,#2@@1DD -> Coefficient@order2Eqm@@1, 2DD, #1D &, RT@@1DDD •. Exp@_ T0 + _D -> 0;
H* r2Rule=MapIndexed@r2,#2@@1DD->Coefficient@order2Eqm@@2,2DD,#1D &,
RT@@1DDD•.Exp@_ T0+_D->0; *L
r3Rule = MapIndexed@r3,#2@@1DD -> Coefficient@order2Eqm@@3, 2DD, #1D &, RT@@2DDD •.
Exp@_ T0 + _D -> 0;
H* r4Rule=MapIndexed@r4,#2@@1DD->Coefficient@order2Eqm@@4,2DD,#1D &,
RT@@2DDD•.Exp@_ T0+_D->0; *L
NRT = Complement@possibleTerms, Join@#, # •. conjugateRuleDD & •ž RT;
RTsymbolList1 = Table@G1,j, 8j, Length@RT@@1DDD<D;
RTsymbolList2 = Table@G3,j, 8j, Length@RT@@2DDD<D;
NRTsymbolList1@i_D = Table@Li,j, 8j, Length@NRT@@1DDD<D;
NRTsymbolList2@i_D = Table@Li+2,j, 8j, Length@NRT@@2DDD<D;
8ratio1, ratio2<= Table@-ccleftVec@@i, 1DD, 8i, 2<D;
sol2Form = 8u1,2 -> Function@8T0, T1, T2<,

```

```

RTsymbolList1.RT@@1DD + HRTsymbolList1.RT@@1DD . conjugateRuleL +
  NRTsymbolList1@1D.NRT@@1DD ** Evaluated, v1,2 -> Function@8T0, T1, T2<,
ratio1 * RTsymbolList1.RT@@1DD + Hratio1 * RTsymbolList1.RT@@1DD . conjugateRuleL +
  NRTsymbolList1@2D.NRT@@1DD ** Evaluated, u2,2 -> Function@8T0, T1, T2<,
RTsymbolList2.RT@@2DD + HRTsymbolList2.RT@@2DD . conjugateRuleL +
  NRTsymbolList2@1D.NRT@@2DD ** Evaluated, v2,2 -> Function@8T0, T1, T2<,
ratio2 * RTsymbolList2.RT@@2DD + Hratio2 * RTsymbolList2.RT@@2DD . conjugateRuleL +
  NRTsymbolList2@2D.NRT@@2DD ** Evaluated<;
eqb = HCoefficient@Subtract žž # . sol2Form, NRT@@1DDD . Exp@_ T0 + _ .D -> 0 & . žž
order2Eqm@@81, 2<DD ** FlattenL == 0 ** Thread;
eqc = HCoefficient@Subtract žž # . sol2Form, NRT@@2DDD . Exp@_ T0 + _ .D -> 0 & . žž
order2Eqm@@83, 4<DD ** FlattenL == 0 ** Thread;
coef11 = Solve@eqb, Array@NRTsymbolList1, 2D ** FlattenD@@1DD;
coef12 = Solve@eqc, Array@NRTsymbolList2, 2D ** FlattenD@@1DD;
eqd = MapIndexed@Coefficient@order2Eqm@@1, 1DD . sol2Form, #1D == r1, #2@@1DD &,
RT@@1DDD . Exp@_ T0 + _ .D -> 0;
eqe = MapIndexed@Coefficient@order2Eqm@@3, 1DD . sol2Form, #1D == r3, #2@@1DD &,
RT@@2DDD . Exp@_ T0 + _ .D -> 0;
coef21 = Solve@eqd, RTsymbolList1D@@1DD . r1Rule;
coef22 = Solve@eqe, RTsymbolList2D@@1DD . r3Rule;
coeffs = Join@coef11, coef12, coef21, coef22,
coef21 . conjugateRule, coef22 . conjugateRuleD . HomgRule@@1DD . e -> 0L;
sol2 = sol2Form . Function@8T0, T1, T2<, a_D :=
Function@8T0, T1, T2< ** Evaluate, a . coeffs ** Expand ** Evaluated
E;

```

H* Third-Order Problem *L

```

order3Eq = eqOrder@3D . sol1 . sol2 ** ExpandAll;
ST21 = Coefficient@order3Eq@@#, 2DD . expRule1@1D, Exp@I w1 T0DD & . žž 81, 2<;
ST22 = Coefficient@order3Eq@@#, 2DD . expRule1@2D, Exp@I w2 T0DD & . žž 83, 4<;
SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0< ** ExpandAll;
SCond2Rule1 = SolveASCond2, 9A1H0,1L@T1, T2D, A2H0,1L@T1, T2D=E@@1DD ** ExpandAll;

```

H* Reconstitution *L

```

moduEq =
Table@2 I wk Akε, 8k, 2<D == HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D . SCond1Rule1 .
SCond2Rule1 ** Collect@#, eD &L ** Thread;
Print@"The second-order approximate solution:"D;
PrintATableA
ui@tD == lui žž timeScales . solRule . e3 -> 0 . sol1 . sol2 . displayRuleM, 8i, 2<EE;
IfA0r žž Table@HFi . fi . scalingL === e, 8i, 2<D,
Print@"where"D;

```

```

PrintA"Li== $\frac{f_i}{2lw_1^2 - W^2M}$ "E
E;
Print@"\nThe modulation equations:"D;
Print@moduEq • . displayRuleD
E

```

As an example, we check the case in Section 6.1.1:

```
scaling1 = 9mn -> e mn, F1 -> e f1, F2 -> e2 f2;
```

```
ResonanceCond1 = 8w2 == 2 w1 + e s1, W == w2 + e s2<;
```

```
MMS@scaling1, ResonanceCond1D •• Timing
```

The second-order approximate solution:

```

:u1@tD == e I EI T0 w1 A1 + E-I T0 W - I t1 L1 + EI T0 W + I t1 L1 + E-I T0 w1  $\dot{A}_1$  M +
e2 }k  $\frac{E^{2 I T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2}{4 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_3}{15 w_1^2} - \frac{3 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1}{2 w_1^2} +$ 
 $\frac{3 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_1 L_1}{4 w_1^2} - \frac{2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_2 L_1}{w_1^2} + \frac{2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_2 L_1}{15 w_1^2} -$ 
 $\frac{6 d_1 L_1^2}{w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1} d_1 L_1^2}{5 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1} d_1 L_1^2}{5 w_1^2} - \frac{I E^{I T_0 w_1} A_1 m_1}{2 w_1} - \frac{4 I E^{-I T_0 W - I t_1} L_1 m_1}{3 w_1} +$ 
 $\frac{4 I E^{I T_0 W + I t_1} L_1 m_1}{3 w_1} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{2 w_1^2} + \frac{3 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{4 w_1^2} -$ 
 $\frac{3 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{2 w_1^2} + \frac{I E^{-I T_0 w_1} m_1 \dot{A}_1}{2 w_1} + \frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} - \frac{E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{2 w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} +$ 
 $\frac{2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{15 w_1^2} - \frac{2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{w_1^2} + \frac{E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{4 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \dot{A}_2^2}{15 w_1^2} \frac{y}{z},$ 
u2@tD == e I EI T0 w2 A2 + E-I T0 w2  $\dot{A}_2$  M + e2 }k  $\frac{E^{-I T_0 W - I t_2} f_2}{32 w_1^2} + \frac{E^{I T_0 W + I t_2} f_2}{32 w_1^2} - \frac{E^{2 I T_0 w_1} A_1^2 d_2}{16 w_1^2} +$ 
 $\frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3}{5 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_4}{4 w_1^2} - \frac{2 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_2 L_1}{3 w_1^2} +$ 
 $\frac{2 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_2 L_1}{5 w_1^2} - \frac{E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_3 L_1}{2 w_1^2} + \frac{E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_3 L_1}{6 w_1^2} -$ 
 $\frac{d_2 L_1^2}{2 w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1} d_2 L_1^2}{12 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1} d_2 L_1^2}{12 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{4 w_1} - \frac{A_1 d_2 \dot{A}_1}{2 w_1^2} -$ 
 $\frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1}{3 w_1^2} + \frac{2 E^{-I T_0 W - I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1}{5 w_1^2} - \frac{2 E^{I T_0 W + I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1}{3 w_1^2} -$ 
 $\frac{E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{16 w_1^2} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2}{3 w_1^2} - \frac{3 A_2 d_4 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W - I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2}{6 w_1^2} -$ 
 $\frac{E^{I T_0 W + I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2}{2 w_1^2} + \frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{4 w_1} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{4 w_1^2} \frac{y}{z} \}$ 

```

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

$$\begin{aligned} 2 I w_1 A_1^{\dot{\epsilon}} &= 2 I e w_1 \left\{ -A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1}{w_1} \right\} + \\ 2 I e^2 w_1 \left\{ -\frac{27 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{4 w_1^3} - \frac{23 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{30 w_1^3} - \right. \\ &\frac{81 I A_1 d_1^2 L_1^2}{4 w_1^3} - \frac{23 I A_1 d_2^2 L_1^2}{30 w_1^3} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} - \\ &\frac{15 I A_2^2 d_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \dot{A}_1}{32 w_1^3} - \frac{9 I A_2^2 d_2^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^3} - \\ &\frac{7 E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^3} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^3} + \frac{6 I A_2^2 a_1 \dot{A}_1}{w_1} - \frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \\ &\frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{4 w_1^3} - \\ &\left. \frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{30 w_1^3} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2}{w_1} \right\}, \\ 2 I w_2 A_2^{\dot{\epsilon}} &= 2 I e \left\{ -A_2 m_2 - \frac{I E^{I T_1 s_2 + I t_2} f_2}{4 w_2} + \frac{I E^{-I T_1 s_1} A_1^2 d_2}{2 w_2} \right\} w_2 + \\ 2 I e^2 w_2 \left\{ \frac{E^{I T_1 s_2 + I t_2} f_2 m_2}{32 w_1^2} - \frac{E^{-I T_1 s_1} A_1^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1}{2 w_2} + \frac{2 I A_2 a_3 L_2^2}{w_2} + \right. \\ &\frac{3 I E^{I T_1 s_2 + I t_1} a_2 L_1^3}{2 w_2} - \frac{29 I E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2 w_2} - \frac{5 I E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{4 w_1^2 w_2} - \\ &\frac{28 I A_2 d_2^2 L_1^2}{15 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \frac{I A_2 d_2^2 L_1^2}{3 w_1^2 w_2} - \frac{3 I A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2} - \frac{29 I E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2 w_2} - \\ &\frac{5 I E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{12 w_1^2 w_2} + \frac{E^{-I T_1 s_1} A_1^2 d_2 m_1}{2 w_1 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1}{w_2} - \\ &\frac{I A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2} - \\ &\frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{4 w_1^2 w_2} - \frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{30 w_1^2 w_2} + \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} + \\ &\frac{3 I E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2}{w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2}{w_2} - \frac{29 I A_2^2 d_2^2 \dot{A}_2}{15 w_1^2 w_2} - \frac{15 I A_2^2 d_2^2 \dot{A}_2}{4 w_1^2 w_2} - \\ &\frac{58 I E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2 w_2} - \frac{5 I E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{2 w_1^2 w_2} - \frac{2 I E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2 w_2} + \\ &\left. \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2 w_2} - \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{2 w_1^2 w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{4 w_1^2 w_2} \right\} \end{aligned}$$

882.98 Second, Null<

¶ 6.1.3 $W \gg w_1$ and $w_2 \gg 2w_1$

In this case, we have a combination of a primary resonance of the first mode, a superharmonic resonance of the second mode, and a two-to-one internal resonance. In order to bring the effects of damping, forcing, and nonlinearity at the same order, we let

$$\text{scaling3} = \epsilon m_n \rightarrow \epsilon m_n, F_1 \rightarrow \epsilon^2 f_1, F_2 \rightarrow \epsilon f_2;$$

To describe quantitatively the nearness of the resonances, we introduce the detuning parameters s_1 and s_2 defined according to

$$\text{ResonanceCond3} = 8w_2 == 2w_1 + \epsilon s_1, W == w_1 + \epsilon s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling3, ResonanceCond3D •• Timing

The second-order approximate solution:

$$\begin{aligned}
:u_1@tD &= e^{-I E^{I T_0} w_1} A_1 + E^{-I T_0 w_1} \dot{A}_1 M + \\
&e^2 \int_k \left\{ \frac{E^{-I T_0 W - I t_2} f_1}{8 w_1^2} + \frac{E^{I T_0 W + I t_2} f_1}{8 w_1^2} + \frac{E^{2 I T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2}{4 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_3}{15 w_1^2} - \right. \\
&\frac{2 E^{-I T_0 W - I t_2 + I T_0 w_1} A_1 d_2 L_2}{w_1^2} + \frac{2 E^{I T_0 W + I t_2 + I T_0 w_1} A_1 d_2 L_2}{3 w_1^2} - \frac{E^{-I T_0 W - I t_2 + I T_0 w_2} A_2 d_3 L_2}{2 w_1^2} + \\
&\frac{E^{I T_0 W + I t_2 + I T_0 w_2} A_2 d_3 L_2}{4 w_1^2} - \frac{2 d_3 L_2^2}{w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_2} d_3 L_2^2}{3 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_2} d_3 L_2^2}{3 w_1^2} - \\
&\frac{I E^{I T_0 w_1} A_1 m_1}{2 w_1} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{2 w_1^2} + \frac{2 E^{-I T_0 W - I t_2 - I T_0 w_1} d_2 L_2 \dot{A}_1}{3 w_1^2} - \\
&\frac{2 E^{I T_0 W + I t_2 - I T_0 w_1} d_2 L_2 \dot{A}_1}{w_1^2} + \frac{I E^{-I T_0 w_1} m_1 \dot{A}_1}{2 w_1} + \frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} - \frac{E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{2 w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} + \\
&\frac{E^{-I T_0 W - I t_2 - I T_0 w_2} d_3 L_2 \dot{A}_2}{4 w_1^2} - \frac{E^{I T_0 W + I t_2 - I T_0 w_2} d_3 L_2 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{4 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \dot{A}_2^2}{15 w_1^2} \Bigg\}, \\
u_2@tD &= e^{-I E^{I T_0} w_2} A_2 + E^{-I T_0 W - I t_2} L_2 + E^{I T_0 W + I t_2} L_2 + E^{-I T_0 w_2} \dot{A}_2 M + \\
&e^2 \int_k \left\{ -\frac{E^{2 I T_0 w_1} A_2^2 d_2}{16 w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3}{5 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_4}{4 w_1^2} - \right. \\
&\frac{E^{-I T_0 W - I t_2 + I T_0 w_1} A_1 d_3 L_2}{2 w_1^2} - \frac{E^{I T_0 W + I t_2 + I T_0 w_1} A_1 d_3 L_2}{8 w_1^2} - \frac{2 E^{-I T_0 W - I t_2 + I T_0 w_2} A_2 d_4 L_2}{w_1^2} + \\
&\frac{6 E^{I T_0 W + I t_2 + I T_0 w_2} A_2 d_4 L_2}{5 w_1^2} - \frac{3 d_4 L_2^2}{2 w_1^2} - \frac{3 E^{-2 I T_0 W - 2 I t_2} d_4 L_2^2}{16 w_1^2} - \frac{3 E^{2 I T_0 W + 2 I t_2} d_4 L_2^2}{16 w_1^2} - \\
&\frac{I E^{I T_0 w_2} A_2 m_2}{4 w_1} + \frac{2 I E^{-I T_0 W - I t_2} L_2 m_2}{3 w_1} - \frac{2 I E^{I T_0 W + I t_2} L_2 m_2}{3 w_1} - \frac{A_1 d_2 \dot{A}_1}{2 w_1^2} - \\
&\frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1}{3 w_1^2} - \frac{E^{-I T_0 W - I t_2 - I T_0 w_1} d_3 L_2 \dot{A}_1}{8 w_1^2} - \frac{E^{I T_0 W + I t_2 - I T_0 w_1} d_3 L_2 \dot{A}_1}{2 w_1^2} - \\
&\frac{E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{16 w_1^2} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2}{3 w_1^2} - \frac{3 A_2 d_4 \dot{A}_2}{2 w_1^2} + \frac{6 E^{-I T_0 W - I t_2 - I T_0 w_2} d_4 L_2 \dot{A}_2}{5 w_1^2} - \\
&\frac{2 E^{I T_0 W + I t_2 - I T_0 w_2} d_4 L_2 \dot{A}_2}{w_1^2} + \frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{4 w_1} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 A_1 \dot{A}_2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{4 w_1^2} \Bigg\}
\end{aligned}$$

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

$$:2 I w_1 A_1^c ==$$

$$\begin{aligned}
& 2 I e w_1 \left\{ \begin{aligned} & -A_1 m_1 - \frac{I E^{I T_1 s_2+I t_1} f_1}{4 w_1} + \frac{I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2 d_3 L_2}{w_1} + \frac{I E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1} \frac{y}{z} + 2 I e^2 w_1 \\ & \left\{ \frac{I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2 f_1 d_2}{8 w_1^3} - \frac{5 I E^{-I T_1 s_2-I t_2} A_2^2 d_1 d_2 L_2}{w_1^3} - \frac{9 I E^{-I T_1 s_2-I t_2} A_2^2 d_2 d_3 L_2}{16 w_1^3} - \frac{4 I A_1 d_2^2 L_2^2}{3 w_1^3} - \right. \\ & \frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} - \frac{5 I A_1 d_2^2 L_2^2}{8 w_1^3} - \frac{3 I A_1 d_2 d_4 L_2^2}{2 w_1^3} - \frac{5 I E^{I T_1 s_2+I t_2} d_2 d_3 L_2^2}{3 w_1^3} - \frac{27 I E^{I T_1 s_2+I t_2} d_3 d_4 L_2^3}{16 w_1^3} + \\ & \frac{E^{I T_1 s_2+I t_1} f_1 m_1}{8 w_1^3} - \frac{E^{I T_1 s_1-I T_1 s_2-I t_2} A_2 d_3 L_2 m_1}{2 w_1^3} - \frac{5 E^{I T_1 s_1-I T_1 s_2-I t_2} A_2 d_3 L_2 m_2}{12 w_1^3} + \\ & \frac{3 I E^{-I T_1 s_2-I t_2} A_2^2 a_2 L_2}{2 w_1} + \frac{2 I A_1 a_3 L_2^2}{w_1} + \frac{3 I E^{I T_1 s_2+I t_2} a_4 L_2^3}{2 w_1} - \frac{I A_1 m_2^2}{2 w_1} - \frac{15 I A_2^2 d_2^2 \dot{A}_1}{w_1} - \\ & \frac{9 I A_2^2 d_2^2 \dot{A}_1}{16 w_1^3} - \frac{10 I E^{I T_1 s_2+I t_2} A_1 d_1 d_2 L_2 \dot{A}_1}{w_1^3} - \frac{9 I E^{I T_1 s_2+I t_2} A_1 d_2 d_3 L_2 \dot{A}_1}{8 w_1^3} - \\ & \frac{2 I E^{2 I T_1 s_2+2 I t_2} d_2^2 L_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_1 s_2+2 I t_2} d_1 d_3 L_2^2 \dot{A}_1}{w_1^3} - \frac{I E^{2 I T_1 s_2+2 I t_2} d_2^2 L_2^2 \dot{A}_1}{2 w_1^3} - \\ & \frac{3 I E^{2 I T_1 s_2+2 I t_2} d_2 d_4 L_2^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2} + \\ & \frac{6 I A_2^2 a_1 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_2+I t_2} A_1 a_2 L_2 \dot{A}_1}{w_1} + \frac{I E^{2 I T_1 s_2+2 I t_2} a_3 L_2^2 \dot{A}_1}{w_1} - \frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \\ & \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{9 I E^{I T_1 s_2+I t_2} A_2 d_2 d_3 L_2 \dot{A}_2}{4 w_1^3} - \\ & \frac{23 I E^{I T_1 s_2+I t_2} A_2 d_3 d_4 L_2 \dot{A}_2}{10 w_1^3} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 s_2+I t_2} A_2 a_4 L_2 \dot{A}_2}{w_1} \frac{y}{z} \left. \right\} 2 I w_2 A_2^c == \\ & 2 I e \left\{ \begin{aligned} & -A_2 m_2 + \frac{I E^{-I T_1 s_1} A_2^2 d_2}{2 w_2} + \frac{I E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_3 L_2}{w_2} + \frac{3 I E^{-I T_1 s_1+2 I T_1 s_2+2 I t_2} d_4 L_2^2}{2 w_2} \frac{y}{z} w_2 + \\ & 2 I e^2 w_2 \left\{ \begin{aligned} & -\frac{E^{-I T_1 s_1} A_2^2 d_2 m_2}{16 w_1^2} - \frac{E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_3 L_2 m_2}{8 w_1^2} - \frac{3 E^{-I T_1 s_1+2 I T_1 s_2+2 I t_2} d_4 L_2^2 m_2}{16 w_1^2} - \\ & \frac{I A_2 m_2^2}{4 w_1} + \frac{3 I E^{-I T_1 s_2-I t_2} A_1 A_2 a_4 L_2}{w_2} + \frac{12 I A_2 a_5 L_2^2}{w_2} + \frac{I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 f_1 d_2}{8 w_1^2 w_2} + \\ & \frac{I E^{-I T_1 s_1+2 I T_1 s_2+I t_1+I t_2} f_1 d_3 L_2}{8 w_1^2 w_2} - \frac{9 I E^{-I T_1 s_2-I t_2} A_1 A_2 d_2 d_3 L_2}{4 w_1^2 w_2} - \\ & \frac{23 I E^{-I T_1 s_2-I t_2} A_1 A_2 d_3 d_4 L_2}{10 w_1^2 w_2} - \frac{9 I A_2 d_2^2 L_2^2}{4 w_1^2 w_2} - \frac{69 I A_2 d_2^2 L_2^2}{10 w_1^2 w_2} + \frac{E^{-I T_1 s_1} A_2^2 d_2 m_1}{2 w_1 w_2} + \\ & \frac{E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_3 L_2 m_1}{2 w_1 w_2} + \frac{2 E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_3 L_2 m_2}{3 w_1 w_2} + \frac{2 E^{-I T_1 s_1+2 I T_1 s_2+2 I t_2} d_4 L_2^2 m_2}{w_1 w_2} + \\ & \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \frac{3 I E^{I T_1 s_2+I t_2} A_2 a_4 L_2 \dot{A}_1}{w_2} - \frac{I A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \\ & \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2} - \frac{9 I E^{I T_1 s_2+I t_2} A_2 d_2 d_3 L_2 \dot{A}_1}{4 w_1^2 w_2} - \\ & \frac{23 I E^{I T_1 s_2+I t_2} A_2 d_3 d_4 L_2 \dot{A}_1}{10 w_1^2 w_2} + \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} - \frac{29 I A_2^2 d_2^2 \dot{A}_2}{15 w_1^2 w_2} - \frac{15 I A_2^2 d_2^2 \dot{A}_2}{4 w_1^2 w_2} \frac{y}{z} \left. \right\} > \end{aligned} \right.
\end{aligned}
\end{aligned}$$

860.36 Second, Null<

¶ 6.1.4 $W \gg w_1 + w_2$ and $w_2 \gg 2w_1$

In this case, we have a combination resonance and a two-to-one internal resonance. We let

$$\text{scaling4} = 8m_n \rightarrow e m_n, F_1 \rightarrow e f_1, F_2 \rightarrow e f_2;$$

$$\text{ResonanceCond4} = 8w_2 == 2w_1 + e s_1, W == w_1 + w_2 + e s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling4, ResonanceCond4D •• Timing

The second-order approximate solution:

$$\begin{aligned} u_1 @ t D &= e^{i E^{i T_0 w_1} A_1 + E^{-i T_0 W - i T_1} L_1 + E^{i T_0 W + i T_1} L_1 + E^{-i T_0 w_1} \dot{A}_1 M} + \\ &e^2 \left\{ \begin{aligned} &\frac{E^{2 i T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{E^{i T_0 w_1 + i T_0 w_2} A_1^2 d_2}{4 w_1^2} + \frac{E^{2 i T_0 w_2} A_1^2 d_3}{15 w_1^2} + \frac{2 E^{-i T_0 W - i T_1 + i T_0 w_1} A_1 d_1 L_1}{w_1^2} + \\ &\frac{2 E^{i T_0 W + i T_1 + i T_0 w_1} A_1 d_1 L_1}{5 w_1^2} - \frac{E^{-i T_0 W - i T_1 + i T_0 w_2} A_1 d_2 L_1}{2 w_1^2} + \frac{E^{i T_0 W + i T_1 + i T_0 w_2} A_1 d_2 L_1}{12 w_1^2} - \\ &\frac{6 d_1 L_1^2}{w_1^2} + \frac{3 E^{-2 i T_0 W - 2 i T_1} d_1 L_1^2}{35 w_1^2} + \frac{3 E^{2 i T_0 W + 2 i T_1} d_1 L_1^2}{35 w_1^2} + \frac{2 E^{-i T_0 W - i T_2 + i T_0 w_1} A_1 d_2 L_2}{3 w_1^2} + \\ &\frac{2 E^{i T_0 W + i T_2 + i T_0 w_1} A_1 d_2 L_2}{15 w_1^2} - \frac{E^{-i T_0 W - i T_2 + i T_0 w_2} A_1 d_3 L_2}{2 w_1^2} + \frac{E^{i T_0 W + i T_2 + i T_0 w_2} A_1 d_3 L_2}{12 w_1^2} + \\ &\frac{2 E^{-2 i T_0 W - i T_1 - i T_2} d_2 L_1 L_2}{35 w_1^2} - \frac{2 E^{i T_1 - i T_2} d_2 L_1 L_2}{w_1^2} - \frac{2 E^{-i T_1 + i T_2} d_2 L_1 L_2}{w_1^2} + \\ &\frac{2 E^{2 i T_0 W + i T_1 + i T_2} d_2 L_1 L_2}{35 w_1^2} - \frac{2 d_3 L_2^2}{w_1^2} + \frac{E^{-2 i T_0 W - 2 i T_2} d_3 L_2^2}{35 w_1^2} + \frac{E^{2 i T_0 W + 2 i T_2} d_3 L_2^2}{35 w_1^2} - \frac{I E^{i T_0 w_1} A_1 m_1}{2 w_1} - \\ &\frac{3 I E^{-i T_0 W - i T_1} L_1 m_1}{4 w_1} + \frac{3 I E^{i T_0 W + i T_1} L_1 m_1}{4 w_1} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{E^{i T_0 w_1 - i T_0 w_2} A_1 d_2 \dot{A}_1}{2 w_1^2} - \\ &\frac{E^{-i T_0 w_1 + i T_0 w_2} A_1 d_2 \dot{A}_1}{2 w_1^2} - \frac{2 A_1 d_3 \dot{A}_1}{w_1^2} + \frac{2 E^{-i T_0 W - i T_1 - i T_0 w_1} d_1 L_1 \dot{A}_1}{5 w_1^2} + \frac{2 E^{i T_0 W + i T_1 - i T_0 w_1} d_1 L_1 \dot{A}_1}{w_1^2} + \\ &\frac{E^{-i T_0 W - i T_1 + i T_0 w_2} d_2 L_1 \dot{A}_1}{12 w_1^2} - \frac{E^{i T_0 W + i T_1 - i T_0 w_2} d_2 L_1 \dot{A}_1}{2 w_1^2} + \frac{2 E^{-i T_0 W - i T_2 - i T_0 w_1} d_2 L_2 \dot{A}_1}{15 w_1^2} + \\ &\frac{2 E^{i T_0 W + i T_2 - i T_0 w_1} d_2 L_2 \dot{A}_1}{3 w_1^2} + \frac{E^{-i T_0 W - i T_2 - i T_0 w_2} d_3 L_2 \dot{A}_1}{12 w_1^2} - \frac{E^{i T_0 W + i T_2 - i T_0 w_2} d_3 L_2 \dot{A}_1}{2 w_1^2} + \\ &\frac{I E^{-i T_0 w_1} m_1 \dot{A}_1}{2 w_1} + \frac{E^{-2 i T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} + \frac{E^{-i T_0 w_1 - i T_0 w_2} d_2 \dot{A}_1^2}{4 w_1^2} + \frac{E^{-2 i T_0 w_2} d_3 \dot{A}_1^2}{15 w_1^2} \frac{y}{z}, \end{aligned} \right\} \\ u_2 @ t D &= e^{i E^{i T_0 w_2} A_2 + E^{-i T_0 W - i T_2} L_2 + E^{i T_0 W + i T_2} L_2 + E^{-i T_0 w_2} \dot{A}_2 M} + \\ &e^2 \left\{ \begin{aligned} &\frac{E^{2 i T_0 w_1} A_2^2 d_2}{16 w_1^2} + \frac{2 E^{i T_0 w_1 + i T_0 w_2} A_2^2 d_3}{5 w_1^2} + \frac{E^{2 i T_0 w_2} A_2^2 d_4}{4 w_1^2} - \frac{E^{-i T_0 W - i T_1 + i T_0 w_1} A_2 d_2 L_1}{8 w_1^2} + \\ &\frac{E^{i T_0 W + i T_1 + i T_0 w_1} A_2 d_2 L_1}{6 w_1^2} - \frac{2 E^{-i T_0 W - i T_1 + i T_0 w_2} A_2 d_3 L_1}{3 w_1^2} + \frac{2 E^{i T_0 W + i T_1 + i T_0 w_2} A_2 d_3 L_1}{21 w_1^2} - \\ &\frac{d_2 L_1^2}{2 w_1^2} + \frac{E^{-2 i T_0 W - 2 i T_1} d_2 L_1^2}{32 w_1^2} + \frac{E^{2 i T_0 W + 2 i T_1} d_2 L_1^2}{32 w_1^2} - \frac{E^{-i T_0 W - i T_2 + i T_0 w_1} A_2 d_3 L_2}{8 w_1^2} + \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned}
& \frac{E^{I T_0 W+I t_2+I T_0 w_1} A_2 d_3 L_2}{6 w_1^2} - \frac{2 E^{-I T_0 W-I t_2+I T_0 w_2} A_2 d_4 L_2}{w_1^2} + \frac{2 E^{I T_0 W+I t_2+I T_0 w_2} A_2 d_4 L_2}{7 w_1^2} + \\
& \frac{E^{-2 I T_0 W-I t_1-I t_2} d_3 L_1 L_2}{16 w_1^2} - \frac{E^{I t_1-I t_2} d_3 L_1 L_2}{2 w_1^2} - \frac{E^{-I t_1+I t_2} d_3 L_1 L_2}{2 w_1^2} + \frac{E^{2 I T_0 W+I t_1+I t_2} d_3 L_1 L_2}{16 w_1^2} - \\
& \frac{3 d_4 L_2^2}{2 w_1^2} + \frac{3 E^{-2 I T_0 W-2 I t_2} d_4 L_2^2}{32 w_1^2} + \frac{3 E^{2 I T_0 W+2 I t_2} d_4 L_2^2}{32 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{4 w_1} - \\
& \frac{6 I E^{-I T_0 W-I t_2} L_2 m_2}{5 w_1} + \frac{6 I E^{I T_0 W+I t_2} L_2 m_2}{5 w_1} - \frac{A_2 d_2 \dot{A}_2}{2 w_1^2} - \frac{2 E^{I T_0 w_1-I T_0 w_2} A_2 d_3 \dot{A}_2}{3 w_1^2} - \\
& \frac{2 E^{-I T_0 w_1+I T_0 w_2} A_2 d_3 \dot{A}_2}{3 w_1^2} - \frac{3 A_2 d_4 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W-I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{6 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{8 w_1^2} + \\
& \frac{2 E^{-I T_0 W-I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{21 w_1^2} - \frac{2 E^{I T_0 W+I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-I T_0 W-I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{6 w_1^2} - \\
& \frac{E^{I T_0 W+I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{8 w_1^2} + \frac{2 E^{-I T_0 W-I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{7 w_1^2} - \frac{2 E^{I T_0 W+I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{w_1^2} + \\
& \frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{4 w_1} - \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_2^2}{16 w_1^2} + \frac{2 E^{-I T_0 w_1-I T_0 w_2} d_3 \dot{A}_2^2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{4 w_1^2} \frac{y}{2} > \{
\end{aligned}$$

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

$$: 2 I w_1 A_1^i =$$

$$\begin{aligned}
& 2 I e^2 w_1 \left\{ \begin{aligned} & - \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_2^2 L_1}{2 w_1^3} + \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_1 d_3 L_1}{5 w_1^3} - \frac{2 I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_3^2 L_1}{3 w_1^3} + \\ & \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_2 d_4 L_1}{4 w_1^3} - \frac{54 I A_1 d_1^2 L_1^2}{5 w_1^3} - \frac{11 I A_1 d_3^2 L_1^2}{24 w_1^3} - \frac{13 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 d_2 d_3 L_2}{30 w_1^3} - \\ & \frac{7 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 d_3 d_4 L_2}{4 w_1^3} - \frac{18 I E^{I t_1-I t_2} A_1 d_1 d_2 L_1 L_2}{5 w_1^3} - \frac{18 I E^{-I t_1+I t_2} A_1 d_1 d_2 L_1 L_2}{5 w_1^3} - \\ & \frac{11 I E^{I t_1-I t_2} A_1 d_2 d_3 L_1 L_2}{24 w_1^3} - \frac{11 I E^{-I t_1+I t_2} A_1 d_2 d_3 L_1 L_2}{24 w_1^3} + \frac{4 I A_1 d_2^2 L_2^2}{5 w_1^3} - \frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} + \\ & \frac{I A_1 d_3^2 L_2^2}{24 w_1^3} - \frac{3 I A_1 d_2 d_4 L_2^2}{2 w_1^3} + \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 a_3 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} + \\ & \frac{3 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 a_4 L_2}{2 w_1} + \frac{3 I E^{I t_1-I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{3 I E^{-I t_1+I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{2 I A_1 a_3 L_2^2}{w_1} - \\ & \frac{I A_1 m_1^2}{2 w_1} - \frac{15 I A_2^2 d_2^2 \dot{A}_1}{w_1^3} - \frac{9 I A_2^2 d_3^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2} + \frac{6 I A_2^2 a_1 \dot{A}_1}{w_1} + \\ & \frac{9 I E^{I T_1 s_1+I T_1 s_2+I t_1} d_1^2 L_1 \dot{A}_1^2}{w_1^3} - \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_1} d_3^2 L_1 \dot{A}_1^2}{16 w_1^3} + \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} d_1 d_2 L_2 \dot{A}_1^2}{w_1^3} - \\ & \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} d_2 d_3 L_2 \dot{A}_1^2}{16 w_1^3} + \frac{6 I E^{I T_1 s_1+I T_1 s_2+I t_1} a_1 L_1 \dot{A}_1^2}{w_1} + \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} a_2 L_2 \dot{A}_1^2}{2 w_1} - \\ & \frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{5 E^{I T_1 s_2+I t_1} d_2 L_1 m_1 \dot{A}_2}{4 w_1^2} - \end{aligned} \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_1 \dot{A}_2}{2 w_1^2} - \frac{E^{I T_1 s_2 + I t_1} d_2 L_1 m_2 \dot{A}_2}{4 w_1^2} - \frac{29 E^{I T_1 s_2 + I t_2} d_2 L_2 m_2 \dot{A}_2}{20 w_1^2} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} \frac{y}{z} + \\
& 2 I e w_1 \Bigg\}_K - A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1} + \frac{I E^{I T_1 s_2 + I t_1} d_2 L_1 \dot{A}_2}{w_1} + \frac{I E^{I T_1 s_2 + I t_2} d_3 L_2 \dot{A}_2}{w_1} \frac{y}{z}, \\
& 2 I w_2 A_2^2 = 2 I e w_2 \Bigg\}_K - A_2 m_2 + \frac{I E^{-I T_1 s_1} A_2^2 d_2}{2 w_2} + \frac{I E^{I T_1 s_2 + I t_1} d_2 L_1 \dot{A}_1}{w_2} + \frac{I E^{I T_1 s_2 + I t_2} d_3 L_2 \dot{A}_1}{w_2} \frac{y}{z} + \\
& 2 I e^2 w_2 \Bigg\}_K - \frac{E^{-I T_1 s_1} A_1^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{2 I A_2 a_3 L_1^2}{w_2} + \frac{3 I E^{I t_1 - I t_2} A_2 a_4 L_1 L_2}{w_2} + \\
& \frac{3 I E^{-I t_1 + I t_2} A_2 a_4 L_1 L_2}{w_2} + \frac{12 I A_2 a_5 L_2^2}{w_2} - \frac{5 I A_2 d_3^2 L_1^2}{12 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \frac{4 I A_2 d_3^2 L_1^2}{7 w_1^2 w_2} - \\
& \frac{3 I A_2 d_2 d_4 L_2^2}{2 w_1^2 w_2} - \frac{29 I E^{I t_1 - I t_2} A_2 d_2 d_3 L_1 L_2}{12 w_1^2 w_2} - \frac{29 I E^{-I t_1 + I t_2} A_2 d_2 d_3 L_1 L_2}{12 w_1^2 w_2} - \\
& \frac{45 I E^{I t_1 - I t_2} A_2 d_3 d_4 L_1 L_2}{14 w_1^2 w_2} - \frac{45 I E^{-I t_1 + I t_2} A_2 d_3 d_4 L_1 L_2}{14 w_1^2 w_2} - \frac{29 I A_2 d_3^2 L_2^2}{12 w_1^2 w_2} - \\
& \frac{135 I A_2 d_4^2 L_2^2}{14 w_1^2 w_2} + \frac{E^{-I T_1 s_1} A_2^2 d_2 m_1}{2 w_1 w_2} - \frac{E^{I T_1 s_2 + I t_1} d_2 L_1 m_2 \dot{A}_1}{8 w_1^2} - \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_2 \dot{A}_1}{8 w_1^2} + \\
& \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} - \frac{I A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2} - \\
& \frac{5 E^{I T_1 s_2 + I t_1} d_2 L_1 m_1 \dot{A}_1}{4 w_1 w_2} - \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_1 \dot{A}_1}{2 w_1 w_2} - \frac{6 E^{I T_1 s_2 + I t_2} d_3 L_2 m_2 \dot{A}_1}{5 w_1 w_2} + \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} + \\
& \frac{2 I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} A_1 a_3 L_1 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1 + I T_1 s_2 + I t_2} A_1 a_4 L_2 \dot{A}_2}{w_2} - \frac{29 I A_2^2 d_3^2 \dot{A}_2}{15 w_1^2 w_2} - \\
& \frac{15 I A_2^2 d_4^2 \dot{A}_2}{4 w_1^2 w_2} - \frac{I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} A_1 d_3^2 L_1 \dot{A}_2}{w_1^2 w_2} + \frac{2 I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} A_1 d_1 d_3 L_1 \dot{A}_2}{5 w_1^2 w_2} - \\
& \frac{4 I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} A_1 d_3^2 L_1 \dot{A}_2}{3 w_1^2 w_2} + \frac{I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} A_1 d_2 d_4 L_1 \dot{A}_2}{2 w_1^2 w_2} - \\
& \frac{13 I E^{-I T_1 s_1 + I T_1 s_2 + I t_2} A_1 d_2 d_3 L_2 \dot{A}_2}{15 w_1^2 w_2} - \frac{7 I E^{-I T_1 s_1 + I T_1 s_2 + I t_2} A_1 d_3 d_4 L_2 \dot{A}_2}{2 w_1^2 w_2} \frac{y}{z} > \\
& \{
\end{aligned}$$

8192.266 Second, Null<

¶ 6.1.5 $W \gg w_2$ and $w_2 \gg 3 w_1$

In this case, we have a primary resonance of the second mode, a subharmonic resonance of the first mode, and a three-to-one internal resonance. We let

$$\text{scaling5} = 9 m_n \rightarrow e^2 m_n, F_1 \rightarrow e f_1, F_2 \rightarrow e^3 f_2;$$

$$\text{ResonanceCond5} = 9 w_2 == 3 w_1 + e^2 s_1, W == w_2 + e^2 s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling5, ResonanceCond5D •• Timing

The second-order approximate solution:

$$\begin{aligned}
:u_1@tD == & e \left| E^{IT_0 w_1} A_1 + E^{-IT_0 W - I t_1} L_1 + E^{IT_0 W + I t_1} L_1 + E^{-IT_0 w_1} \dot{A}_1 M + \right. \\
& e^2 \left\{ \frac{E^{2IT_0 w_1} A_1^2 d_1}{w_1^2} + \frac{2 E^{IT_0 w_1 + IT_0 w_2} A_1 A_2 d_2}{15 w_1^2} + \frac{E^{2IT_0 w_2} A_2^2 d_3}{35 w_1^2} + \frac{2 E^{-IT_0 W - I t_1 + IT_0 w_1} A_1 d_1 L_1}{w_1^2} + \right. \\
& \frac{2 E^{IT_0 W + I t_1 + IT_0 w_1} A_1 d_1 L_1}{5 w_1^2} - \frac{2 E^{-IT_0 W - I t_1 + IT_0 w_2} A_2 d_2 L_1}{w_1^2} + \frac{2 E^{IT_0 W + I t_1 + IT_0 w_2} A_2 d_2 L_1}{35 w_1^2} - \\
& \frac{6 d_1 L_1^2}{w_1^2} + \frac{3 E^{-2IT_0 W - 2I t_1} d_1 L_1^2}{35 w_1^2} + \frac{3 E^{2IT_0 W + 2I t_1} d_1 L_1^2}{35 w_1^2} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} + \\
& \frac{2 E^{-IT_0 w_1 + IT_0 w_2} A_2 d_2 \dot{A}_1}{3 w_1^2} + \frac{2 E^{-IT_0 W - I t_1 - IT_0 w_1} d_1 L_1 \dot{A}_1}{5 w_1^2} + \frac{2 E^{IT_0 W + I t_1 - IT_0 w_1} d_1 L_1 \dot{A}_1}{w_1^2} + \\
& \frac{E^{-2IT_0 w_1} d_1 \dot{A}_1^2}{w_1^2} + \frac{2 E^{IT_0 w_1 - IT_0 w_2} A_1 d_2 \dot{A}_2}{3 w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} + \frac{2 E^{-IT_0 W - I t_1 - IT_0 w_2} d_2 L_1 \dot{A}_2}{35 w_1^2} - \\
& \frac{2 E^{IT_0 W + I t_1 - IT_0 w_2} d_2 L_1 \dot{A}_2}{w_1^2} + \frac{2 E^{-IT_0 w_1 - IT_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{15 w_1^2} + \frac{E^{-2IT_0 w_2} d_3 \dot{A}_2^2}{35 w_1^2} \Big\}, \\
u_2@tD == & e \left| E^{IT_0 w_2} A_2 + E^{-IT_0 w_2} \dot{A}_2 M + e^2 \right\{ - \frac{E^{2IT_0 w_1} A_1^2 d_2}{5 w_1^2} + \frac{2 E^{IT_0 w_1 + IT_0 w_2} A_1 A_2 d_3}{7 w_1^2} + \frac{E^{2IT_0 w_2} A_2^2 d_4}{9 w_1^2} - \\
& \frac{2 E^{-IT_0 W - I t_1 + IT_0 w_1} A_1 d_2 L_1}{5 w_1^2} + \frac{2 E^{IT_0 W + I t_1 + IT_0 w_1} A_1 d_2 L_1}{7 w_1^2} - \frac{2 E^{-IT_0 W - I t_1 + IT_0 w_2} A_2 d_3 L_1}{9 w_1^2} + \\
& \frac{2 E^{IT_0 W + I t_1 + IT_0 w_2} A_2 d_3 L_1}{27 w_1^2} - \frac{2 d_2 L_1^2}{9 w_1^2} + \frac{E^{-2IT_0 W - 2I t_1} d_2 L_1^2}{27 w_1^2} + \frac{E^{2IT_0 W + 2I t_1} d_2 L_1^2}{27 w_1^2} - \frac{2 A_1 d_2 \dot{A}_1}{9 w_1^2} - \\
& \frac{2 E^{-IT_0 w_1 + IT_0 w_2} A_2 d_3 \dot{A}_1}{5 w_1^2} + \frac{2 E^{-IT_0 W - I t_1 - IT_0 w_1} d_2 L_1 \dot{A}_1}{7 w_1^2} - \frac{2 E^{IT_0 W + I t_1 - IT_0 w_1} d_2 L_1 \dot{A}_1}{5 w_1^2} - \\
& \frac{E^{-2IT_0 w_1} d_2 \dot{A}_1^2}{5 w_1^2} - \frac{2 E^{IT_0 w_1 - IT_0 w_2} A_1 d_3 \dot{A}_2}{5 w_1^2} - \frac{2 A_2 d_4 \dot{A}_2}{3 w_1^2} + \frac{2 E^{-IT_0 W - I t_1 - IT_0 w_2} d_3 L_1 \dot{A}_2}{27 w_1^2} - \\
& \frac{2 E^{IT_0 W + I t_1 - IT_0 w_2} d_3 L_1 \dot{A}_2}{9 w_1^2} + \frac{2 E^{-IT_0 w_1 - IT_0 w_2} d_3 \dot{A}_1 \dot{A}_2}{7 w_1^2} + \frac{E^{-2IT_0 w_2} d_4 \dot{A}_2^2}{9 w_1^2} \Big\}
\end{aligned}$$

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

: 2 I w₁ A₁^ε ==

$$\begin{aligned}
 & 2 I e^2 w_1 \Big|_k - A_1 m_1 - \frac{18 I E^{-I T_2 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{5 w_1^3} - \frac{106 I E^{-I T_2 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{315 w_1^3} - \frac{54 I A_1 d_1^2 L_1^2}{5 w_1^3} - \\
 & \frac{106 I A_1 d_2^2 L_1^2}{315 w_1^3} + \frac{3 I E^{-I T_2 s_2 - I t_1} A_1 A_2 a_2 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} - \frac{15 I A_2^2 d_1^2 \dot{A}_1}{w_1^3} - \frac{19 I A_2^2 d_2^2 \dot{A}_1}{45 w_1^3} + \\
 & \frac{6 I A_2^2 a_1 \dot{A}_1}{w_1} + \frac{3 I E^{I T_2 s_1} A_2 d_1 d_2 \dot{A}_2}{w_1^3} - \frac{3 I E^{I T_2 s_1} A_2 d_2 d_3 \dot{A}_2}{5 w_1^3} + \frac{9 I E^{I T_2 s_1 + I T_2 s_2 + I t_1} d_1^2 L_1 \dot{A}_2}{w_1^3} - \\
 & \frac{3 I E^{I T_2 s_1 + I T_2 s_2 + I t_1} d_2^2 L_1 \dot{A}_2}{5 w_1^3} + \frac{3 I E^{I T_2 s_1} A_2 a_2 \dot{A}_1}{2 w_1} + \frac{6 I E^{I T_2 s_1 + I T_2 s_2 + I t_1} a_1 L_1 \dot{A}_1}{w_1} + \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{5 w_1^3} - \\
 & \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{35 w_1^3} - \frac{2 I A_1 A_2 d_2 d_4 \dot{A}_2}{3 w_1^3} - \frac{18 I E^{I T_2 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{5 w_1^3} - \\
 & \frac{106 I E^{I T_2 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{315 w_1^3} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_2 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2}{w_1} \Big\}, \\
 & 2 I w_2 A_2^\epsilon = 2 I e^2 w_2 \Big|_k - A_2 m_2 - \frac{I E^{I T_2 s_2 + I t_1} f_2}{4 w_2} + \frac{I E^{-I T_2 s_1} A_3^3 a_2}{2 w_2} + \frac{3 I E^{-I T_2 s_2 - I t_1} A_2^2 a_4 L_1}{2 w_2} + \\
 & \frac{2 I A_2 a_3 L_1^2}{w_2} + \frac{3 I E^{I T_2 s_2 + I t_1} a_2 L_1^3}{2 w_2} + \frac{I E^{-I T_2 s_1} A_3^3 d_1 d_2}{w_1^2 w_2} - \frac{I E^{-I T_2 s_1} A_3^3 d_2 d_3}{5 w_1^2 w_2} - \\
 & \frac{69 I E^{-I T_2 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{35 w_1^2 w_2} - \frac{5 I E^{-I T_2 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{9 w_1^2 w_2} - \frac{68 I A_2 d_2^2 L_1^2}{35 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \\
 & \frac{4 I A_2 d_2^2 L_1^2}{27 w_1^2 w_2} - \frac{2 I A_2 d_2 d_4 L_1^2}{3 w_1^2 w_2} - \frac{207 I E^{I T_2 s_2 + I t_1} d_1 d_2 L_1^3}{35 w_1^2 w_2} - \frac{5 I E^{I T_2 s_2 + I t_1} d_2 d_3 L_1^3}{27 w_1^2 w_2} + \\
 & \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \frac{3 I E^{I T_2 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1}{w_2} + \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{5 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{35 w_1^2 w_2} - \\
 & \frac{2 I A_1 A_2 d_2 d_4 \dot{A}_1}{3 w_1^2 w_2} - \frac{18 I E^{I T_2 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{5 w_1^2 w_2} - \frac{106 I E^{I T_2 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{315 w_1^2 w_2} + \\
 & \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} + \frac{3 I E^{I T_2 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2}{w_2} + \frac{I E^{2 I T_2 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2}{w_2} - \frac{69 I A_2^2 d_2^2 \dot{A}_2}{35 w_1^2 w_2} - \frac{5 I A_2^2 d_2^2 \dot{A}_2}{3 w_1^2 w_2} - \\
 & \frac{138 I E^{I T_2 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{35 w_1^2 w_2} - \frac{10 I E^{I T_2 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{9 w_1^2 w_2} - \frac{2 I E^{2 I T_2 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2 w_2} + \\
 & \frac{3 I E^{2 I T_2 s_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{35 w_1^2 w_2} - \frac{2 I E^{2 I T_2 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{9 w_1^2 w_2} + \frac{I E^{2 I T_2 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{9 w_1^2 w_2} \Big\}
 \end{aligned}$$

866.906 Second, Null<

§ 6.1.6 W » w₂ - 2 w₁ and w₂ » 3 w₁

In this case, we have a primary resonance of the first mode, a superharmonic resonance of the second mode, and a three-to-one internal resonance. We let

$$\text{scaling6} = 9 m_n \rightarrow e^2 m_n, \quad F_1 \rightarrow e^3 f_1, \quad F_2 \rightarrow e f_2;$$

$$\text{ResonanceCond6} = 9 w_2 == 3 w_1 + e^2 s_1, \quad W == w_2 - 2 w_1 + e^2 s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling6, ResonanceCond6D •• Timing

The second-order approximate solution:

$$\begin{aligned}
 u_1 @ t D &= e^{i E^{I T_0} w_1} A_1 + E^{-I T_0 w_1} \dot{A}_1 M + \\
 &e^2 \int_k \left\{ \frac{E^{2 I T_0} w_1 A_1^2 d_1}{w_1^2} + \frac{2 E^{I T_0} w_1 + I T_0 w_2 A_1^2 d_2}{15 w_1^2} + \frac{E^{2 I T_0} w_2 A_1^2 d_3}{35 w_1^2} - \frac{2 E^{-I T_0} W - I t_2 + I T_0 w_1 A_1 d_2 L_2}{w_1^2} + \right. \\
 &\frac{2 E^{I T_0} W + I t_2 + I T_0 w_1 A_1 d_2 L_2}{3 w_1^2} + \frac{2 E^{-I T_0} W - I t_2 + I T_0 w_2 A_1 d_3 L_2}{3 w_1^2} + \frac{2 E^{I T_0} W + I t_2 + I T_0 w_2 A_1 d_3 L_2}{15 w_1^2} - \\
 &\frac{2 d_3 L_2^2}{w_1^2} + \frac{E^{-2 I T_0} W - 2 I t_2 d_3 L_2^2}{3 w_1^2} + \frac{E^{2 I T_0} W + 2 I t_2 d_3 L_2^2}{3 w_1^2} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} + \\
 &\frac{2 E^{I T_0} w_1 - I T_0 w_2 A_1 d_2 \dot{A}_1}{3 w_1^2} + \frac{2 E^{-I T_0} w_1 + I T_0 w_2 A_1 d_2 \dot{A}_1}{3 w_1^2} - \frac{2 A_1 d_3 \dot{A}_1}{w_1^2} + \\
 &\frac{2 E^{-I T_0} W - I t_2 - I T_0 w_1 d_2 L_2 \dot{A}_1}{3 w_1^2} - \frac{2 E^{I T_0} W + I t_2 - I T_0 w_1 d_2 L_2 \dot{A}_1}{w_1^2} + \frac{2 E^{-I T_0} W - I t_2 - I T_0 w_2 d_3 L_2 \dot{A}_1}{15 w_1^2} + \\
 &\frac{2 E^{I T_0} W + I t_2 - I T_0 w_2 d_3 L_2 \dot{A}_1}{3 w_1^2} + \frac{E^{-2 I T_0} w_1 d_1 \dot{A}_1^2}{w_1^2} + \frac{2 E^{-I T_0} w_1 - I T_0 w_2 d_2 \dot{A}_1^2}{15 w_1^2} + \frac{E^{-2 I T_0} w_2 d_3 \dot{A}_1^2}{35 w_1^2} \frac{y}{z} \Bigg\} \\
 u_2 @ t D &= e^{i E^{I T_0} w_2} A_2 + E^{-I T_0 W - I t_2} L_2 + E^{I T_0 W + I t_2} L_2 + E^{-I T_0 w_2} \dot{A}_2 M + \\
 &e^2 \int_k \left\{ -\frac{E^{2 I T_0} w_1 A_2^2 d_2}{5 w_1^2} + \frac{2 E^{I T_0} w_1 + I T_0 w_2 A_2^2 d_3}{7 w_1^2} + \frac{E^{2 I T_0} w_2 A_2^2 d_4}{9 w_1^2} - \right. \\
 &\frac{2 E^{-I T_0} W - I t_2 + I T_0 w_1 A_2 d_3 L_2}{9 w_1^2} - \frac{2 E^{I T_0} W + I t_2 + I T_0 w_1 A_2 d_3 L_2}{5 w_1^2} - \frac{6 E^{-I T_0} W - I t_2 + I T_0 w_2 A_2 d_4 L_2}{5 w_1^2} + \\
 &\frac{6 E^{I T_0} W + I t_2 + I T_0 w_2 A_2 d_4 L_2}{7 w_1^2} - \frac{2 d_4 L_2^2}{3 w_1^2} - \frac{3 E^{-2 I T_0} W - 2 I t_2 d_4 L_2^2}{5 w_1^2} - \frac{3 E^{2 I T_0} W + 2 I t_2 d_4 L_2^2}{5 w_1^2} - \\
 &\frac{2 A_2 d_2 \dot{A}_2}{9 w_1^2} - \frac{2 E^{I T_0} w_1 - I T_0 w_2 A_2 d_3 \dot{A}_2}{5 w_1^2} - \frac{2 E^{-I T_0} w_1 + I T_0 w_2 A_2 d_3 \dot{A}_2}{5 w_1^2} - \frac{2 A_2 d_4 \dot{A}_2}{3 w_1^2} - \\
 &\frac{2 E^{-I T_0} W - I t_2 - I T_0 w_1 d_3 L_2 \dot{A}_2}{5 w_1^2} - \frac{2 E^{I T_0} W + I t_2 - I T_0 w_1 d_3 L_2 \dot{A}_2}{9 w_1^2} + \frac{6 E^{-I T_0} W - I t_2 - I T_0 w_2 d_4 L_2 \dot{A}_2}{7 w_1^2} - \\
 &\frac{6 E^{I T_0} W + I t_2 - I T_0 w_2 d_4 L_2 \dot{A}_2}{5 w_1^2} - \frac{E^{-2 I T_0} w_1 d_2 \dot{A}_2^2}{5 w_1^2} + \frac{2 E^{-I T_0} w_1 - I T_0 w_2 d_3 \dot{A}_2^2}{7 w_1^2} + \frac{E^{-2 I T_0} w_2 d_4 \dot{A}_2^2}{9 w_1^2} \frac{y}{z} \Bigg\}
 \end{aligned}$$

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

$$\begin{aligned}
 2 I w_1 A_1^i &= \\
 2 I e^2 w_1 \int_k &\left\{ -A_1 m_1 - \frac{5 I E^{-I T_2} s_1 - I T_2 s_2 - I t_2 A_1^2 d_1 d_2 L_2}{w_1^3} - \frac{19 I E^{-I T_2} s_1 - I T_2 s_2 - I t_2 A_1^2 d_2 d_3 L_2}{45 w_1^3} - \frac{4 I A_1 d_1^2 L_2^2}{3 w_1^3} - \right. \\
 &\frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} + \frac{I E^{-I T_2} s_1 - I T_2 s_2 - 2 I t_2 A_2 d_2 d_3 L_2^2}{w_1^3} - \frac{28 I A_1 d_3^2 L_2^2}{45 w_1^3} -
 \end{aligned}$$

872.154 Second, Null<

§ 6.1.7 $W \gg w_1 + w_2$ and $w_2 \gg w_1$

In this case, we have a subharmonic resonance of either mode and a one-to-one internal resonance. We let

$$\text{scaling7} = 8m_n \rightarrow e m_n, F_1 \rightarrow e f_1, F_2 \rightarrow e f_2;$$

$$\text{ResonanceCond7} = 8w_2 == w_1 + e s_1, W == w_1 + w_2 + e s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling7, ResonanceCond7D •• Timing

The second-order approximate solution:

$$\begin{aligned} u_1 @ tD &= e^{i E T_0 w_1} A_1 + E^{-i T_0 W - i t_1} L_1 + E^{i T_0 W + i t_1} L_1 + E^{-i T_0 w_1} \dot{A}_1 M + \\ &e^2 \left\{ \begin{aligned} &\frac{E^2 i T_0 w_1 A_1^2 d_1}{w_1^2} + \frac{2 E i T_0 w_1 + i T_0 w_2 A_1^2 d_2}{3 w_1^2} + \frac{E^2 i T_0 w_2 A_1^2 d_3}{3 w_1^2} - \frac{3 E^{-i T_0 W - i t_1 + i T_0 w_1} A_1 d_1 L_1}{2 w_1^2} + \\ &\frac{3 E i T_0 W + i t_1 + i T_0 w_1 A_1 d_1 L_1}{4 w_1^2} - \frac{E^{-i T_0 W - i t_1 + i T_0 w_2} A_1 d_2 L_1}{2 w_1^2} + \frac{E i T_0 W + i t_1 + i T_0 w_2 A_1 d_2 L_1}{4 w_1^2} - \frac{6 d_1 L_1^2}{w_1^2} + \\ &\frac{E^{-2 i T_0 W - 2 i t_1} d_1 L_1^2}{5 w_1^2} + \frac{E^2 i T_0 W + 2 i t_1 d_1 L_1^2}{5 w_1^2} - \frac{E^{-i T_0 W - i t_2 + i T_0 w_1} A_1 d_2 L_2}{2 w_1^2} + \frac{E i T_0 W + i t_2 + i T_0 w_1 A_1 d_2 L_2}{4 w_1^2} - \\ &\frac{E^{-i T_0 W - i t_2 + i T_0 w_2} A_1 d_3 L_2}{2 w_1^2} + \frac{E i T_0 W + i t_2 + i T_0 w_2 A_1 d_3 L_2}{4 w_1^2} + \frac{2 E^{-2 i T_0 W - i t_1 - i t_2} d_2 L_1 L_2}{15 w_1^2} - \\ &\frac{2 E i t_1 - i t_2 d_2 L_1 L_2}{w_1^2} - \frac{2 E^{-i t_1 + i t_2} d_2 L_1 L_2}{w_1^2} + \frac{2 E^2 i T_0 W + i t_1 + i t_2 d_2 L_1 L_2}{15 w_1^2} - \frac{2 d_3 L_1^2}{w_1^2} + \\ &\frac{E^{-2 i T_0 W - 2 i t_2} d_3 L_1^2}{15 w_1^2} + \frac{E^2 i T_0 W + 2 i t_2 d_3 L_1^2}{15 w_1^2} - \frac{I E i T_0 w_1 A_1 m_1}{2 w_1} - \frac{4 I E^{-i T_0 W - i t_1} L_1 m_1}{3 w_1} + \\ &\frac{4 I E i T_0 W + i t_1 L_1 m_1}{3 w_1} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{2 E i T_0 w_1 - i T_0 w_2 A_1 d_2 \dot{A}_1}{w_1^2} - \frac{2 E^{-i T_0 w_1 + i T_0 w_2} A_1 d_2 \dot{A}_1}{w_1^2} - \\ &\frac{2 A_1 d_3 \dot{A}_1}{w_1^2} + \frac{3 E^{-i T_0 W - i t_1 - i T_0 w_1} d_1 L_1 \dot{A}_1}{4 w_1^2} - \frac{3 E i T_0 W + i t_1 - i T_0 w_1 d_1 L_1 \dot{A}_1}{2 w_1^2} + \\ &\frac{E^{-i T_0 W - i t_1 - i T_0 w_2} d_2 L_1 \dot{A}_1}{4 w_1^2} - \frac{E i T_0 W + i t_1 - i T_0 w_2 d_2 L_1 \dot{A}_1}{2 w_1^2} + \frac{E^{-i T_0 W - i t_2 - i T_0 w_1} d_2 L_2 \dot{A}_1}{4 w_1^2} - \\ &\frac{E i T_0 W + i t_2 - i T_0 w_1 d_2 L_2 \dot{A}_1}{2 w_1^2} + \frac{E^{-i T_0 W - i t_2 - i T_0 w_2} d_3 L_2 \dot{A}_1}{4 w_1^2} - \frac{E i T_0 W + i t_2 - i T_0 w_2 d_3 L_2 \dot{A}_1}{2 w_1^2} + \\ &\frac{I E^{-i T_0 w_1} m_1 \dot{A}_1}{2 w_1} + \frac{E^{-2 i T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} + \frac{2 E^{-i T_0 w_1 - i T_0 w_2} d_2 \dot{A}_1^2}{3 w_1^2} + \frac{E^{-2 i T_0 w_2} d_3 \dot{A}_1^2}{3 w_1^2} \frac{1}{2}, \end{aligned} \right\} \\ u_2 @ tD &= e^{i E T_0 w_2} A_2 + E^{-i T_0 W - i t_2} L_2 + E^{i T_0 W + i t_2} L_2 + E^{-i T_0 w_2} \dot{A}_2 M + \\ &e^2 \left\{ \begin{aligned} &\frac{E^2 i T_0 w_1 A_2^2 d_2}{3 w_1^2} + \frac{2 E i T_0 w_1 + i T_0 w_2 A_2^2 d_3}{3 w_1^2} + \frac{E^2 i T_0 w_2 A_2^2 d_4}{w_1^2} - \frac{E^{-i T_0 W - i t_1 + i T_0 w_1} A_2 d_2 L_1}{2 w_1^2} + \\ &\frac{E i T_0 W + i t_1 + i T_0 w_1 A_2 d_2 L_1}{4 w_1^2} - \frac{E^{-i T_0 W - i t_1 + i T_0 w_2} A_2 d_3 L_1}{2 w_1^2} + \frac{E i T_0 W + i t_1 + i T_0 w_2 A_2 d_3 L_1}{4 w_1^2} - \\ &\frac{2 d_2 L_1^2}{w_1^2} + \frac{E^{-2 i T_0 W - 2 i t_1} d_2 L_1^2}{15 w_1^2} + \frac{E^2 i T_0 W + 2 i t_1 d_2 L_1^2}{15 w_1^2} - \frac{E^{-i T_0 W - i t_2 + i T_0 w_1} A_2 d_3 L_2}{2 w_1^2} + \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned}
& \frac{E^{I T_0 W+I t_2+I T_0 w_1} A_2 d_3 L_2}{4 w_1^2} - \frac{3 E^{-I T_0 W-I t_2+I T_0 w_2} A_2 d_4 L_2}{2 w_1^2} + \frac{3 E^{I T_0 W+I t_2+I T_0 w_2} A_2 d_4 L_2}{4 w_1^2} + \\
& \frac{2 E^{-2 I T_0 W-I t_1-I t_2} d_3 L_1 L_2}{15 w_1^2} - \frac{2 E^{I t_1-I t_2} d_3 L_1 L_2}{w_1^2} - \frac{2 E^{-I t_1+I t_2} d_3 L_1 L_2}{w_1^2} + \\
& \frac{2 E^{2 I T_0 W+I t_1+I t_2} d_3 L_1 L_2}{15 w_1^2} - \frac{6 d_4 L_2^2}{w_1^2} + \frac{E^{-2 I T_0 W-2 I t_2} d_4 L_2^2}{5 w_1^2} + \frac{E^{2 I T_0 W+2 I t_2} d_4 L_2^2}{5 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{2 w_1} - \\
& \frac{4 I E^{-I T_0 W-I t_2} L_2 m_2}{3 w_1} + \frac{4 I E^{I T_0 W+I t_2} L_2 m_2}{3 w_1} - \frac{2 A_2 d_2 \dot{A}_2}{w_1^2} - \frac{2 E^{I T_0 w_1-I T_0 w_2} A_2 d_3 \dot{A}_2}{w_1^2} - \\
& \frac{2 E^{-I T_0 w_1+I T_0 w_2} A_2 d_3 \dot{A}_2}{w_1^2} - \frac{6 A_2 d_4 \dot{A}_2}{w_1^2} + \frac{E^{-I T_0 W-I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{4 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{2 w_1^2} + \\
& \frac{E^{-I T_0 W-I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{4 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W-I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{4 w_1^2} - \\
& \frac{E^{I T_0 W+I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{2 w_1^2} + \frac{3 E^{-I T_0 W-I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{4 w_1^2} - \frac{3 E^{I T_0 W+I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{2 w_1^2} + \\
& \frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{2 w_1} + \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_2^2}{3 w_1^2} + \frac{2 E^{-I T_0 w_1-I T_0 w_2} d_3 \dot{A}_2^2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{w_1^2} \Bigg\} >
\end{aligned}$$

where

$$L_i = \frac{f_i}{2 H w_1^2 - W^2 L}$$

The modulation equations:

$$\begin{aligned}
: 2 I w_1 A_1^i &= 2 I e^{w_1} \Bigg\{ - \frac{81 I A_1 d_2^2 L_2^2}{4 w_1^2} - \frac{27 I E^{I T_1 s_1} A_2 d_1 d_2 L_2^2}{4 w_1^2} - \frac{9 I A_1 d_2^2 L_2^2}{4 w_1^2} - \frac{9 I E^{I T_1 s_1} A_2 d_2 d_3 L_1^2}{4 w_1^2} - \\
& \frac{27 I E^{I t_1-I t_2} A_1 d_1 d_2 L_1 L_2}{4 w_1^2} - \frac{27 I E^{-I t_1+I t_2} A_1 d_1 d_2 L_1 L_2}{4 w_1^2} - \frac{7 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_2^2 L_1 L_2}{4 w_1^2} - \\
& \frac{5 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_2^2 L_1 L_2}{2 w_1^3} - \frac{3 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_1 d_3 L_1 L_2}{2 w_1^3} + \frac{3 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_1 d_3 L_1 L_2}{4 w_1^3} - \\
& \frac{9 I E^{I t_1-I t_2} A_1 d_2 d_3 L_1 L_2}{4 w_1^3} - \frac{9 I E^{-I t_1+I t_2} A_1 d_2 d_3 L_1 L_2}{4 w_1^3} - \frac{7 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_2^2 L_1 L_2}{4 w_1^3} - \\
& \frac{5 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_2^2 L_1 L_2}{2 w_1^3} - \frac{3 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_2 d_4 L_1 L_2}{2 w_1^3} + \frac{3 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_2 d_4 L_1 L_2}{4 w_1^3} - \\
& \frac{I A_1 d_2^2 L_2^2}{4 w_1^3} - \frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} - \frac{9 I E^{I T_1 s_1} A_2 d_2 d_3 L_2^2}{4 w_1^3} - \frac{I A_1 d_3^2 L_2^2}{4 w_1^3} - \frac{6 I A_1 d_2 d_4 L_2^2}{w_1^3} - \\
& \frac{27 I E^{I T_1 s_1} A_2 d_3 d_4 L_2^2}{4 w_1^3} + \frac{12 I A_1 a_1 L_1^2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2 a_2 L_1^2}{w_1} + \frac{3 I E^{I t_1-I t_2} A_1 a_2 L_1 L_2}{w_1} + \\
& \frac{3 I E^{-I t_1+I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{2 I E^{I T_1 s_1+I t_1-I t_2} A_2 a_3 L_1 L_2}{w_1} + \frac{2 I E^{I T_1 s_1-I t_1+I t_2} A_2 a_3 L_1 L_2}{w_1} + \\
& \frac{2 I A_1 a_3 L_2^2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2 a_4 L_2^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} - \frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^2} - \frac{10 I E^{I T_1 s_1} A_1 A_2 d_1 d_2 \dot{A}_1}{w_1^2} - \\
& \frac{5 I A_1^2 d_2^2 \dot{A}_1}{3 w_1^3} - \frac{2 I E^{2 I T_1 s_1} A_2^2 d_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_1 s_1} A_2^2 d_1 d_2 \dot{A}_1}{w_1^3} - \frac{10 I E^{I T_1 s_1} A_1 A_2 d_2 d_3 \dot{A}_1}{3 w_1^3} - \\
& \frac{2 I E^{2 I T_1 s_1} A_2^2 d_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_1 s_1} A_2^2 d_2 d_4 \dot{A}_1}{w_1^3} - \frac{7 E^{I T_1 s_1+I T_1 s_2+I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^3} -
\end{aligned}$$

$$\begin{aligned}
& \frac{E^{I T_1 s_1 + I T_1 s_2 + I t_2} d_2 L_2 m_1 \dot{A}_1}{w_1^3} - \frac{4 E^{I T_1 s_1 + I T_1 s_2 + I t_2} d_2 L_2 m_2 \dot{A}_1}{3 w_1^3} + \frac{6 I A_2^2 a_1 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1} A_1 A_2 a_2 \dot{A}_1}{w_1} + \\
& \frac{I E^{2 I T_1 s_1} A_2^2 a_3 \dot{A}_1}{w_1} - \frac{5 I E^{-I T_1 s_1} A_1^2 d_1 d_2 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \\
& \frac{5 I E^{-I T_1 s_1} A_1^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{5 I E^{I T_1 s_1} A_2^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^3} - \\
& \frac{5 I E^{I T_1 s_1} A_2^2 d_3 d_4 \dot{A}_2}{w_1^3} - \frac{11 E^{I T_1 s_2 + I t_1} d_2 L_1 m_1 \dot{A}_2}{6 w_1^2} - \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_1 \dot{A}_2}{2 w_1^2} - \frac{E^{I T_1 s_2 + I t_1} d_2 L_1 m_2 \dot{A}_2}{2 w_1^2} - \\
& \frac{11 E^{I T_1 s_2 + I t_2} d_3 L_2 m_2 \dot{A}_2}{6 w_1^2} + \frac{3 I E^{-I T_1 s_1} A_1^2 a_2 \dot{A}_2}{2 w_1} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2^2 a_4 \dot{A}_2}{2 w_1} \frac{y}{z} + \\
& 2 I e w_1 \Bigg|_k - A_1 m_1 + \frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \dot{A}_1}{w_1} + \frac{I E^{I T_1 s_1 + I T_1 s_2 + I t_2} d_2 L_2 \dot{A}_1}{w_1} + \\
& \frac{I E^{I T_1 s_2 + I t_1} d_2 L_1 \dot{A}_2}{w_1} + \frac{I E^{I T_1 s_2 + I t_2} d_3 L_2 \dot{A}_2}{w_1} \frac{y}{z}, \\
& 2 I w_2 A_2^2 = 2 I e w_2 \Bigg|_k - A_2 m_2 + \frac{I E^{I T_1 s_2 + I t_1} d_2 L_1 \dot{A}_1}{w_2} + \frac{I E^{I T_1 s_2 + I t_2} d_3 L_2 \dot{A}_1}{w_2} + \\
& \frac{I E^{-I T_1 s_1 + I T_1 s_2 + I t_1} d_3 L_1 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1 + I T_1 s_2 + I t_2} d_4 L_2 \dot{A}_2}{w_2} \frac{y}{z} + \\
& 2 I e^2 w_2 \Bigg|_k - \frac{I A_2 m_2^2}{2 w_1} + \frac{3 I E^{-I T_1 s_1} A_1 a_2 L_2^2}{w_2} + \frac{2 I A_2 a_3 L_2^2}{w_2} + \frac{2 I E^{-I T_1 s_1 + I t_1 - I t_2} A_1 a_3 L_1 L_2}{w_2} + \\
& \frac{2 I E^{-I T_1 s_1 - I t_1 + I t_2} A_1 a_3 L_1 L_2}{w_2} + \frac{3 I E^{I t_1 - I t_2} A_2 a_4 L_1 L_2}{w_2} + \frac{3 I E^{-I t_1 + I t_2} A_2 a_4 L_1 L_2}{w_2} + \\
& \frac{3 I E^{-I T_1 s_1} A_1 a_4 L_2^2}{w_2} + \frac{12 I A_2 a_5 L_2^2}{w_2} - \frac{27 I E^{-I T_1 s_1} A_1 d_1 d_2 L_2^2}{4 w_1^2 w_2} - \frac{I A_2 d_2^2 L_2^2}{4 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_2^2}{w_1^2 w_2} - \\
& \frac{9 I E^{-I T_1 s_1} A_1 d_2 d_3 L_1^2}{4 w_1^2 w_2} - \frac{I A_2 d_2^2 L_1^2}{4 w_1^2 w_2} - \frac{6 I A_2 d_2 d_4 L_1^2}{w_1^2 w_2} - \frac{5 I E^{-I T_1 s_1 + I t_1 - I t_2} A_1 d_2^2 L_1 L_2}{2 w_1^2 w_2} - \\
& \frac{7 I E^{-I T_1 s_1 - I t_1 + I t_2} A_1 d_2^2 L_1 L_2}{4 w_1^2 w_2} + \frac{3 I E^{-I T_1 s_1 + I t_1 - I t_2} A_1 d_1 d_3 L_1 L_2}{4 w_1^2 w_2} - \\
& \frac{3 I E^{-I T_1 s_1 - I t_1 + I t_2} A_1 d_1 d_3 L_1 L_2}{2 w_1^2 w_2} - \frac{9 I E^{I t_1 - I t_2} A_2 d_2 d_3 L_1 L_2}{4 w_1^2 w_2} - \frac{9 I E^{-I t_1 + I t_2} A_2 d_2 d_3 L_1 L_2}{4 w_1^2 w_2} - \\
& \frac{5 I E^{-I T_1 s_1 + I t_1 - I t_2} A_1 d_2^2 L_1 L_2}{2 w_1^2 w_2} - \frac{7 I E^{-I T_1 s_1 - I t_1 + I t_2} A_1 d_2^2 L_1 L_2}{4 w_1^2 w_2} + \frac{3 I E^{-I T_1 s_1 + I t_1 - I t_2} A_1 d_2 d_4 L_1 L_2}{4 w_1^2 w_2} - \\
& \frac{3 I E^{-I T_1 s_1 - I t_1 + I t_2} A_1 d_2 d_4 L_1 L_2}{2 w_1^2 w_2} - \frac{27 I E^{I t_1 - I t_2} A_2 d_3 d_4 L_1 L_2}{4 w_1^2 w_2} - \frac{27 I E^{-I t_1 + I t_2} A_2 d_3 d_4 L_1 L_2}{4 w_1^2 w_2} - \\
& \frac{9 I E^{-I T_1 s_1} A_1 d_2 d_3 L_2^2}{4 w_1^2 w_2} - \frac{9 I A_2 d_2^2 L_2^2}{4 w_1^2 w_2} - \frac{27 I E^{-I T_1 s_1} A_1 d_3 d_4 L_2^2}{4 w_1^2 w_2} - \frac{81 I A_2 d_2^2 L_2^2}{4 w_1^2 w_2} - \\
& \frac{E^{I T_1 s_2 + I t_1} d_2 L_1 m_2 \dot{A}_1}{2 w_1^2} - \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_2 \dot{A}_1}{2 w_1^2} + \frac{3 I E^{-I T_1 s_1} A_2^2 a_2 \dot{A}_1}{2 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \\
& \frac{3 I E^{I T_1 s_1} A_2^2 a_4 \dot{A}_1}{2 w_2} - \frac{5 I E^{-I T_1 s_1} A_1^2 d_1 d_2 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \\
& \frac{5 I E^{-I T_1 s_1} A_1^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{5 I E^{I T_1 s_1} A_2^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2 w_2} - \\
& \frac{5 I E^{I T_1 s_1} A_2^2 d_3 d_4 \dot{A}_1}{w_1^2 w_2} - \frac{11 E^{I T_1 s_2 + I t_1} d_2 L_1 m_1 \dot{A}_1}{6 w_1 w_2} - \frac{E^{I T_1 s_2 + I t_2} d_3 L_2 m_1 \dot{A}_1}{2 w_1 w_2} -
\end{aligned}$$

$$\begin{aligned}
& \frac{4 E^{I T_1 s_2 + I t_2} d_3 L_2 m_2 \dot{A}_1}{3 w_1 w_2} - \frac{E^{-I T_1 s_1 + I T_1 s_2 + I t_1} d_3 L_1 m_2 \dot{A}_2}{2 w_1^2} - \frac{3 E^{-I T_1 s_1 + I T_1 s_2 + I t_2} d_4 L_2 m_2 \dot{A}_2}{2 w_1^2} + \\
& \frac{I E^{-2 I T_1 s_1} A_1^2 a_3 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1} A_1 A_2 a_4 \dot{A}_2}{w_2} + \frac{6 I A_1^2 a_5 \dot{A}_2}{w_2} - \frac{2 I E^{-2 I T_1 s_1} A_1^2 d_3^2 \dot{A}_2}{w_1^2 w_2} + \\
& \frac{I E^{-2 I T_1 s_1} A_1^2 d_1 d_3 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_1 s_1} A_1 A_2 d_2 d_3 \dot{A}_2}{3 w_1^2 w_2} - \frac{2 I E^{-2 I T_1 s_1} A_1^2 d_3^2 \dot{A}_2}{w_1^2 w_2} - \\
& \frac{5 I A_1^2 d_3^2 \dot{A}_2}{3 w_1^2 w_2} + \frac{I E^{-2 I T_1 s_1} A_1^2 d_2 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_1 s_1} A_1 A_2 d_3 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{15 I A_1^2 d_4^2 \dot{A}_2}{w_1^2 w_2} - \\
& \frac{4 E^{-I T_1 s_1 + I T_1 s_2 + I t_1} d_3 L_1 m_1 \dot{A}_2}{3 w_1 w_2} - \frac{E^{-I T_1 s_1 + I T_1 s_2 + I t_1} d_3 L_1 m_2 \dot{A}_2}{2 w_1 w_2} - \frac{11 E^{-I T_1 s_1 + I T_1 s_2 + I t_2} d_4 L_2 m_2 \dot{A}_2}{2 w_1 w_2} \Bigg\}
\end{aligned}$$

8194.159 Second, Null<

§ 6.1.8 $W \gg 2 w_2 - w_1$ and $w_2 \gg w_1$

In this case, we have a primary resonance of either mode and a one-to-one internal resonance. We let

$$\text{scaling8} = 9 m_n \rightarrow e^2 m_n, F_1 \rightarrow e^3 f_1, F_2 \rightarrow e^3 f_2;$$

$$\text{ResonanceCond8} = 9 w_2 == w_1 + e^2 s_1, W == 2 w_2 - w_1 + e^2 s_2;$$

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k :

MMS@scaling8, ResonanceCond8D •• Timing

The second-order approximate solution:

$$\begin{aligned}
& : u_1 @ t D == e^{I E^{I T_0 w_1} A_1 + E^{-I T_0 w_1} \dot{A}_1 M} + \\
& e^2 \Bigg\{ \frac{E^{2 I T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2}{3 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_3}{3 w_1^2} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{w_1^2} + \\
& \frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \dot{A}_2^2}{3 w_1^2} \Bigg\}, \\
& u_2 @ t D == e^{I E^{I T_0 w_2} A_2 + E^{-I T_0 w_2} \dot{A}_2 M} + e^2 \Bigg\{ \frac{E^{2 I T_0 w_1} A_1^2 d_2}{3 w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3}{3 w_1^2} + \\
& \frac{E^{2 I T_0 w_2} A_2^2 d_4}{w_1^2} - \frac{2 A_1 d_2 \dot{A}_1}{w_1^2} - \frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1}{w_1^2} + \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{3 w_1^2} - \\
& \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2}{w_1^2} - \frac{6 A_2 d_4 \dot{A}_2}{w_1^2} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{w_1^2} \Bigg\}
\end{aligned}$$

The modulation equations:

$$\begin{aligned}
& : 2 I w_1 A_1^{\epsilon} == \\
& 2 I e^2 w_1 \left\{ -A_1 m_1 - \frac{I E^{2 I T_2 s_1 + I T_2 s_2 + I t_1} f_1}{4 w_1} - \frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^3} - \frac{10 I E^{I T_2 s_1} A_1 A_2 d_1 d_2 \dot{A}_1}{w_1^3} - \frac{5 I A_1^2 d_2^2 \dot{A}_1}{3 w_1^3} - \right. \\
& \frac{2 I E^{2 I T_2 s_1} A_2^2 d_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_2 s_1} A_2^2 d_1 d_3 \dot{A}_1}{w_1^3} - \frac{10 I E^{I T_2 s_1} A_1 A_2 d_2 d_3 \dot{A}_1}{3 w_1^3} - \\
& \frac{2 I E^{2 I T_2 s_1} A_2^2 d_3^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_2 s_1} A_2^2 d_2 d_4 \dot{A}_1}{w_1^3} + \frac{6 I A_1^2 a_1 \dot{A}_1}{w_1} + \frac{3 I E^{I T_2 s_1} A_1 A_2 a_2 \dot{A}_1}{w_1} + \\
& \frac{I E^{2 I T_2 s_1} A_2^2 a_3 \dot{A}_1}{w_1} - \frac{5 I E^{-I T_2 s_1} A_1^2 d_1 d_2 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \\
& \frac{5 I E^{-I T_2 s_1} A_1^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{5 I E^{I T_2 s_1} A_2^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^3} - \\
& \frac{5 I E^{I T_2 s_1} A_2^2 d_3 d_4 \dot{A}_2}{w_1^3} + \frac{3 I E^{-I T_2 s_1} A_2^2 a_2 \dot{A}_2}{2 w_1} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_2 s_1} A_2^2 a_4 \dot{A}_2}{2 w_1} \Big\}, \\
& 2 I w_2 A_2^{\epsilon} == 2 I e^2 w_2 \left\{ -A_2 m_2 - \frac{I E^{I T_2 s_1 + I T_2 s_2 + I t_2} f_2}{4 w_2} + \frac{3 I E^{-I T_2 s_1} A_1^2 a_2 \dot{A}_1}{2 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} + \right. \\
& \frac{3 I E^{I T_2 s_1} A_2^2 a_4 \dot{A}_1}{2 w_2} - \frac{5 I E^{-I T_2 s_1} A_1^2 d_1 d_2 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} - \\
& \frac{5 I E^{-I T_2 s_1} A_1^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{5 I E^{I T_2 s_1} A_2^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2 w_2} - \\
& \frac{5 I E^{I T_2 s_1} A_2^2 d_3 d_4 \dot{A}_1}{w_1^2 w_2} + \frac{I E^{-2 I T_2 s_1} A_2^2 a_3 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_2 s_1} A_1 A_2 a_4 \dot{A}_2}{w_2} + \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} - \\
& \frac{2 I E^{-2 I T_2 s_1} A_1^2 d_2^2 \dot{A}_2}{w_1^2 w_2} + \frac{I E^{-2 I T_2 s_1} A_2^2 d_1 d_3 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_2 s_1} A_1 A_2 d_2 d_3 \dot{A}_2}{3 w_1^2 w_2} - \frac{2 I E^{-2 I T_2 s_1} A_2^2 d_2^2 \dot{A}_2}{w_1^2 w_2} - \\
& \frac{5 I A_2^2 d_3^2 \dot{A}_2}{3 w_1^2 w_2} + \frac{I E^{-2 I T_2 s_1} A_2^2 d_2 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_2 s_1} A_1 A_2 d_3 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{15 I A_2^2 d_4^2 \dot{A}_2}{w_1^2 w_2} \Big\}
\end{aligned}$$

816.664 Second, Null<

à 6.2 Parametrically Excited Linearly Coupled Systems

¶ 6.2.1 Two-Pendulum Oscillator

As an example, we consider a parametrically excited two-pendulum oscillator with a three-to-one internal resonance:

$$\begin{aligned}
\text{eq621a} &= 9 q_1^2 @tD + a q_2^2 @tD + q_1 @tD == \frac{1}{2} a H q_2 @tD - q_1 @tD L^2 q_2^2 @tD + \frac{1}{6} q_1 @tD^3 + \\
& a H q_2 @tD - q_1 @tD L q_2^{\epsilon} @tD^2 - 2 F_1 W^2 q_1 @tD \text{Cos}@W tD, q_2^2 @tD + q_1^2 @tD + q_2 @tD == \\
& \frac{1}{2} H q_2 @tD - q_1 @tD L^2 q_1^2 @tD + \frac{1}{6} q_2 @tD^3 - H q_2 @tD - q_1 @tD L q_1^{\epsilon} @tD^2 - 2 F_1 W^2 q_2 @tD \text{Cos}@W tD =;
\end{aligned}$$

where

$$\text{params} = 9a \rightarrow \frac{16}{25} + e^2 s_1 =;$$

for small ϵ . Here, the excitation frequency, W , is assumed to be close to twice the natural frequency of the second mode, w_2 , which is three times the natural frequency of the first mode, w_1 . Hence, we define the following lists:

```
omgList = {w1, w2};

ResonanceCond = {W == 2 w2 + e^2 s2, w2 == 3 w1};

omgRule = Solve[ResonanceCond, W, # < Flatten[D@@1DD & # omgList & Reverse
: 8W @ e^2 s2 + 6 w1, w2 @ 3 w1 <, : W @ e^2 s2 + 2 w2, w1 @  $\frac{w_2}{3}$  >>
```

In order to bring the effects of forcing and nonlinearity at the same order, we let

```
scaling = {F1 -> e^2 f1};
```

For a consistent expansion, we first transform [eq621a](#) into a system of four first-order equations. To this end, we introduce the two states $v_1(t)$ and $v_2(t)$ defined by

```
vel = {q1'[t] -> v1[t], q2'[t] -> v2[t];
```

Substituting the velocity and acceleration terms, using [vel](#), into [eq621a](#) and combining the result with [vel](#), we transform [eq621a](#) into the following set of four first-order equations:

```
eq621b = {vel . Rule -> Equal, eq621a . D@vel, t . vel < . Transpose . Flatten
9 q1'[t] == v1[t], q1[t] + v1'[t] + a v2'[t] ==
- 2 W^2 Cos[t] W D F1 q1[t] +  $\frac{1}{6}$  q1[t]^3 + a v2[t] H - q1[t] + q2[t] DL +  $\frac{1}{2}$  a H - q1[t] + q2[t] DL^2 v2[t],
q2'[t] == v2[t], q2[t] + v2'[t] + v2[t] ==
- 2 W^2 Cos[t] W D F1 q2[t] +  $\frac{1}{6}$  q2[t]^3 - v1[t] D^2 H - q1[t] + q2[t] DL +  $\frac{1}{2}$  H - q1[t] + q2[t] DL^2 v1[t]
```

We seek a uniform second-order expansion of the solution of [eq621b](#) in the form

```
solRule =
9 qi_ -> ISum[Ae^j qi,j@{#1, #2, #3D, 8j, 3<E &M, vi_ -> ISum[Ae^j vi,j@{#1, #2, #3D, 8j, 3<E &M=;

maxOrder = 2;
```

Using the time scales T_0 , T_1 , and T_2 , we express the dependent variables and their time derivatives as

```
multiScales = {qi_@t -> qi_ @@ timeScales, vi_@t -> vi_ @@ timeScales,
Derivative@1D@u_D@t -> dt@1D@u @@ timeScalesD, t -> T0};
```

Substituting the [params](#), [scaling](#), [multiScales](#), and [solRule](#) into [eq621b](#), expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq621c = {eq621b . params . scaling . multiScales . solRule . TrigToExp . ExpandAll .
e^n_ ; n>3 -> 0};
```

Equating coefficients of like powers of e , we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq621c •• Transpose;
```

To place the linear operator on one side and the nonhomogeneous terms on the otherside, we define

```
eqOrder@i_D := I#@1DD & •ž eqEps@1DD •. f_s_ -> 0 •. u_{k,1} -> u_{k,i}M ==  
I#@1DD & •ž eqEps@1DD •. f_s_ -> 0 •. u_{k,1} -> u_{k,i}M - H#@1DD & •ž eqEps@iDDL •• Thread
```

Using the `eqOrder[i]` and `displayRule`, we rewrite `eqEps` in a concise way as

```
eqOrder@1D •. displayRule  
eqOrder@2D •. displayRule  
eqOrder@3D •. displayRule
```

$$9D_0q_{1,1} - v_{1,1} == 0, D_0v_{1,1} + \frac{16}{25}HD_0v_{2,1}L + q_{1,1} == 0, D_0q_{2,1} - v_{2,1} == 0, D_0v_{1,1} + D_0v_{2,1} + q_{2,1} == 0=$$

$$9D_0q_{1,2} - v_{1,2} == -HD_1q_{1,1}L, D_0v_{1,2} + \frac{16}{25}HD_0v_{2,2}L + q_{1,2} == -HD_1v_{1,1}L - \frac{16}{25}HD_1v_{2,1}L,$$

$$D_0q_{2,2} - v_{2,2} == -HD_1q_{2,1}L, D_0v_{1,2} + D_0v_{2,2} + q_{2,2} == -HD_1v_{1,1}L - D_1v_{2,1}=$$

$$:D_0q_{1,3} - v_{1,3} == -HD_1q_{1,2}L - D_2q_{1,1},$$

$$D_0v_{1,3} + \frac{16}{25}HD_0v_{2,3}L + q_{1,3} == -HD_1v_{1,2}L - \frac{16}{25}HD_1v_{2,2}L - D_2v_{1,1} - \frac{16}{25}HD_2v_{2,1}L -$$

$$HD_0v_{2,1}L s_1 - E^{-IT_0W}W^2f_1q_{1,1} - E^{IT_0W}W^2f_1q_{1,1} - \frac{16}{25}v_{2,1}^2q_{1,1} + \frac{8}{25}HD_0v_{2,1}Lq_{1,1}^2 + \frac{q_{1,1}^3}{6} +$$

$$\frac{16}{25}v_{2,1}^2q_{2,1} - \frac{16}{25}HD_0v_{2,1}Lq_{1,1}q_{2,1} + \frac{8}{25}HD_0v_{2,1}Lq_{2,1}^2, D_0q_{2,3} - v_{2,3} == -HD_1q_{2,2}L - D_2q_{2,1},$$

$$D_0v_{1,3} + D_0v_{2,3} + q_{2,3} == -HD_1v_{1,2}L - D_1v_{2,2} - D_2v_{1,1} - D_2v_{2,1} + v_{1,1}^2q_{1,1} + \frac{1}{2}HD_0v_{1,1}Lq_{1,1}^2 -$$

$$E^{-IT_0W}W^2f_1q_{2,1} - E^{IT_0W}W^2f_1q_{2,1} - v_{1,1}^2q_{2,1} - HD_0v_{1,1}Lq_{1,1}q_{2,1} + \frac{1}{2}HD_0v_{1,1}Lq_{2,1}^2 + \frac{q_{2,1}^3}{6} >$$

Ÿ First-Order Problem: Linear System

The first-order problem, `eqOrder[1]`, consists of a set of coupled linear homogeneous differential equations. Hence, the general solution is the solution of the homogeneous equations. To determine the general solution of the homogeneous set, we rewrite `eqOrder[1]` as

```
linearSys = #@@1DD & •ž eqOrder@1D;  
linearSys •. displayRule
```

$$9D_0q_{1,1} - v_{1,1}, D_0v_{1,1} + \frac{16}{25}HD_0v_{2,1}L + q_{1,1}, D_0q_{2,1} - v_{2,1}, D_0v_{1,1} + D_0v_{2,1} + q_{2,1}=$$

Next, we seek a solution of the `linearSys` in the form

```
assumedForm = 9qi-1 -> Ici EIw# &M, vi-1 -> Idi EIw# &M;
```

Substituting **assumedForm** into **linearSys** and collecting coefficients of c_i and d_i , we obtain the coefficient matrix as

```
coefMat = OuterAD, E-IwT0 linearSys •. assumedForm •• Expand, 8c1, d1, c2, d2<E
```

```
98I w, -1, 0, 0<, 91, I w, 0,  $\frac{16 I w}{25}$  =, 80, 0, I w, -1<, 80, I w, 1, I w<=
```

whose adjoint is defined by

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

The natural frequencies of the two modes involved are

```
evals = Cases@w •. Solve@Det@coefMatD == 0, wD, n_ •; n > 0D
```

```
:  $\frac{5 \sqrt{3}}{3}$ ,  $\frac{5 \sqrt{3}}{5}$ >
```

Hence, the right and left eigenvectors of **coefMat** corresponding to **evals** are

```
rightVec = # •. #@@1DD & •ž HNullSpace@coefMat •. w -> #D@@1DD & •ž evalsL
```

```
:: 1,  $\frac{I \sqrt{5}}{3}$ ,  $\frac{5}{4}$ ,  $\frac{5 I \sqrt{5}}{12}$ >, : 1,  $I \sqrt{5}$ ,  $-\frac{5}{4}$ ,  $-\frac{5 I \sqrt{5}}{4}$ >>
```

```
leftVec = NullSpace@hermitian@coefMatD •. w -> #D@@1DD & •ž evals
```

```
:: -  $\frac{3 I \sqrt{5}}{4}$ ,  $\frac{5}{4}$ ,  $-\frac{3 I}{5}$ , 1>, :  $\frac{I \sqrt{5}}{4}$ ,  $-\frac{5}{4}$ ,  $-\frac{I}{5}$ , 1>>
```

whose complex conjugates are

```
ccleftVec = leftVec •. conjugateRule
```

```
::  $\frac{3 I \sqrt{5}}{4}$ ,  $\frac{5}{4}$ ,  $\frac{3 I}{5}$ , 1>, :  $\frac{I \sqrt{5}}{4}$ ,  $-\frac{5}{4}$ ,  $\frac{I}{5}$ , 1>>
```

Based on the **rightVec**, we can assume the general solution form as

```
sol1Form = Transpose@rightVecD.9A1@T1, T2D EI w1 T0, A2@T1, T2D EI w2 T0=
```

```
9EI T0 w1 A1@T1, T2D + EI T0 w2 A2@T1, T2D,  $\frac{1}{3}$  I  $\frac{5 \sqrt{3}}{5}$  EI T0 w1 A1@T1, T2D + I  $\frac{5 \sqrt{3}}{5}$  EI T0 w2 A2@T1, T2D,
```

```
 $\frac{5}{4}$  EI T0 w1 A1@T1, T2D -  $\frac{5}{4}$  EI T0 w2 A2@T1, T2D,  $\frac{5}{12}$  I  $\frac{5 \sqrt{3}}{5}$  EI T0 w1 A1@T1, T2D -  $\frac{5}{4}$  I  $\frac{5 \sqrt{3}}{5}$  EI T0 w2 A2@T1, T2D=
```

where $w_1 = \frac{5 \sqrt{3}}{3}$ and $w_2 = \frac{5 \sqrt{3}}{5}$. Therefore, the general solution of **eqOrder[1]** can be expressed in pure function form as


```

sol1 = 8q1,1, v1,1, q2,1, v2,1 <->
  HFunction@8T0, T1, T2<, # + H# •. conjugateRuleL •. Evaluated & •ž sol1FormL •. Thread;
sol1 •. displayRule

9q1,1 ® Function@8T0, T1, T2<, EI T0 w1 A1 + EI T0 w2 A2 + E-I T0 w1  $\dot{A}_1$  + E-I T0 w2  $\dot{A}_2$  D, v1,1 ®
  Function@8T0, T1, T2<,  $\frac{1}{3}$  I •  $\frac{1}{5}$  EI T0 w1 A1 + I •  $\frac{1}{5}$  EI T0 w2 A2 -  $\frac{1}{3}$  I •  $\frac{1}{5}$  E-I T0 w1  $\dot{A}_1$  - I •  $\frac{1}{5}$  E-I T0 w2  $\dot{A}_2$  E,

q2,1 ® Function@8T0, T1, T2<,  $\frac{5}{4}$  EI T0 w1 A1 -  $\frac{5}{4}$  EI T0 w2 A2 +  $\frac{5}{4}$  E-I T0 w1  $\dot{A}_1$  -  $\frac{5}{4}$  E-I T0 w2  $\dot{A}_2$  E,

v2,1 ® Function@8T0, T1, T2<,
 $\frac{5}{12}$  I •  $\frac{1}{5}$  EI T0 w1 A1 -  $\frac{5}{4}$  I •  $\frac{1}{5}$  EI T0 w2 A2 -  $\frac{5}{12}$  I •  $\frac{1}{5}$  E-I T0 w1  $\dot{A}_1$  +  $\frac{5}{4}$  I •  $\frac{1}{5}$  E-I T0 w2  $\dot{A}_2$  E=

```

where A_1 and A_2 are to be determined from the solvability conditions at the next levels of approximation.

Ÿ Second-Order Problem

Substituting the first-order solution **sol1** into the second-order problem, **eqOrder[2]**, yields

```

order2Eq = eqOrder@2D •. sol1 •. ExpandAll;
order2Eq •. displayRule

: D0q1,2 - v1,2 == - EI T0 w1 HD1A1L - EI T0 w2 HD1A2L - E-I T0 w1 HD1 $\dot{A}_1$ L - E-I T0 w2 HD1 $\dot{A}_2$ L,

D0v1,2 +  $\frac{16}{25}$  HD0v2,2L + q1,2 ==
-  $\frac{4}{3}$  I EI T0 w1 HD1A1L -  $\frac{1}{3}$  I •  $\frac{1}{5}$  EI T0 w1 HD1A1L +  $\frac{4}{5}$  I EI T0 w2 HD1A2L - I •  $\frac{1}{5}$  EI T0 w2 HD1A2L +
 $\frac{4}{3}$  I E-I T0 w1 HD1 $\dot{A}_1$ L +  $\frac{1}{3}$  I •  $\frac{1}{5}$  E-I T0 w1 HD1 $\dot{A}_1$ L -  $\frac{4}{5}$  I E-I T0 w2 HD1 $\dot{A}_2$ L + I •  $\frac{1}{5}$  E-I T0 w2 HD1 $\dot{A}_2$ L,

D0q2,2 - v2,2 == -  $\frac{5}{4}$  EI T0 w1 HD1A1L +  $\frac{5}{4}$  EI T0 w2 HD1A2L -  $\frac{5}{4}$  E-I T0 w1 HD1 $\dot{A}_1$ L +  $\frac{5}{4}$  E-I T0 w2 HD1 $\dot{A}_2$ L,

D0v1,2 + D0v2,2 + q2,2 ==
-  $\frac{3}{4}$  I •  $\frac{1}{5}$  EI T0 w1 HD1A1L +  $\frac{1}{4}$  I •  $\frac{1}{5}$  EI T0 w2 HD1A2L +  $\frac{3}{4}$  I •  $\frac{1}{5}$  E-I T0 w1 HD1 $\dot{A}_1$ L -  $\frac{1}{4}$  I •  $\frac{1}{5}$  E-I T0 w2 HD1 $\dot{A}_2$ L>

```

For a uniform expansion, we choose $D_1 A_1$ and $D_1 A_2$ to eliminate the secular or small-divisor terms. To accomplish this, we first convert the small-divisor terms into secular terms using the rule

```
expRule1@i_D := Exp@a_D > ExpAExpand@a •. omgRule@@iDDD •. e2 T0 -> T2E
```

To eliminate the terms that produce secular terms (i.e., determine the solvability conditions) from **order2Eq**, we collect the terms proportional to $E^{I w_1 T_0}$ and $E^{I w_2 T_0}$ and obtain

```

ST11 = CoefficientAorder2Eq@@#, 2DD •. expRule1@1D, EI w1 T0E & •ž Range@4D

9-A1H1,0L@T1, T2D, -  $\frac{3}{5}$  I A1H1,0L@T1, T2D, -  $\frac{5}{4}$  A1H1,0L@T1, T2D, -  $\frac{3}{4}$  I •  $\frac{1}{5}$  A1H1,0L@T1, T2D=

```

ST12 = CoefficientAorder2Eq@@#, 2DD •. expRule1@2D, E^{I w₂ T₀}E & •ž Range@4D

$$9 - A_2^{H1, \theta L} @ T_1, T_2 D, - \frac{I A_2^{H1, \theta L} @ T_1, T_2 D}{5}, \frac{5}{4} A_2^{H1, \theta L} @ T_1, T_2 D, \frac{1}{4} I \cdot \frac{I I I}{5} A_2^{H1, \theta L} @ T_1, T_2 D =$$

Then, the solvability conditions demand that **ST11** and **ST12** be orthogonal to every solution of the corresponding adjoint problem, namely, the components of **ccleftVec**. The result is

SCond11 = ccleftVec@@1DD.ST11 == 0

$$-3 I \cdot \frac{I I I}{5} A_1^{H1, \theta L} @ T_1, T_2 D == 0$$

SCond12 = ccleftVec@@2DD.ST12 == 0

$$I \cdot \frac{I I I}{5} A_2^{H1, \theta L} @ T_1, T_2 D == 0$$

Solving **SCond11** and **SCond12** for $D_1 A_1$ and $D_1 A_2$, respectively, yields

SCond11Rule = SolveASCond11, A₁^{H1, θL}@T₁, T₂DE@@1DD

$$9 A_1^{H1, \theta L} @ T_1, T_2 D \otimes 0 =$$

SCond12Rule = SolveASCond12, A₂^{H1, θL}@T₁, T₂DE@@1DD

$$9 A_2^{H1, \theta L} @ T_1, T_2 D \otimes 0 =$$

SCond1Rule = Join@SCond11Rule, SCond12RuleD;

whose complex conjugate is

ccSCond1Rule = SCond1Rule •. conjugateRule;

Substituting **SCond1Rule** and **ccSCond1Rule** into **order2Eq**, we obtain

order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule •• ExpandAll;

order2Eqm •. displayRule

$$9 D_0 q_{1,2} - v_{1,2} == 0, D_0 v_{1,2} + \frac{16}{25} H D_0 v_{2,2} L + q_{1,2} == 0, D_0 q_{2,2} - v_{2,2} == 0, D_0 v_{1,2} + D_0 v_{2,2} + q_{2,2} == 0 =$$

which is a set of homogeneous equations. Then, we express the solution of the second-order equations in pure function form as

sol2 = 8q_{1,2} -> H0 &L, v_{1,2} -> H0 &L, q_{2,2} -> H0 &L, v_{2,2} -> H0 &L<;

Ÿ Third-Order Problem

Substituting the first- and second-order solutions into the third-order problem, **eqOrder[3]**, yields

order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;

Substituting the **expRule1** into the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, the terms proportional to $E^{I w_1 T_0}$ and $E^{I w_2 T_0}$, we have

ST21 = CoefficientA#@@2DD . expRule1@1D, $E^{I w_1 T_0} E$ & $\cdot \dot{z}$ order3Eq;

ST21 . displayRule

$$\begin{aligned}
 & 9 - \text{HD}_2 A_1 L, - \frac{3}{5} I \text{HD}_2 A_1 L + \frac{5}{12} \cdot \frac{I}{5} A_1 s_1 w_1 + \frac{23}{36} A_1^2 \dot{A}_1 - \frac{A_1^2 w_1 \dot{A}_1}{8 \cdot \frac{I}{5}} + \frac{11}{12} A_2 \dot{A}_1^2 + \\
 & \frac{3}{4} A_2 w_1 \dot{A}_1^2 + \frac{A_2 w_2 \dot{A}_1^2}{8 \cdot \frac{I}{5}} + \frac{7}{2} A_1 A_2 \dot{A}_2 - \frac{27}{4} A_1 A_2 w_1 \dot{A}_2 - \frac{9}{2} A_1 A_2 w_2 \dot{A}_2, - \frac{5}{4} \text{HD}_2 A_1 L, \\
 & - \frac{3}{4} I \cdot \frac{I}{5} \text{HD}_2 A_1 L + \frac{965}{1152} A_1^2 \dot{A}_1 - \frac{1}{32} \cdot \frac{I}{5} A_1^2 w_1 \dot{A}_1 - \frac{1175}{384} A_2 \dot{A}_1^2 + \frac{3}{16} \cdot \frac{I}{5} A_2 w_1 \dot{A}_1^2 - \\
 & \frac{1}{32} \cdot \frac{I}{5} A_2 w_2 \dot{A}_1^2 - \frac{35}{64} A_1 A_2 \dot{A}_2 - \frac{27}{16} \cdot \frac{I}{5} A_1 A_2 w_1 \dot{A}_2 + \frac{9}{8} \cdot \frac{I}{5} A_1 A_2 w_2 \dot{A}_2 =
 \end{aligned}$$

ST22 = CoefficientA#@@2DD . expRule1@2D, $E^{I w_2 T_0} E$ & $\cdot \dot{z}$ order3Eq;

ST22 . displayRule

$$\begin{aligned}
 & : - \text{HD}_2 A_2 L, - \frac{I}{5} \text{HD}_2 A_2 L + \frac{A_2^3}{36} - \frac{A_2^3 w_1}{24 \cdot \frac{I}{5}} - \frac{5}{4} \cdot \frac{I}{5} A_2 s_1 w_2 - \frac{3}{2} A_1 A_2 \dot{A}_1 + \\
 & \frac{3}{2} A_1 A_2 w_1 \dot{A}_1 + \frac{A_1 A_2 w_2 \dot{A}_1}{4 \cdot \frac{I}{5}} - \frac{43}{4} A_2^2 \dot{A}_2 - E^{I T_2 s_2} W^2 f_1 \dot{A}_2 + \frac{243}{8} A_2^2 w_2 \dot{A}_2, \frac{5}{4} \text{HD}_2 A_2 L, \\
 & \frac{1}{4} I \cdot \frac{I}{5} \text{HD}_2 A_2 L + \frac{535}{1152} A_1^3 - \frac{1}{96} \cdot \frac{I}{5} A_1^3 w_1 + \frac{35}{64} A_1 A_2 \dot{A}_1 + \frac{3}{8} \cdot \frac{I}{5} A_1 A_2 w_1 \dot{A}_1 - \\
 & \frac{1}{16} \cdot \frac{I}{5} A_1 A_2 w_2 \dot{A}_1 + \frac{1315}{128} A_2^2 \dot{A}_2 + \frac{5}{4} E^{I T_2 s_2} W^2 f_1 \dot{A}_2 - \frac{243}{32} \cdot \frac{I}{5} A_2^2 w_2 \dot{A}_2 >
 \end{aligned}$$

Then, solvability conditions demand that **ST21** and **ST22** be orthogonal to their corresponding adjoints. The result is

SCond21 = SolveAccleftVec@@1DD.ST21 == 0, $A_1^{H_0, 1L}$ @T₁, T₂DE@@1DD .

9w₁ -> $\frac{I}{5}$, 3, w₂ -> $\frac{I}{5}$ = .. ExpandAll;

SCond21 . displayRule

$$: D_2 A_1 \otimes - \frac{25}{432} I \cdot \frac{I}{5} A_1 s_1 - \frac{353}{3456} I \cdot \frac{I}{5} A_2^2 \dot{A}_1 + \frac{11}{128} I \cdot \frac{I}{5} A_2 \dot{A}_1^2 + \frac{23}{192} I \cdot \frac{I}{5} A_1 A_2 \dot{A}_2 >$$

SCond22 = SolveAccleftVec@@2DD.ST22 == 0, $A_2^{H_0, 1L}$ @T₁, T₂DE@@1DD .

9W -> 2 $\frac{I}{5}$, w₁ -> $\frac{I}{5}$, 3, w₂ -> $\frac{I}{5}$ = .. ExpandAll;

SCond22 . displayRule

$$9 D_2 A_2 \otimes \frac{11}{128} I \cdot \frac{I}{5} A_1^3 + \frac{25}{16} I \cdot \frac{I}{5} A_2 s_1 + \frac{23}{64} I \cdot \frac{I}{5} A_1 A_2 \dot{A}_1 - \frac{1337}{128} I \cdot \frac{I}{5} A_2^2 \dot{A}_2 + 10 I \cdot \frac{I}{5} E^{I T_2 s_2} f_1 \dot{A}_2 =$$

SCond2Rule = Join@SCond21, SCond22D;

Reconstitution

Using the method of reconstitution, $A_k^\epsilon = e D_1 A_k + e^2 D_2 A_k + \dots$, we combine the partial-differential equations **SCond1Rule** and **SCond2Rule** into the following two ordinary-differential equations governing the modulation of the complex-valued functions A_k :

$$\begin{aligned} \text{moduEq} &= \text{Table} @ 2 \text{ I } w_k A_k^\epsilon, 8k, 2 < D == \\ &\quad \text{HTable} @ 2 \text{ I } w_k \text{ dt} @ 1 D @ A_k @ T_1, T_2 D D, 8k, 2 < D \cdot \text{SCond1Rule} \cdot \text{SCond2RuleL} \cdot \cdot \text{Thread}; \\ \text{moduEq} &\cdot \cdot \text{displayRule} \\ 2 \text{ I } w_1 A_1^\epsilon &= 2 \text{ I } e^2 w_1 \left\{ -\frac{25}{432} \text{ I } \cdot \frac{111}{5} A_1 s_1 - \frac{353}{3456} \text{ I } \cdot \frac{111}{5} A_1^2 \dot{A}_1 + \frac{11}{128} \text{ I } \cdot \frac{111}{5} A_2 \dot{A}_1 + \frac{23}{192} \text{ I } \cdot \frac{111}{5} A_1 A_2 \dot{A}_2 \right\}, \\ 2 \text{ I } w_2 A_2^\epsilon &= 2 \text{ I } e^2 w_2 \left\{ \frac{11}{128} \text{ I } \cdot \frac{111}{5} A_1^3 + \frac{25}{16} \text{ I } \cdot \frac{111}{5} A_2 s_1 + \frac{23}{64} \text{ I } \cdot \frac{111}{5} A_1 A_2 \dot{A}_1 - \frac{1337}{128} \text{ I } \cdot \frac{111}{5} A_2^2 \dot{A}_2 + 10 \text{ I } \cdot \frac{111}{5} E^{I T_2 s_2} f_1 A_2 \dot{A}_2 \right\} \end{aligned}$$

6.2.2 The function **MMSC**

According to the procedures described in the previous section, we build a function named **MMSC** (**M**ethod of **M**ultiple **S**cales for Linearly **C**oupled Systems) to automate the process.

```
MMSC@eqs_List, depVar_List, scaling_List, ResonanceCond : 8__Equal<D :=
ModuleA8<,
  omgList = 8w1, w2<;
  omgRule = Solve@ResonanceCond, 8W, #<.. FlattenD@@1DD & .> omgList .. Reverse;
  mydepVar = 8u1, v1, u2, v2<;
  solRule = 8u_i_ -> HSum@e^j u_i,j@#1, #2, #3D, 8j, 3<D &L,
  v_i_ -> HSum@e^j v_i,j@#1, #2, #3D, 8j, 3<D &L<;
  maxOrder = 2;
  multiScales = 8u_i_@tD -> u_i žž timeScales, v_i_@tD -> v_i žž timeScales,
  Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
  eqa = Heqs .. Thread@depVar -> mydepVarD .. scaling .. multiScales .. solRule ..
  TrigToExp .. ExpandAllL .. e^Hn_ ; n > 3L -> 0;
  eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & .> eqa .. Transpose;
  eqOrder@i_D :=
  l#@@1DD & .> eqEps@@1DD .. f_s_ -> 0 .. u_-k_,1 -> u_k,iM == l#@@1DD & .> eqEps@@1DD .. f_s_ -> 0 ..
  u_-k_,1 -> u_k,iM - H#@@1DD & .> eqEps@@iDDL .. Thread;

H* First-Order Problem *L
linearSys = #@@1DD & .> eqOrder@1D;
assumedForm = 8u_i_,1 -> Hc_i E^HI w #L &L, v_i_,1 -> Hd_i E^HI w #L &L<;
coefMat =
```

```

Outer@D, E^H-I w T_0 L linearSys •. assumedForm •• Expand, 8c_1, d_1, c_2, d_2<D;
  hermitian@mat_?MatrixQD := mat •. conjugateRule •• Transpose;
  evals = Cases@w •. Solve@Det@coefMatD == 0, wD, n_ •; n > 0D;
  values = Append@omgRule@@1DD •. e -> 0 •. w_1 -> evals@@1DD, w_1 -> evals@@1DDD;
  rightVec = # •@@1DD & •ž HNullSpace@coefMat •. w -> #D@@1DD & •ž evalsL;
leftVec = NullSpace@hermitian@coefMatD •. w -> #D@@1DD & •ž evals;
  ccleftVec = leftVec •. conjugateRule;
  order1Eq = eqOrder@1D •. u_{i,1} -> Hu_{i,1}@#1D &L;
  sol1p =
DSolve@order1Eq, 8u_{1,1}@T_0D, v_{1,1}@T_0D, u_{2,1}@T_0D, v_{2,1}@T_0D<, T_0D@@1DD •. C@_D -> 0 •• Simplify;
  fRule = 9f_{i_} -> 2 L_{i_} I w_{i_}^2 - W^2 M=;
  sol1Form = H#@@2DD & •ž sol1pL + HH# + H# •. conjugateRuleLL & •ž
HTranspose@rightVecD.8A_1@T_1, T_2D E^HI w_1 T_0 L, A_2@T_1, T_2D E^HI w_2 T_0 L<LL;
  sol1 = 8u_{1,1}, v_{1,1}, u_{2,1}, v_{2,1}< -> HFunction@8T_0, T_1, T_2<, # •• Evaluated & •ž
sol1FormL •• Thread;

H* Second-Order Problem *L
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
expRule1@i_D :=
Exp@a_D := Exp@Expand@a •. omgRule@@iDDD •. e^a_{n_}. T_0 := timeScales@@n + 1DDD;
ST11 = Coefficient@#@@2DD •. expRule1@1D, E^HI w_1 T_0 LD & •ž order2Eq;
ST12 = Coefficient@#@@2DD •. expRule1@2D, E^HI w_2 T_0 LD & •ž order2Eq;
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
SCond1Rule1 = SolveASCond1, 9A_1^{H1,0L}@T_1, T_2D, A_2^{H1,0L}@T_1, T_2D=E@@1DD •• ExpandAll;
H* sigRule=Solve@ResonanceCond, 8s_1, s_2<D@@1DD; *L
sigRule = Solve@ResonanceCond, s_2D@@1DD;
expRule2 = Exp@a_D := Exp@a •. 8T_1 -> e T_0, T_2 -> e^2 T_0< •. sigRule •• ExpandD;
SCond1Rule2 = SCond1Rule1 •. expRule2;
ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
order2Eqm = order2Eq •. SCond1Rule2 •. ccSCond1Rule2 •• ExpandAll;
IfAUnion@#@@2DD & •ž order2EqmD === 80<,
  sol2 = 8u_{1,2} -> H0 &L, v_{1,2} -> H0 &L, u_{2,2} -> H0 &L, v_{2,2} -> H0 &L<,
  Return@"The particular solution of the second order
equations needs to be solved, which is not included in this function."D
H*
  basicH=
TableA9A_i@T_1, T_2D E^HI w_i T_0 L, A_i@T_1, T_2D E^H-I w_i T_0 L=, 8i, 2<E••Flatten;
  collectForm=Join@basicH,
If@list1=List žž Plus žž H#@@2DD & •ž eqOrder@1DL•.c_ f_{i_} E^a_{i_}->L_{i_} E^a_{i_};
  Head@list1D===List, list1, 8<DD;
  possibleTerms=JoinAcollectForm,
IfAlist1=List žž Plus žž I#@@2DD & •ž eqOrder@2D•.u_{i,j_}->H0 &LM•.

```

```

c_ f_i_ E^a_ -> E^a; Head@list1D===List,list1,8<E,
Outer@Times,collectForm,collectFormD•Flatten•UnionE;
    ResonantTerms@i_D:=H#•.8a_•;a!=0->1< & •ž
    HE^H-I w_i T_0L possibleTerms•.expRule1@iD•.Exp@_ T_0+_•D->0LL
possibleTerms••Union••Rest;
    RT=8Array@ResonantTerms,2D,Array@ResonantTerms,2D•.conjugateRule<••
Flatten;
    NRT=Complement@possibleTerms,RTD;
    rRule=Table@MapIndexed@
r_i,#2@@1DD->Coefficient@order2Eqm@@i,2DD,#1D &,RTD,8i,4<D••Flatten;
    list1=Join@ccleftVec,Conjugate@ccleftVecDD;
    r4Rule=
Table@Solve@list1@@jDD.Table@r_i,j,8i,4<D==0,r_4,jD@@1DD,8j,4<D••Flatten;
    RTsymbolList@i_D=Table@G_i,j,8j,Length@RTD<D;
    NRTsymbolList@i_D=Table@L_i,j,8j,Length@NRTD<D;
    sol2Form=8u_1,2,v_1,2,u_2,2,v_2,2<->
HFunction@8T_0,T_1,T_2<,RTsymbolList@#D.RT+NRTsymbolList@#D.NRT••Evaluated & •ž
    Range@4DL••Thread;
    coef1=Solve@Coefficient@Subtract žž #•.sol2Form,NRTD==0•.
    Exp@_ T_0+_•D->0 & •ž order2Eqm,
    Array@NRTsymbolList,4D••FlattenD@@1DD•.values••ExpandAll;
    eq1=Table@MapIndexed@Coefficient@order2Eqm@@k,1DD•.sol2Form,#1D==
    r_k,#2@@1DD &,RTD•.Exp@_ T_0+_•D->0,8k,4<D•.r4Rule••Flatten;
    coef2=HSolve@eq1,Join žž Array@RTsymbolList,4DD@@1DD••SimplifyL•.values•.
rRule••ExpandAll;
    sol2=sol2Form•.Function@8T_0,T_1,T_2<,a_D:>
Function@8T_0,T_1,T_2<••Evaluate,a•.coef1•.coef2••Expand••Evaluated *L
E;

```

H* Third-Order Problem *L

```

order3Eq = eq0Order@3D •. sol1 •. sol2 •• ExpandAll;
ST21 = Coefficient@#@2DD •. expRule1@1D, Exp@I w_1 T_0DD & •ž order3Eq;
ST22 = Coefficient@#@2DD •. expRule1@2D, Exp@I w_2 T_0DD & •ž order3Eq;
SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0< •• ExpandAll;
SCond2Rule1 =
SolveASCond2, 9A_1^{H_0,1L}@T_1, T_2D, A_2^{H_0,1L}@T_1, T_2D=E@@1DD •. values •• ExpandAll;

```

H* Reconstitution *L

```

moduEq = Table@2 I w_k A_k^e, 8k, 2<D ==
HTable@2 I w_k dt@1D@ A_k@T_1, T_2DD, 8k, 2<D •. HSCond1Rule1 •. valuesL •. SCond2Rule1 ••
Collect@#, eD &L •• Thread;
Print@"The second-order approximate solution:"D;
Print@

```

```

Table@ui@tD == Hui žž timeScales •. solRule •. e^3 -> 0 •. sol1 •. sol2 •. displayRuleL,
8i, 2<D •. Thread@mydepVar -> depVarDD;
IfAOr žž Table@HFi • fi •. scalingL === e, 8i, 2<D,
Print@"where"D;
PrintA"Li==fi•H2Hwi2-W^2LL"E
E;
Print@"\nThe modulation equations:"D;
Print@moduEq •. displayRuleD
E

```

As an example, we check the case in Section 6.2.1:

```

eqtest = 9
q1ε@tD == v1@tD,
v1ε@tD + a v2ε@tD + q1@tD ==
- 2 W2 Cos@t WD F1 q1@tD +  $\frac{1}{6}$  q1@tD3 + a v2@tD2 H- q1@tD + q2@tDL +  $\frac{1}{2}$  a H- q1@tD + q2@tDL2 v2ε@tD,
q2ε@tD == v2@tD,
v2ε@tD + v1ε@tD + q2@tD ==
- 2 W2 Cos@t WD F1 q2@tD +  $\frac{1}{6}$  q2@tD3 - v1@tD2 H- q1@tD + q2@tDL +  $\frac{1}{2}$  H- q1@tD + q2@tDL2 v1ε@tD
= •. 9a @  $\frac{16}{25}$  + e2 s1=;

scaling1 = 9F1 -> e2 f1=;

ResonanceCond1 = 9w2 == 3 w1, W == 2 w2 + e2 s2=;

MMSC@eqtest, 8q1, v1, q2, v2<, scaling1, ResonanceCond1D •• Timing

The second-order approximate solution:

: q1@tD == e I EI T0 w1 A1 + EI T0 w2 A2 + E-I T0 w1  $\dot{A}_1$  + E-I T0 w2  $\dot{A}_2$  M,
q2@tD == e  $\int_k \frac{5}{4} E^{I T_0 w_1} A_1 - \frac{5}{4} E^{I T_0 w_2} A_2 + \frac{5}{4} E^{-I T_0 w_1} \dot{A}_1 - \frac{5}{4} E^{-I T_0 w_2} \dot{A}_2 \frac{y}{\{}$ 

The modulation equations:

: 2 I w1 A1ε == 2 I e2 w1  $\left\{ -\frac{25}{432} I \cdot \frac{!!!}{5} A_1 s_1 - \frac{353 I \cdot \frac{!!!}{5} A_2 \dot{A}_1}{3456} + \frac{11}{128} I \cdot \frac{!!!}{5} A_2 \dot{A}_1 + \frac{23}{192} I \cdot \frac{!!!}{5} A_1 A_2 \dot{A}_2 \right\}$ ,
2 I w2 A2ε == 2 I e2 w2
 $\int_k \frac{11}{128} I \cdot \frac{!!!}{5} A_1^3 + \frac{25}{16} I \cdot \frac{!!!}{5} A_2 s_1 + \frac{23}{64} I \cdot \frac{!!!}{5} A_1 A_2 \dot{A}_1 - \frac{1337}{128} I \cdot \frac{!!!}{5} A_2^2 \dot{A}_2 + 10 I \cdot \frac{!!!}{5} E^{I T_2 s_2} f_1 A_2 \frac{y}{\{}$ 

87.691 Second, Null<

```

Chapter 7

Continuous Systems with Cubic Nonlinearities

Elastic systems, such as beams, plates, and shells, are usually modeled by partial-differential equations with specified boundary conditions. For small oscillations, the responses of such deformable bodies, [continuous](#) or [distributed-parameter systems](#), can be adequately described by linear equations and boundary conditions. However, as the amplitude of oscillations increases, nonlinear effects in the governing equations, boundary conditions, or both come into play. The sources of nonlinearities may be [geometric](#), [inertial](#), [material](#), or [damping](#) in nature.

[Geometric nonlinearities](#) may be due to one or more of the following: nonlinear relationships among the strains and the displacements, large rotations, free surfaces in fluids, time-dependent constraints, mid-plane stretching, and large curvatures. Nonlinear stretching of the midplane of a deformable body accompanies its transverse vibrations if it is supported in such a way as to restrict the movement of its ends and / or edges. If large-amplitude vibrations are accompanied by large changes in the curvature, it is necessary to employ a nonlinear relationship between the curvature and the displacement. [Inertial nonlinearities](#) are caused by concentrated and / or distributed masses, convective accelerations, and Coriolis and centripetal accelerations. [Material nonlinearities](#) occur whenever the constitutive relations are nonlinear, such as the case when the stresses are nonlinear functions of the strains; the resistive, inductive, and capacitive circuit elements are nonlinear; and the feedback control forces and moments in servomechanisms are nonlinear. [Damping nonlinearities](#) occur due to form drag and hysteresis.

Since exact solutions are, in general, not available for determining the dynamic responses of nonlinear continuous systems to external or parametric excitations, recourse has been made to approximate analyses by using either purely numerical techniques, or purely analytical techniques, or a combination of numerical and analytical techniques. Application of purely numerical techniques to such problems may be costly in terms of computation time and may not reveal some of their intricate and complicated responses. With purely analytical methods, it may be difficult to treat systems with inhomogeneities or complicated geometries. With a combination of numerical and analytical techniques, one can determine some of the complicated responses of systems with inhomogeneities and complicated shapes.

The numerical-analytic approaches can be divided into two groups: discretization and direct methods. In the discretization methods, one postulates the solution in the form

$$w[x,t] = \sum_{m=1}^M f_m(x) q_m(t)$$

where M is a finite integer. Then, one assumes the spatial functions $f_m(x)$, space discretization, or the temporal functions $q_m(t)$, time discretization. With this discretization, the $q_m(t)$ are usually taken to be harmonic and the method of harmonic balance is used to obtain a set of nonlinear boundary-value problems for the $f_m(x)$.

With space discretization, the $f_m(x)$ (and thus the spatial dependence) are assumed a priori. If the boundary conditions are homogeneous, the $f_m(x)$ are usually taken to be the eigenfunctions of the linearized problem. The method of weighted residuals or variational principles can then be used to determine a set of ordinary-differential equations governing the $q_m(t)$.

The obtained set of ordinary-differential equations can be studied using any of a great number of methods developed for discrete systems.

The most common implementation of weighted residuals is the Galerkin method. The great majority of recent studies of forced vibrations assume that the response can be expressed in terms of only the linear modes that are directly or indirectly excited (Nayfeh and Mook, 1979). For example, if a system is driven near the natural frequency of a linear mode and that mode is not involved in an internal resonance with other modes, the response is assumed to consist of only that mode. Such an approach is usually referred to as a single-mode approximation.

In the direct approach, a reduction method, such as the method of multiple scales, is applied directly to the governing partial-differential equations and associated boundary conditions and no assumptions are made a priori regarding the spatial or temporal dependence of the response. The principal advantage of this approach is in the treatment of the boundary conditions at higher orders. This approach has been used recently by Nayfeh and Nayfeh (1979), Nayfeh (1975, 1996), Nayfeh and Asfar (1986), Nayfeh and Bouguerra (1990), Pai and Nayfeh (1990), Raouf and Nayfeh (1990), Nayfeh, Nayfeh, and Mook (1992), Nayfeh and Nayfeh (1993, 1995), Nayfeh, Nayfeh, and Pakdemirli (1995), Pakdemirli, Nayfeh, and Nayfeh (1995), Chin and Nayfeh (1996), Nayfeh and Lacarbonara (1997, 1998), Lacarbonara, Nayfeh, and Kreider (1998), Nayfeh, Lacarbonara, and Chin (1999), and Rega, et al. (1999).

Some of the aforementioned studies show that the discretization and direct approaches yield the same results for systems with cubic nonlinearities provided that first-order approximations are sought. For systems with quadratic and cubic nonlinearities, the discretization approach might produce quantitative, and in some case qualitative, errors unless many modes are included in the discretized model. In other words, one has to include as many terms in the discretized model as needed for convergence. Since we are dealing with systems with cubic nonlinearities in this chapter, we will obtain first-order results using both the discretization and direct approaches.

à Preliminaries

Off@General::spell1, Integrate::generD

Needs@"Utilities`Notation`"D

To use the method of multiple scales, we introduce different time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, symbolize them as

Symbolize@T₀D; Symbolize@T₁D; Symbolize@T₂D;

and form a list of them as follows:

timeScales = {T₀, T₁, T₂};

In terms of the time scales T_n , the time derivatives become

dt@0D@expr_D := expr; dt@1D@expr_D := SumAeⁱ D@expr, timeScales@@i + 1DDD, {i, 0, 1} < E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •• eⁱ •; i>1 -> 0;

In the course of the analysis, we need the complex conjugates of A and G . We define them using the following rule:

conjugateRule = {A -> Ā, Ā -> A, G -> Ġ, Ġ -> G, Complex@0, n_D -> Complex@0, -nD=};

To manipulate some complicated integrals without *Mathematica* being choked, we define the following rules:

```
intRule1 = 8int@fun_, arg2__D := int@Expand@funD, arg2D<;
intRule2 = 8int@a_ + b_, arg2__D := int@a, arg2D + int@b, arg2D,
int@e^n_ fun_, arg2__D := e^n int@fun, arg2D,
int@a_ fun_, a1_, b1__D := a int@fun, a1, b1D •; FreeQ@a, First@a1DD,
int@int@a1_, a2_D fun_, a3__D := int@a1, a2D int@fun, a3D •; FreeQ@Rest@a2D, First@a2DD<;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule = 9Derivative@a_, b__DAw_i_E@x, __D :=
SequenceFormAIfAarg1 = Times @@ MapIndexedAD^#1_#2@1DD-1 &, 8b<E; arg1 != 1, arg1, ""E,
w_i SequenceForm @@ Table["", 8a<D_E,
Derivative@a__D@A_i_D@__D := SequenceFormATimes @@ MapIndexedAD^#1_#2@1DD &, 8a<E, A_iE,
Derivative@a__D@A_i_D@__D := SequenceFormATimes @@ MapIndexedAD^#1_#2@1DD &, 8a<E, A_iE,
w_i @x, __D -> w_i, A_i @__D -> A_i, A_i @__D -> A_i,
Exp@a_. + b_. Complex@0, m_D T_0 + c_. Complex@0, n_D T_0D := Exp@a + Hm * b + n * cL I T_0D,
int -> Integrate=;
```

à 7.1 Solvability Conditions and the Concept of Adjoint

In directly attacking continuous systems, one often encounters nonhomogeneous boundary-value problems whose homogeneous parts have nontrivial solutions. Consequently, the nonhomogeneous problems have solutions only if [solvability](#) or [consistency conditions](#) are satisfied (Nayfeh, 1981). In this section, we describe how to determine such solvability conditions.

We start the discussion with the simple problem

```
eq1 = y^2 @xD + p^2 y@xD == p Sin@p xD;
bc1 = 8y@0D == b_1, y@1D == b_2<;
```

The corresponding homogeneous problem has the nontrivial solution

```
DSolve@8eq1@@1DD == 0<~Join~bc1 •. b_i_ -> 0, y@xD, xD@@1DD
8y@xD ® - C@1D Sin@p xD<
```

Hence, the nonhomogeneous problem will not have a solution unless a solvability condition is satisfied. To determine this solvability condition, we use two approaches.

In the first approach, we find the general solution of [eq1](#) using the function **DSolve** as

```
yRule = DSolve@eq1, y@xD, xD@@1DD •• Simplify
:y@xD ® J - X/2 + C@2DN Cos@p xD - C@1D Sin@p xD>
```

where $C[1]$ and $C[2]$ are arbitrary constants. Imposing the boundary conditions **bc1**, we have

$$\text{bc2} = \text{bc1} \cdot \text{Flatten}[\text{yRule} \cdot \text{88x} \rightarrow 0, \text{8x} \rightarrow 1] < \text{D}$$

$$9C[2]D == b_1, \frac{1}{2} - C[2]D == b_2 =$$

These equations are inconsistent unless

$$\text{Equal}[\text{22 Plus 22 HList 22 \# \& \cdot 22 bc2L}$$

$$\frac{1}{2} == b_1 + b_2$$

which is the desired solvability condition.

In the second approach, instead of determining the general solution of the homogeneous differential equations and then enforcing the boundary conditions to determine the solvability conditions, we use the concept of **adjoint** as described next. This approach is attractive if one is not interested in determining the solution of the nonhomogeneous problem but interested only in determining the solvability conditions, as is the case in many applications of perturbation methods.

To determine the adjoint and then the solvability conditions, we need to perform integration by parts of some products. To accomplish this with *Mathematica*, we define a function named **intByParts** according to

```
intByParts@expr_ ; Head@exprD != Equal, depVar_D :=
Module@8intRule1, intRule2, intRule3<,
  intRule1 = f_@terms_, x_D := int@Expand@termsD, xD;
  intRule2 = int@term1_ + term2_, x_D := int@term1, xD + int@term2, xD;
  intRule3 = c1_. int@term1_, x_D + c2_. int@term2_, x_D := int@c1 term1 + c2 term2, xD;
  IBP@u_, v_, n_ ; n >= 1, range : 8x_, a_, b_<D := Hu Derivative@n - 1D@vD@xD . x -> bL -
    Hu Derivative@n - 1D@vD@xD . x -> aL - IBP@D@u, xD, v, n - 1, rangeD;
  IBP@u_, v_, 0, range : 8x_, a_, b_<D := int@u v@xD, rangeD;

  expr . intRule1 . . intRule2 . int@u_ Derivative@n_D@depVarD@xD, arg2_D :=
    IBP@u, depVar, n, arg2D . . intRule3 . int -> Integrate
D
```

To determine the solvability condition of **eq1** and **bc1** with this approach, we multiply **eq1** by $u[x]$, integrate the result by parts from $x = 0$ to $x = 1$, and obtain

$$\text{eq1a} = \text{intByParts}[\text{int}[\text{u}[\text{x}], \text{8x}, 0, 1] < \text{D}, \text{yD} \& \cdot \text{22 eq1}$$

$$\int_0^1 \text{Hp}^2 u[x] y[x] + y[x] u^2[x] \text{d}x + y[0] u^c[0] - y[1] u^c[1] - u[0] y^c[0] + u[1] y^c[1] ==$$

$$p \int_0^1 \text{Sin}[p x] u[x] \text{d}x$$

To determine the adjoint, we set the coefficient of $y[x]$ in the integrand on the left-hand side of **eq1a** equal to zero; that is,

```
adjointEq1 = Cases@eq1a@@1DD, Integrate@expr_, _D :=> Coefficient@expr, y@xDDD@@1DD == 0
```

$$p^2 u(xD + u'(xD) == 0$$

To determine the adjoint boundary conditions, we consider the homogeneous problem (i.e., $f(xD) = 0$, $b_1 = 0$, and $b_2 = 0$), use the **adjointEq1**, and obtain from **eq1a** that

```
bc1a = eq1a •. Integrate -> H0 &L •. Hbc1 •. Equal -> Rule •. b_i_ -> 0L
```

$$-u(0D) y'(0D) + u(1D) y'(1D) == 0$$

To determine the adjoint boundary conditions, we set each of the coefficients of $y'(0D)$ and $y'(1D)$ in **bc1a** equal to zero and obtain

```
adjointBC1 =
```

```
Solve@Coefficient@bc1a@@1DD, 8y'(0D), y'(1D)<D == 0, 8u(0D), u(1D)<D@@1DD •. Rule -> Equal
```

$$8u(0D) == 0, u(1D) == 0<$$

Therefore, the adjoint u is defined by the adjoint system consisting of **adjointEq1** and **adjointBC1**. Since they are the same as the homogeneous parts of **eq1** and **bc1**, the problem is said to be self-adjoint. Hence, $u(xD) = \sin(xD)$.

```
adjoint = 8u -> Function@x, Sin@p xDD<
```

$$8u \otimes \text{Function}@x, \sin@p xDD<$$

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc1** into **eq1a** yields the solvability condition

```
SolvCond1 = eq1a@@2DD ==
```

```
Heq1a@@1DD •. Integrate -> H0 &L •. Hbc1 •. Equal -> RuleL •. HadjointBC1 •. Equal -> RuleLL
```

$$p \int_0^1 \sin@p xD u(xD) dx == b_1 u'(0D) - b_2 u'(1D)$$

```
SolvCondf = SolvCond1 •. adjoint
```

$$\frac{p}{2} == p b_1 + p b_2$$

which is the same as the solvability condition obtained with the other method.

7.1.1 Hinged-Clamped Beam

In treating a hinged-clamped beam, we need to determine the adjoint of the boundary-value problem

$$\text{eq1a} = -w^2 f(xD) - 2 P f'(xD) + f^{H4L}(xD) == f(xD);$$

$$\text{bc1a} = 8f(0D) == 0, f'(0D) == 0, f(1D) == 0, f'(1D) == 0<;$$

Multiplying **eq1a** by $u(x)$ and integrating the result by parts from $x = 0$ to $x = 1$, we obtain

$$\begin{aligned} \text{eq1b} = \text{intByParts}[\text{int}[u(x) \#, 8x, 0, 1], fD \& \cdot \# \text{eq1a} \\ \int_0^1 H - w^2 u(x) f(x) - 2 P f(x) u'(x) + f(x) u^{H4L}(x) dx - 2 P f(0) u'(0) + 2 P f(1) u'(1) + \\ 2 P u(0) f'(0) - 2 P u(1) f'(1) - f'(0) u^2(0) + f'(1) u^2(1) + u(0) f^2(0) - u(1) f^2(1) + \\ f(0) u^{H3L}(0) - f(1) u^{H3L}(1) - u(0) f^{H3L}(0) + u(1) f^{H3L}(1) == \int_0^1 f(x) u(x) dx \end{aligned}$$

To determine the adjoint equation, we set the coefficient of $f(x)$ in the integrand on the left-hand side of **eq1b** equal to zero; that is,

$$\begin{aligned} \text{adjointEq1} = \text{Cases}[\text{eq1b} @ 1 \text{DD}, \text{Integrate}[\text{expr}_, _D] \> \text{Coefficient}[\text{expr}, f @ x \text{DDD} @ 1 \text{DD}] == 0 \\ -w^2 u(x) - 2 P u'(x) + u^{H4L}(x) == 0 \end{aligned}$$

To determine the adjoint boundary conditions, we consider the homogeneous problem, use the **adjointEq1**, and obtain from **eq1b** that

$$\begin{aligned} \text{bc1b} = \text{eq1b} \cdot \text{Integrate} \rightarrow H0 \&L \cdot \text{Hbc1a} \cdot \text{Equal} \rightarrow \text{RuleL} \\ 2 P u(0) f'(0) - f'(0) u^2(0) - u(1) f^2(1) - u(0) f^{H3L}(0) + u(1) f^{H3L}(1) == 0 \end{aligned}$$

To determine the adjoint boundary conditions, we set each of the coefficients of $f'(0)$, $f^{H3L}(0)$, $f^2(1)$, and $f^{H3L}(1)$ in **bc1b** equal to zero and obtain

$$\begin{aligned} \text{adjointBC1} = \text{Solve}[\text{Coefficient}[\text{bc1b} @ 1 \text{DD}, 9 f'(0), f^{H3L}(0), f^2(1), f^{H3L}(1) = E == 0, \\ 8 u(0), u^2(0), u(1), u'(1) = E @ 1 \text{DD}] \cdot \text{Rule} \rightarrow \text{Equal} \\ 8 u^2(0) == 0, u(1) == 0, u'(1) == 0, u(0) == 0 < \end{aligned}$$

Therefore, the adjoint u is defined by the adjoint system consisting of **adjointEq1** and **adjointBC1**. Since they are the same as the homogeneous parts of **eq1a** and **bc1a**, the problem is said to be self-adjoint.

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc1a** into **eq1b** yields the solvability condition

$$\begin{aligned} \text{SolvCond1} = \text{eq1b} @ 2 \text{DD} == \\ \text{Heq1b} @ 1 \text{DD} \cdot \text{Integrate} \rightarrow H0 \&L \cdot \text{Hbc1a} \cdot \text{Equal} \rightarrow \text{RuleL} \cdot \text{HadjointBC1} \cdot \text{Equal} \rightarrow \text{RuleLL} \\ \int_0^1 f(x) u(x) dx == 0 \end{aligned}$$

7.1.2 Cantilever Beam

The boundary-value problem for a cantilever beam can be written as

$$\text{eq2a} = -w^2 f(x) + f^{H4L}(x) == f(x);$$

$$bc2a = 9f @ 0D == 0, f' @ 0D == 0, f^2 @ 1D == 0, f^{H3L} @ 1D == 0;$$

To determine the adjoint of this problem, we multiply **eq2a** by $u @ xD$, integrate the result by parts from $x = 0$ to $x = 1$, and obtain

$$eq2b = \text{intByParts} @ \text{int} @ u @ xD \#, 8x, 0, 1 < D, fD \& \cdot \checkmark eq2a$$

$$\int_0^1 H - w^2 u @ xD f @ xD + f @ xD u^{H4L} @ xDL \hat{a} x - f' @ 0D u^2 @ 0D + f' @ 1D u^2 @ 1D + u' @ 0D f^2 @ 0D - \\ u' @ 1D f^2 @ 1D + f @ 0D u^{H3L} @ 0D - f @ 1D u^{H3L} @ 1D - u @ 0D f^{H3L} @ 0D + u @ 1D f^{H3L} @ 1D == \int_0^1 f @ xD u @ xD \hat{a} x$$

We set the coefficient of $f @ xD$ in the integrand on the left-hand side of **eq2b** equal to zero and obtain the adjoint equation as

$$\text{adjointEq2} = \text{Cases} @ eq2b @ 1DD, \text{Integrate} @ \text{expr}_, _D := \text{Coefficient} @ \text{expr}, f @ xDDD @ 1DD == 0 \\ - w^2 u @ xD + u^{H4L} @ xD == 0$$

To determine the adjoint boundary conditions, we consider the homogeneous problem, use the **adjointEq2**, and obtain from **eq2b** that

$$bc2b = eq2b \cdot \cdot \text{Integrate} -> H0 \&L \cdot \cdot Hbc2a \cdot \cdot \text{Equal} -> \text{RuleL} \\ f' @ 1D u^2 @ 1D + u' @ 0D f^2 @ 0D - f @ 1D u^{H3L} @ 1D - u @ 0D f^{H3L} @ 0D == 0$$

To determine the adjoint boundary conditions, we set each of the coefficients of $f^2 @ 0D$, $f^{H3L} @ 0D$, $f @ 1D$, and $f' @ 1D$ in **bc2b** equal to zero and obtain

$$\text{adjointBC2} = \text{SolveACoefficient} @ bc2b @ 1DD, 9f^2 @ 0D, f^{H3L} @ 0D, f @ 1D, f' @ 1D = E == 0, \\ 9u @ 0D, u' @ 0D, u^2 @ 1D, u^{H3L} @ 1D = E @ 1DD \cdot \cdot \text{Rule} -> \text{Equal} \\ 8u @ 0D == 0, u' @ 0D == 0, u^2 @ 1D == 0, u^{H3L} @ 1D == 0 <$$

Therefore, the adjoint u is defined by the adjoint system consisting of **adjointEq2** and **adjointBC2**. Since they are the same as the homogeneous parts of **eq2a** and **bc2a**, the problem is said to be self-adjoint.

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc2a** into **eq2b** yields the solvability condition

$$\text{SolvCond2} = eq2b @ 2DD == \\ \text{Heq2b} @ 1DD \cdot \cdot \text{Integrate} -> H0 \&L \cdot \cdot Hbc2a \cdot \cdot \text{Equal} -> \text{RuleL} \cdot \cdot \text{HadjointBC2} \cdot \cdot \text{Equal} -> \text{RuleLL} \\ \int_0^1 f @ xD u @ xD \hat{a} x == 0$$

7.2 Hinged-Clamped Beam

7.2.1 EOM and BC's

We consider the nonlinear planar response of a hinged-clamped uniform prismatic beam to a harmonic axial load. The beam is subjected to a static axial load and one of its ends is restrained by a linear spring. We assume that the curvature and inertia nonlinearities are much smaller than the nonlinearity caused by the mid-plane stretching. The transverse deflection $w(x, t)$ of the beam at the position x and time t is governed by the nondimensional integral-partial-differential equation (Nayfeh and Mook, 1979)

$$\begin{aligned} \text{EOM} = & \rho A \frac{\partial^2 w}{\partial t^2} + P \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \rho A \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial w}{\partial x} \right)^2 - 2 e m \omega_D^2 \frac{\partial^2 w}{\partial t^2} + 4 e F \cos \omega t \frac{\partial^2 w}{\partial x^2} + e a \int_0^1 H_1(x) \frac{\partial^2 w}{\partial x^2} dx \\ & - 2 e m \omega_D^2 \frac{\partial^2 w}{\partial t^2} + 4 e F \cos \omega t \frac{\partial^2 w}{\partial x^2} + e a \int_0^1 H_1(x) \frac{\partial^2 w}{\partial x^2} dx \end{aligned}$$

and homogeneous boundary conditions

$$\begin{aligned} \text{BC} = & w(0, t) = 0, \quad \frac{\partial w}{\partial x}(0, t) = 0, \quad w(1, t) = 0, \quad \frac{\partial w}{\partial x}(1, t) = 0 \\ & w(0, t) = 0, \quad \frac{\partial w}{\partial x}(0, t) = 0, \quad w(1, t) = 0, \quad \frac{\partial w}{\partial x}(1, t) = 0 \end{aligned}$$

We assume that the static axial load is such that the lowest two natural frequencies of the beam are in the ratio of three-to-one; that is, there is a three-to-one internal resonance between the second and first modes. Moreover, we assume that neither of these two modes is involved in an internal resonance with any other mode. We consider three parametric resonances, namely, principal parametric resonance of the first mode, principal parametric resonance of the second mode, and combination parametric resonance of the first and second modes.

7.2.2 Direct Attack of the Continuous Problem

In this section, we directly attack the integral-partial-differential equation **EOM** and associated boundary conditions **BC** and seek a first-order uniform expansion of their solution in the form

$$\begin{aligned} \text{solRule} = & w \rightarrow \text{EvaluateASum}[w_j(x, t), \{j, 0, 1\}] \\ & w(x, t) = H_0(x, t) + e w_1(x, t) + \dots \end{aligned}$$

where the first independent variable stands for x and the last two independent variables stand for the two time scales T_0 and T_1 . Substituting the **solRule** into **EOM**, transforming the total time derivatives into partial derivatives in terms of T_0 and T_1 , expanding the result for small e , discarding terms of order higher than e , and using the **intRule2** to simplify the expansions of the integrands, we obtain

```

eq722a =
  HHJoin@8EOM<, BCD •. Integrate -> int •. 8w@x_, tD -> w@x, T0, T1D, Derivative@m_, n_D@wD@
    x_, tD -> dt@nD@D@w@x, T0, T1D, 8x, m<DD, t -> T0< •.
  solRule •• ExpandAllL ••. intRule2 •• ExpandAllL •. en_•; n>1 -> 0;

```

Equating coefficients of like powers of e in **eq722a**, we obtain

```

eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq722a •• Transpose;
eqEps •. displayRule

: 9 - 2 P Hw0'L + w0'' + D02w0 == 0, w0@0, T0, T1D == 0,
  w0H2,0,0L@0, T0, T1D == 0, w0@1, T0, T1D == 0, w0H1,0,0L@1, T0, T1D == 0=,
: - 4 F Cos@T0 WD Hw0'L - a Hw0'L2 Hw0'L - 2 P Hw1'L + w1'' + D02w1 + 2 HD0 D1w0L + 2 HD0w0L m@xD == 0,
  w1@0, T0, T1D == 0, w1H2,0,0L@0, T0, T1D == 0, w1@1, T0, T1D == 0, w1H1,0,0L@1, T0, T1D == 0>>

```

Because in the presence of damping, all modes that are not directly excited by the forcing or indirectly excited by the internal resonance will decay with time (Nayfeh and Mook, 1979), the solution of **eqEps[[1]]** can be expressed in terms of the lowest two linear free-vibration modes; that is,

```

sol0 =
  w0 -> FunctionA8x, T0, T1<, SumAAi@T1D fi@xD Exp@I wi T0D + Āi@T1D fi@xD Exp@- I wi T0D, 8i, 2<EE;

```

where w_1 and w_2 are the natural frequencies of these modes. For later use, we define the list

```
omgList = 8w1, w2<;
```

One can easily show that the mode shapes $f_i@xD$ are orthogonal. We assume that these modes are normalized so that

$$\int_0^1 f_i@xD f_j@xD \hat{a} x \rightarrow d_{ij}$$

where d_{ij} is the Kronecker delta function.

Substituting **sol0** into the first-order equation, **eqEps[[2,1]]**, and using **intRule1** and **intRule2**, we obtain


```

order1Eq = HeqEps@@1, 1, 1DD •. w0 -> w1L ==
  HHeqEps@@1, 1, 1DD •. w0 -> w1L - HSubtract žž eqEps@@2, 1DD •. sol0 •• TrigToExp ••
  ExpandL •. intRule1 ••. intRule2 •• ExpandL;

```

```

order1Eq •. displayRule

```

$$\begin{aligned}
& -2 P H w_1' L + w_1'' + D_0^2 w_1 = -2 I E^{I T_0 w_1} H D_1 A_1 L w_1 f_1 @ x D + 2 I E^{-I T_0 w_1} H D_1 \dot{A}_1 L w_1 f_1 @ x D - \\
& 2 I E^{I T_0 w_1} A_1 w_1 m @ x D f_1 @ x D + 2 I E^{-I T_0 w_1} w_1 \dot{A}_1 m @ x D f_1 @ x D - 2 I E^{I T_0 w_2} H D_1 A_2 L w_2 f_2 @ x D + \\
& 2 I E^{-I T_0 w_2} H D_1 \dot{A}_2 L w_2 f_2 @ x D - 2 I E^{I T_0 w_2} A_2 w_2 m @ x D f_2 @ x D + 2 I E^{-I T_0 w_2} w_2 \dot{A}_2 m @ x D f_2 @ x D + \\
& 2 E^{I T_0 H - W + w_1 L} F A_1 f_1^2 @ x D + 2 E^{I T_0 H W + w_1 L} F A_1 f_1^2 @ x D + E^{3 I T_0 w_1} a \int_{k=0}^1 \ddot{a} f_1 @ x D^2 \hat{a} x \frac{Y}{\{ } A_1^3 f_1^2 @ x D + \\
& 2 E^{I T_0 H 2 w_1 + w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1^2 A_2 f_1^2 @ x D + E^{I T_0 H w_1 + 2 w_2 L} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 A_2^2 f_1^2 @ x D + \\
& 2 E^{I T_0 H - W - w_1 L} F \dot{A}_1 f_1^2 @ x D + 2 E^{I T_0 H W - w_1 L} F \dot{A}_1 f_1^2 @ x D + 3 E^{I T_0 w_1} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1^2 \dot{A}_1 f_1^2 @ x D + \\
& 4 E^{I T_0 w_2} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 A_2 \dot{A}_1 f_1^2 @ x D + E^{I T_0 H - w_1 + 2 w_2 L} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2^2 \dot{A}_1 f_1^2 @ x D + \\
& 3 E^{-I T_0 w_1} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_1 f_1^2 @ x D + 2 E^{I T_0 H - 2 w_1 + w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_2 \dot{A}_1 f_1^2 @ x D + \\
& E^{-3 I T_0 w_1} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1^3 f_1^2 @ x D + 2 E^{I T_0 H 2 w_1 - w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1^2 \dot{A}_2 f_1^2 @ x D + \\
& 2 E^{I T_0 w_1} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 A_2 \dot{A}_2 f_1^2 @ x D + 4 E^{-I T_0 w_2} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_1 \dot{A}_2 f_1^2 @ x D + \\
& 2 E^{-I T_0 w_1} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2 \dot{A}_1 \dot{A}_2 f_1^2 @ x D + 2 E^{I T_0 H - 2 w_1 - w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } \dot{A}_1^2 \dot{A}_2 f_1^2 @ x D + \\
& E^{I T_0 H w_1 - 2 w_2 L} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_2^2 f_1^2 @ x D + E^{I T_0 H - w_1 - 2 w_2 L} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_2^2 f_1^2 @ x D + \\
& 2 E^{I T_0 H - W + w_2 L} F A_2 f_2^2 @ x D + 2 E^{I T_0 H W + w_2 L} F A_2 f_2^2 @ x D + E^{I T_0 H 2 w_1 + w_2 L} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1^2 A_2 f_2^2 @ x D + \\
& 2 E^{I T_0 H w_1 + 2 w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 A_2^2 f_2^2 @ x D + E^{3 I T_0 w_2} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2^3 f_2^2 @ x D + \\
& 2 E^{I T_0 w_2} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 A_2 \dot{A}_1 f_2^2 @ x D + 2 E^{I T_0 H - w_1 + 2 w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_2^2 \dot{A}_1 f_2^2 @ x D + \\
& E^{I T_0 H - 2 w_1 + w_2 L} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2 \dot{A}_1^2 f_2^2 @ x D + 2 E^{I T_0 H - W - w_2 L} F \dot{A}_2 f_2^2 @ x D + 2 E^{I T_0 H W - w_2 L} F \dot{A}_2 f_2^2 @ x D + \\
& E^{I T_0 H 2 w_1 - w_2 L} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1^2 \dot{A}_2 f_2^2 @ x D + 4 E^{I T_0 w_1} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 A_2 \dot{A}_2 f_2^2 @ x D + \\
& 3 E^{I T_0 w_2} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2^2 \dot{A}_2 f_2^2 @ x D + 2 E^{-I T_0 w_2} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_1 \dot{A}_2 f_2^2 @ x D + \\
& 4 E^{-I T_0 w_1} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_2 \dot{A}_1 \dot{A}_2 f_2^2 @ x D + E^{I T_0 H - 2 w_1 - w_2 L} a \int_{k=0}^1 f_1^c @ x D^2 \hat{a} x \frac{Y}{\{ } \dot{A}_1^2 \dot{A}_2 f_2^2 @ x D + \\
& 2 E^{I T_0 H w_1 - 2 w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_2^2 f_2^2 @ x D + 3 E^{-I T_0 w_2} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2 \dot{A}_2^2 f_2^2 @ x D + \\
& 2 E^{I T_0 H - w_1 - 2 w_2 L} a \int_{k=0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{Y}{\{ } A_1 \dot{A}_2^2 f_2^2 @ x D + E^{-3 I T_0 w_2} a \int_{k=0}^1 f_2^c @ x D^2 \hat{a} x \frac{Y}{\{ } A_2^3 f_2^2 @ x D
\end{aligned}$$

It follows from [eqEps\[\[2\]\]](#) that the first-order boundary conditions are

order1BC = eqEps@@2DD •• Rest

$$9w_1 @ 0, T_0, T_1 D == 0, w_1^{H2, 0, 0L} @ 0, T_0, T_1 D == 0, w_1 @ 1, T_0, T_1 D == 0, w_1^{H1, 0, 0L} @ 1, T_0, T_1 D == 0 =$$

à Principal Parametric Resonance of the First Mode

In this case, $W \gg 2w_1$. To describe the nearness of the internal and principal parametric resonances, we introduce the two detuning parameters S_1 and S_2 defined by

$$\text{ResonanceConds} = 8w_2 == 3w_1 + e s_1, W == 2w_1 + e s_2;$$

and define the following rules:

$$\text{OmgRule} = \text{Solve}@\text{ResonanceConds}, \text{Drop}@\text{omgList}, 8\#<D \sim \text{Join} \sim 8W<D@@1DD \& \cdot \checkmark 81, 2<$$

$$98w_2 \otimes e s_1 + 3w_1, W \otimes e s_2 + 2w_1<, 9w_1 \otimes \frac{1}{3} H - e s_1 + w_2 L, W \otimes \frac{1}{3} H - 2e s_1 + 3e s_2 + 2w_2 L ==$$

$$\text{expRule}@i_D := \text{Exp}@arg_D \rightarrow \text{Exp}@\text{Expand}@arg \cdot \cdot \text{OmgRule}@@iDDD \cdot \cdot e T_0 \rightarrow T_1 D$$

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

$$\text{ST} = \text{Table}@\text{Coefficient}@\text{order1Eq}@@2DD \cdot \cdot \text{expRule}@iD, \text{Exp}@I w_1 T_0 DD, 8i, 2<D;$$

ST •• displayRule

$$\begin{aligned} &: -2 I HD_1 A_1 L w_1 f_1 @ x D - 2 I A_1 w_1 m @ x D f_1 @ x D + 2 E^{I T_1 S_2} F A_1 f_1^2 @ x D + \\ &3 a \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{V}{\{ A_1^2 \dot{A}_1 f_1^2 @ x D + 2 E^{I T_1 S_1} a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{a} x \frac{V}{\{ A_2 \dot{A}_1^2 f_1^2 @ x D + \\ &2 a \int_{k_0}^1 f_2 @ x D^2 \hat{a} x \frac{V}{\{ A_1 A_2 \dot{A}_2 f_1^2 @ x D + 2 E^{I T_1 S_1 - I T_1 S_2} F A_2 f_2^2 @ x D + \\ &E^{I T_1 S_1} a \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{V}{\{ A_2 \dot{A}_1^2 f_2^2 @ x D + 4 a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{a} x \frac{V}{\{ A_1 A_2 \dot{A}_2 f_2^2 @ x D, \\ &-2 I HD_1 A_2 L w_2 f_2 @ x D - 2 I A_2 w_2 m @ x D f_2 @ x D + 2 E^{-I T_1 S_1 + I T_1 S_2} F A_1 f_1^2 @ x D + \\ &E^{-I T_1 S_1} a \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{V}{\{ A_1^3 f_1^2 @ x D + 4 a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{a} x \frac{V}{\{ A_1 A_2 \dot{A}_1 f_1^2 @ x D + \\ &2 a \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{V}{\{ A_1 A_2 \dot{A}_1 f_2^2 @ x D + 3 a \int_{k_0}^1 f_2 @ x D^2 \hat{a} x \frac{V}{\{ A_2^2 \dot{A}_2 f_2^2 @ x D > \end{aligned}$$

Because the homogeneous part of **order1Eq** and **order1BC** has a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. It follows from Section 7.1.1 that the homogeneous problem is self-adjoint and that the solvability conditions demand that **ST** be orthogonal to $f_1 @ x D$ and $f_2 @ x D$, respectively. Imposing these conditions, we have

SCond =

Table@int@fj@xD ST@@jDD, 8x, 0, 1<D •. intRule1 •. intRule2 •. int -> Integrate, 8j, 2<D ==
0 •• Thread;

SCond •. displayRule

$$\begin{aligned}
 & : 2 E^{I T_1 s_1 - I T_1 s_2} F \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } A_2 - 2 I \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } H D_1 A_1 L w_1 - 2 I \int_{k_0}^1 m @ x D f_1 @ x D^2 \hat{x} \frac{V}{\{ } A_1 w_1 + \\
 & 2 E^{I T_1 s_2} F \int_{k_0}^1 f_1 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } \dot{A}_1 + 3 a \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_1^2 \dot{A}_1 + \\
 & 2 E^{I T_1 s_1} a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_2 \dot{A}_1^2 + \\
 & E^{I T_1 s_1} a \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } A_2 \dot{A}_1^2 + 2 a \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } \\
 & \int_{k_0}^1 f_1 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_1 A_2 \dot{A}_2 + 4 a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } A_1 A_2 \dot{A}_2 == 0, \\
 & 2 E^{-I T_1 s_1 + I T_1 s_2} F \int_{k_0}^1 f_2 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_1 + E^{-I T_1 s_1} a \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_1^3 - \\
 & 2 I \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } H D_1 A_2 L w_2 - 2 I \int_{k_0}^1 m @ x D f_2 @ x D^2 \hat{x} \frac{V}{\{ } A_2 w_2 + \\
 & 4 a \int_{k_0}^1 f_1 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2 @ x D f_1 @ x D \hat{x} \frac{V}{\{ } A_1 A_2 \dot{A}_1 + 2 a \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } \\
 & \int_{k_0}^1 f_2 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } A_1 A_2 \dot{A}_1 + 3 a \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2 @ x D f_2 @ x D \hat{x} \frac{V}{\{ } A_2^2 \dot{A}_2 == 0 >
 \end{aligned}$$

To simplify the notation in the solvability conditions, we use the orthonormality of the mode shapes and define the following parameters:

$$\text{notationRule} = 9 \hat{a}_0^1 f_{i-} @ x D^2 \hat{x} \frac{V}{\{ } x \rightarrow 1, \hat{a}_0^1 m @ x D f_{i-} @ x D^2 \hat{x} \frac{V}{\{ } x \rightarrow m_i, F \hat{a}_0^1 f_{i-} @ x D f_{j-} @ x D \hat{x} \frac{V}{\{ } x \rightarrow -w_i s_{i,j} f;$$

To identify the forms of the nonlinear terms in the solvability conditions, we first identify all of the possible forms of the nonhomogeneous terms in the first-order problem. To this end, we let

basicTerms = TableA9A_i@T₁D E^{I w_i T₀}, $\dot{A}_i$ @T₁D E^{-I w_i T₀}==, 8i, 2<E •• Flatten

$$8 E^{I T_0 w_1} A_1 @ T_1 D, E^{-I T_0 w_1} \dot{A}_1 @ T_1 D, E^{I T_0 w_2} A_2 @ T_1 D, E^{-I T_0 w_2} \dot{A}_2 @ T_1 D <$$

Then, all of the possible forms of the nonlinear terms in the first-order problem are given by

cubicTerms = Nest@Outer@Times, basicTerms, #D &, basicTerms, 2D •• Flatten •• Union;
cubicTerms •. displayRule

$$\begin{aligned}
 & : E^{3 I T_0 w_1} A_1^3, E^{I T_0 H_2 w_1 + w_2 L} A_1^2 A_2, E^{I T_0 H w_1 + 2 w_2 L} A_1 A_2^2, E^{3 I T_0 w_2} A_2^3, E^{I T_0 w_1} A_1^2 \dot{A}_1, \\
 & E^{I T_0 w_2} A_1 A_2 \dot{A}_1, E^{I T_0 H - w_1 + 2 w_2 L} A_2^2 \dot{A}_1, E^{-I T_0 w_1} A_1 \dot{A}_1^2, E^{I T_0 H - 2 w_1 + w_2 L} A_2 \dot{A}_1^2, E^{-3 I T_0 w_1} \dot{A}_1^3, \\
 & E^{I T_0 H_2 w_1 - w_2 L} A_1^2 \dot{A}_2, E^{I T_0 w_1} A_1 A_2 \dot{A}_2, E^{I T_0 w_2} A_2^2 \dot{A}_2, E^{-I T_0 w_2} A_1 \dot{A}_1 \dot{A}_2, E^{-I T_0 w_1} A_2 \dot{A}_1 \dot{A}_2, \\
 & E^{I T_0 H - 2 w_1 - w_2 L} \dot{A}_1^2 \dot{A}_2, E^{I T_0 H w_1 - 2 w_2 L} A_1 \dot{A}_2^2, E^{-I T_0 w_2} A_2 \dot{A}_2^2, E^{I T_0 H - w_1 - 2 w_2 L} \dot{A}_1 \dot{A}_2^2, E^{-3 I T_0 w_2} \dot{A}_2^3 >
 \end{aligned}$$

Out of these terms, only the terms that may lead to secular terms appear in the solvability conditions, which can be identified according to

```
secularTerms =
  I E-I w# T0 cubicTerms •. expRule@#D •. Exp@_ T0 + _ .D -> 0 •• Union •• RestM & •ž 81, 2<

: : A1@T1D2 Ā1@T1D, EI T1 s1 A2@T1D Ā1@T1D2, A1@T1D A2@T1D Ā2@T1D>,
  8 E-I T1 s1 A1@T1D3, A1@T1D A2@T1D Ā1@T1D, A2@T1D2 Ā2@T1D<>
```

Next, we define the following parameters:

```
symbolList = 8 - 8 w1 8 g11, d1, g12<, - 8 w2 8 d2, g21, g22<<

88 - 8 g11 w1, - 8 d1 w1, - 8 g12 w1<, 8 - 8 d2 w2, - 8 g21 w2, - 8 g22 w2<<
```

and express **SCond** in a more concise form as

```
eqMod = ExpandA - 1 HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
  symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<

: 2 I m1 A1@T1D + 2 EI T1 s1 - I T1 s2 fS1,2 A2@T1D + 2 EI T1 s2 fS1,1 Ā1@T1D + 8 g11 A1@T1D2 Ā1@T1D +
  8 EI T1 s1 d1 A2@T1D Ā1@T1D2 + 8 g12 A1@T1D A2@T1D Ā2@T1D + 2 I A1†@T1D == 0,
  2 E-I T1 s1 + I T1 s2 fS2,1 A1@T1D + 8 E-I T1 s1 d2 A1@T1D3 + 2 I m2 A2@T1D +
  8 g21 A1@T1D A2@T1D Ā1@T1D + 8 g22 A2@T1D2 Ā2@T1D + 2 I A2†@T1D == 0>
```

where

```
-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
TableForm

8 g11 w1 ® - 3 a I ð01 f1†@xD2 â xM ð01 f1@xD f1†@xD â x
8 d1 w1 ® - 2 a I ð01 f1†@xD f2†@xD â xM ð01 f1@xD f1†@xD â x - a I ð01 f1†@xD2 â xM ð01 f1@xD f2†@xD â x
8 g12 w1 ® - 2 a I ð01 f2†@xD2 â xM ð01 f1@xD f1†@xD â x - 4 a I ð01 f1†@xD f2†@xD â xM ð01 f1@xD f2†@xD â x

-symbolList@@2DD -> H-Coefficient@SCond@@2, 1DD, #D & •ž secularTerms@@2DDL •• Thread ••
TableForm

8 d2 w2 ® - a I ð01 f1†@xD2 â xM ð01 f2@xD f1†@xD â x
8 g21 w2 ® - 4 a I ð01 f1†@xD f2†@xD â xM ð01 f2@xD f1†@xD â x - 2 a I ð01 f1†@xD2 â xM ð01 f2@xD f2†@xD â x
8 g22 w2 ® - 3 a I ð01 f2†@xD2 â xM ð01 f2@xD f2†@xD â x
```

Modulation Equations in Polar Form

The complex-valued solvability conditions can be expressed in real-valued form by introducing the polar transformation

$$\text{ruleA} = 9A_i \rightarrow \int_{k=2}^1 a_i \# D \text{Exp} @ I q_i \# DD \& \{ \dot{A}_i \rightarrow \int_{k=2}^1 a_i \# D \text{Exp} @ - I q_i \# DD \& \{ \dot{A}_i \} ;$$

into **eqMod** and obtain

$$\begin{aligned} \text{expr1} = & \text{Expand} @ \text{eqMod} @ \{ \#, 1DD \text{Exp} @ - I q \# T_1 DD \cdot \text{ruleAD} \& \cdot \dot{z} \} 81, 2 < \\ 8I m_1 a_1 @ T_1 D + E^{I T_1 s_2 - 2 I q_1 @ T_1 D} f s_{1,1} a_1 @ T_1 D + g_{11} a_1 @ T_1 D^3 + E^{I T_1 s_1 - I T_1 s_2 - I q_1 @ T_1 D + I q_2 @ T_1 D} f s_{1,2} a_2 @ T_1 D + \\ & E^{I T_1 s_1 - 3 I q_1 @ T_1 D + I q_2 @ T_1 D} d_1 a_1 @ T_1 D^2 a_2 @ T_1 D + g_{12} a_1 @ T_1 D a_2 @ T_1 D^2 + I a_1^c @ T_1 D - a_1 @ T_1 D q_1^c @ T_1 D, \\ & E^{-I T_1 s_1 + I T_1 s_2 + I q_1 @ T_1 D - I q_2 @ T_1 D} f s_{2,1} a_1 @ T_1 D + E^{-I T_1 s_1 + 3 I q_1 @ T_1 D - I q_2 @ T_1 D} d_2 a_1 @ T_1 D^3 + \\ & I m_2 a_2 @ T_1 D + g_{21} a_1 @ T_1 D^2 a_2 @ T_1 D + g_{22} a_2 @ T_1 D^3 + I a_2^c @ T_1 D - a_2 @ T_1 D q_2^c @ T_1 D < \end{aligned}$$

Next, we separate the real and imaginary parts of **expr1** to obtain the equations governing the modulation of the amplitudes a_i and phases q_i . To accomplish this, we define the rule

$$\text{realRule} = 8\text{Re} @ s_D \rightarrow s, \text{Im} @ s_D \rightarrow 0 <;$$

Then, the equations governing the amplitudes of motion correspond to the imaginary parts of **expr1**; that is,

$$\begin{aligned} \text{ampEq} = & \text{Solve} @ \text{ComplexExpand} @ \text{Im} @ \# DD == 0 \cdot \text{realRule} \& \cdot \dot{z} \text{expr1}, 8a_1^c @ T_1 D, a_2^c @ T_1 D < D @ 1DD \cdot \\ & \text{Rule} \rightarrow \text{Equal} \cdot \cdot \text{ExpandAll} \\ 8a_1^c @ T_1 D = & -m_1 a_1 @ T_1 D - f \sin @ T_1 s_2 - 2 q_1 @ T_1 DD s_{1,1} a_1 @ T_1 D - \\ & f \sin @ T_1 s_1 - T_1 s_2 - q_1 @ T_1 D + q_2 @ T_1 DD s_{1,2} a_2 @ T_1 D - \sin @ T_1 s_1 - 3 q_1 @ T_1 D + q_2 @ T_1 DD d_1 a_1 @ T_1 D^2 a_2 @ T_1 D, \\ a_2^c @ T_1 D = & f \sin @ T_1 s_1 - T_1 s_2 - q_1 @ T_1 D + q_2 @ T_1 DD s_{2,1} a_1 @ T_1 D + \\ & \sin @ T_1 s_1 - 3 q_1 @ T_1 D + q_2 @ T_1 DD d_2 a_1 @ T_1 D^3 - m_2 a_2 @ T_1 D < \end{aligned}$$

Moreover, the equations governing the phases of motion correspond to the real parts of **expr1**; that is,

$$\begin{aligned} \text{phaseEq} = & \text{Solve} @ \text{ComplexExpand} @ \text{Re} @ \# DD == 0 \cdot \text{realRule} \& \cdot \dot{z} \text{expr1}, 8q_1^c @ T_1 D, q_2^c @ T_1 D < D @ 1DD \cdot \\ & \text{Rule} \rightarrow \text{Equal} \cdot \cdot \text{ExpandAll} \\ : q_1^c @ T_1 D = & \\ & f \cos @ T_1 s_2 - 2 q_1 @ T_1 DD s_{1,1} + g_{11} a_1 @ T_1 D^2 + \frac{f \cos @ T_1 s_1 - T_1 s_2 - q_1 @ T_1 D + q_2 @ T_1 DD s_{1,2} a_2 @ T_1 D}{a_1 @ T_1 D} + \\ & \cos @ T_1 s_1 - 3 q_1 @ T_1 D + q_2 @ T_1 DD d_1 a_1 @ T_1 D a_2 @ T_1 D + g_{12} a_2 @ T_1 D^2, \\ q_2^c @ T_1 D = & g_{21} a_1 @ T_1 D^2 + \frac{f \cos @ T_1 s_1 - T_1 s_2 - q_1 @ T_1 D + q_2 @ T_1 DD s_{2,1} a_1 @ T_1 D}{a_2 @ T_1 D} + \\ & \frac{\cos @ T_1 s_1 - 3 q_1 @ T_1 D + q_2 @ T_1 DD d_2 a_1 @ T_1 D^3}{a_2 @ T_1 D} + g_{22} a_2 @ T_1 D^2 > \end{aligned}$$

These modulation equations are nonautonomous because they depend explicitly on T_1 .

To determine an autonomous set of modulation equations, we start with **expr1** and identify the independent phase arguments in it. To accomplish this, we first identify all possible phase arguments; that is,

expTerms = I Cases@expr1, Exp@arg_D -> arg, InfinityD •• Expand

8- T₁ s₂ + 2 q₁@T₁D, -T₁ s₁ + T₁ s₂ + q₁@T₁D - q₂@T₁D,
 -T₁ s₁ + 3 q₁@T₁D - q₂@T₁D, T₁ s₁ - T₁ s₂ - q₁@T₁D + q₂@T₁D, T₁ s₁ - 3 q₁@T₁D + q₂@T₁D<

Out of these arguments, only two are independent because

Outer@D, expTerms, 8q₁@T₁D, q₂@T₁D<D •• RowReduce

881, 0<, 80, 1<, 80, 0<, 80, 0<, 80, 0<<

Denoting these two independent arguments by g₁ and g₂, we have

gammaList = 82 g₁@T₁D, g₂@T₁D< == expTerms@@81, 2<DD •• Thread

82 g₁@T₁D == -T₁ s₂ + 2 q₁@T₁D, g₂@T₁D == -T₁ s₁ + T₁ s₂ + q₁@T₁D - q₂@T₁D<

which can be solved for q₁ and q₂ to obtain

thetaRule = Solve@gammaList, 8q₁@T₁D, q₂@T₁D<D@@1DD •• ExpandAll

9q₁@T₁D ® $\frac{T_1 s_2}{2}$ + g₁@T₁D, q₂@T₁D ® -T₁ s₁ + $\frac{3 T_1 s_2}{2}$ + g₁@T₁D - g₂@T₁D=

Substituting for the q_i in **expr1** yields

**expr2 = expr1 •. Table@q_i -> HEvaluate@thetaRule@@i, 2DD •. T₁ -> #D &L, 8i, 2<D •.
 Exp@arg_D := Exp@Expand@argDD •• Expand**

9I m₁ a₁@T₁D - $\frac{1}{2}$ s₂ a₁@T₁D + E^{-2 I g₁@T₁D} f s_{1,1} a₁@T₁D + g₁₁ a₁@T₁D³ + E^{-I g₂@T₁D} f s_{1,2} a₂@T₁D +
 E^{-2 I g₁@T₁D - I g₂@T₁D} d₁ a₁@T₁D² a₂@T₁D + g₁₂ a₁@T₁D a₂@T₁D² + I a₁^ε@T₁D - a₁@T₁D g₁^ε@T₁D,
 E^{I g₂@T₁D} f s_{2,1} a₁@T₁D + E^{2 I g₁@T₁D + I g₂@T₁D} d₂ a₁@T₁D³ + I m₂ a₂@T₁D + s₁ a₂@T₁D - $\frac{3}{2}$ s₂ a₂@T₁D +
 g₂₁ a₁@T₁D² a₂@T₁D + g₂₂ a₂@T₁D³ + I a₂^ε@T₁D - a₂@T₁D g₁^ε@T₁D + a₂@T₁D g₂^ε@T₁D=

Then, the equations governing the amplitudes of motion correspond to the imaginary parts of **expr2**; that is,

**HampEq = Solve@ComplexExpand@Im@#DD == 0 •. realRule & •ž expr2, 8a₁^ε@T₁D, a₂^ε@T₁D<D@@1DD •.
 Rule -> Equal •• ExpandAllL •. f_@T₁D -> f**

8a₁^ε == Sin@2 g₁ + g₂D a₁² a₂ d₁ - a₁ m₁ + f Sin@2 g₁D a₁ s_{1,1} + f Sin@g₂D a₂ s_{1,2},
 a₂^ε == -Sin@2 g₁ + g₂D a₁³ d₂ - a₂ m₂ - f Sin@g₂D a₁ s_{2,1}<

Moreover, the equations governing the phases of motion correspond to the real parts of **expr2**; that is,


```
lambdaRule = Solve@list1 + 8m1, m2< 2 p == 0 •• Thread, Table@li@T1D, 8i, 2<DD@@1DD
```

$$9l_1@T_1D \otimes \frac{1}{2} H^2 p m_1 + T_1 s_2 L, l_2@T_1D \otimes \frac{1}{2} H^2 p m_1 + 4 p m_2 - 2 T_1 s_1 + 3 T_1 s_2 L =$$

where the m_i are positive or negative integers. Letting $n_i = l_i^c$ yields

```
D@lambdaRule, T1D •. realRule
```

$$9n_1 \otimes \frac{s_2}{2}, n_2 \otimes \frac{1}{2} H - 2 s_1 + 3 s_2 L =$$

Then, separating the real and imaginary parts of **eqModCart**, we obtain the following Cartesian form of the modulation equations:

```
Heqs = Solve@
```

```
Flatten@Table@ComplexExpand@8Im@#D, Re@#D<D & •ž HeqModCart@kD •. Exp@_D -> 1L ••.
```

```
realRule, 8k, 2<DD == 0 •• Thread, Table@8pkc@T1D, qkc@T1D<, 8k, 2<D ••
```

```
FlattenD@@1DD •. Rule -> EqualL •. f_@T1D -> f •• TableForm
```

$$\begin{aligned} p_1^c &== p_1^2 q_1 g_{11} + q_1^3 g_{11} + p_2^2 q_1 g_{12} + q_1 q_2^2 g_{12} - 2 p_1 p_2 q_1 d_1 + p_1^2 q_2 d_1 - q_1^2 q_2 d_1 - p_1 m_1 - q_1 n_1 - f q_1 s_{1,1} + f q_2 \\ q_1^c &== -p_1^3 g_{11} - p_1 q_1^2 g_{11} - p_1 p_2^2 g_{12} - p_1 q_2^2 g_{12} - p_1^2 p_2 d_1 + p_2 q_1^2 d_1 - 2 p_1 q_1 q_2 d_1 - q_1 m_1 + p_1 n_1 - f p_1 s_{1,1} - f p \\ p_2^c &== p_1^2 q_2 g_{21} + q_1^2 q_2 g_{21} + p_2^2 q_2 g_{22} + q_2^3 g_{22} + 3 p_1^2 q_1 d_2 - q_1^3 d_2 - p_2 m_2 - q_2 n_2 + f q_1 s_{2,1} \\ q_2^c &== -p_1^2 p_2 g_{21} - p_2 q_1^2 g_{21} - p_2^3 g_{22} - p_2 q_2^2 g_{22} - p_1^3 d_2 + 3 p_1 q_1^2 d_2 - q_2 m_2 + p_2 n_2 - f p_1 s_{2,1} \end{aligned}$$

The symmetry property of the system can also be obtained as follows:

```
phase = Flatten@
```

```
Table@Mod@#@@@2DD & •ž lambdaRule •. T1 -> 0, 2 pD, 8m1, 0, 10<, 8m2, 0, 10<D, 1D •• Union
```

```
880, 0<, 8p, p<<
```

```
rotMat@a_D = 88Cos@aD, Sin@aD<, 8-Sin@aD, Cos@aD<<;
```

```
symmetryList = Join žž Table@rotMat@#@@@iDDD.8pi, qi<, 8i, 2<D & •ž phase
```

```
88p1, q1, p2, q2<, 8-p1, -q1, -p2, -q2<<
```

Hence, if $\{p_1, q_1, p_2, q_2\}$ is a solution of the modulation equations, then $\{8-p_1, -q_1, -p_2, -q_2\}$ is also another solution of these equations. If these two sets of solutions are the same, then the system response is symmetric; otherwise, it will be asymmetric.

à The Functions **PolarForm** and **CartesianForm**

Following the procedures described in the preceding section, we build two functions, **PolarForm** and **CartesianForm**, to automate the process provided that the complex dependent variables in the modulation equations are written in the form S_i , $i = 1, \dots, \text{neq}$ and the corresponding complex conjugates are $\bar{S}_i$. The functions can be slightly modified to allow a more general form for the modulation equations. To obtain the polar form or Cartesian form of the complex modulation equations, we simply provide the list of modulation equations and S as two required inputs in **PolarForm** or **CartesianForm**, respectively. If $S = A$, the list of modulation equations is the only required input.

```

PolarForm@eqs_List, S_Symbol: AD :=
Module[A8neq = Length@eqsD, t<,
  t = Cases@eqs, S_i@a_D -> a, ¥D@@1DD;
  polarRule = 9S_k_ ->  $\int_{-\infty}^{\infty} a_k \exp(i b_k t) dt$ , S_k_ ->  $\int_{-\infty}^{\infty} a_k \exp(-i b_k t) dt$ ;
  realRule = 8Im@x_D -> 0, Re@x_D -> x<;
  eqModPolar@k_D := Exp@-I b_k@tDD Subtract @@ eqs@@kDD . polarRule . Expand;
  list1 = I Cases@Array@eqModPolar, neqD, Exp@a_D -> a, ¥D . Expand . Union;
  newList1@list_, n_D :=
    With@8v = Table@Unique@D, 8n<D<,
      ReplaceList@list,
    Append@Flatten@8____, Pattern@#, _D< & . ¥ vD, ____D -> vDD;
    betaList = Table@b_i@tD, 8i, neq<D;
    list2 = 8<;
    Scan@
  If@NullSpace@Outer@Coefficient, #, betaListDD === 8<, list2 = -#; Return@DD &,
  newList1@list1, neqDD;
  If@list2 === 8<, Print@
    "Autonomous system is not possible!!!\nThe phase list is:"D; Return@list1D
  D;
  betaRule1 = Solve@list3 = Table@g_i@tD, 8i, neq<D == list2 . Thread, betaListD@@1DD;
  betaRule2 =
  Table@b_i -> Function@t . Evaluate, b_i@tD . betaRule1 . EvaluateD, 8i, neq<D;
  Do@eq0@iD = eqModPolar@iD . betaRule2 . Exp@a_D :> Exp@Expand@aDD;
  eq@iD =
  Solve@ComplexExpand@Im@eq0@iDDD == 0 . realRule, a_i^@tDD@@1, 1DD . Rule -> Equal;
  eqRe@iD = ComplexExpand@Re@eq0@iDDD == 0 . realRule, 8i, neq<
  D;
  gammapList = Table@g_i^@tD, 8i, neq<D;
  eqList = Solve@Array@eqRe, neqD, gammapListD@@1DD . Rule -> Equal . ExpandAll;
  Do@eq@i + neqD = eqList@@iDD, 8i, neq<D;
  8Array@eq, 2 * neqD, list3<
E

```

```

CartesianForm@eqs_List, S_Symbol: AD :=
Module[A8neq = Length@eqsD, t<,
  t = Cases@eqs, S_i@a_D -> a, ¥D@@1DD;
  cartRule = 9S_k -> H1 • 2 Hp_k@#D - I q_k@#DL Exp@I l_k@#DD &L,
  Š_k -> H1 • 2 Hp_k@#D + I q_k@#DL Exp@-I l_k@#DD &L=;
  realRule = 8Im@x_D -> 0, Re@x_D -> x, l_k^tD -> n_k<;
  eqModCart@k_D := Exp@-I l_k@tDD Subtract žž eqs@@kDD •. cartRule •• Expand;
  eqs1 = 8ComplexExpand@Im@#DD == 0, ComplexExpand@Re@#DD == 0< & •ž
  HArray@eqModCart, neqD •. E^_ -> 1L ••. realRule •• Flatten;
  eqs2 = Solve@eqs1, Table@8p_k^tD, q_k^tD<, 8k, neq<D •• FlattenD@@1DD •.
Rule -> Equal;
  list1 = I Cases@Array@eqModCart, neqD, Exp@a_D -> a, ¥D •• Expand •• Union;
  newList1@list_, n_D :=
    With@8v = Table@Unique@D, 8n<D<,
      ReplaceList@list,
Append@Flatten@8____, Pattern@#, _D< & •ž vD, ____D -> vDD;
  lambdaList = Table@l_i@tD, 8i, neq<D;
  list2 = 8<;
  Scan@
If@NullSpace@Outer@Coefficient, #, lambdaListDD === 8<, list2 = -#; Return@DD &,
newList1@list1, neqDD;
  If@list2 === 8<, Print@
"Autonomous system is not possible!!!\nThe phase list is:"D; Return@list1D
D;
  lambdaRule = Solve@list2 + 2 Table@m_i, 8i, neq<D p == 0 •• Thread, lambdaListD@@1DD;
  iterList = Table@8m_i, 0, 10<, 8i, neq<D;
  phase =
Flatten@Table@Mod@Select@#, !FreeQ@#, pD &D & •ž HlambdaList •. lambdaRule •• ExpandL,
2 pD •• Evaluate, Sequence žž iterList •• Evaluated, neq - 1D •• Union;
  rotMat@q_D := 88Cos@qD, -Sin@qD<, 8Sin@qD, Cos@qD<<;
  symmetryList = Flatten@Table@rotMat@#@@@kDDD.8p_k, q_k<, 8k, 2<DD & •ž phase;
  8eqs2, D@lambdaRule, tD •. realRule, symmetryList<
E

```

Ÿ Principal Parametric Resonance of the Second Mode

In this case, $W \gg w_2$. To describe the nearness of the internal and principal parametric resonances, we introduce the two detuning parameters S_1 and S_2 defined by

$$\text{ResonanceConds} = 8w_2 == 3w_1 + e s_1, W == 2w_2 + e s_2<;$$

and define the following rules:

OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D~Join~8W<D@@1DD & .ž 81, 2<

$$98w_2 \otimes e s_1 + 3 w_1, W \otimes 2 e s_1 + e s_2 + 6 w_1 <, 9w_1 \otimes \frac{1}{3} H - e s_1 + w_2 L, W \otimes e s_2 + 2 w_2 ==$$

expRule@i_D := Exp@arg_D :> Exp@Expand@arg . . OmgRule@@iDDD . . e T₀ -> T₁D

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

ST = Table@Coefficient@order1Eq@@2DD . . expRule@iD, Exp@I w_i T₀DD, 8i, 2<D;

ST . . displayRule

$$\begin{aligned} &: -2 \int_0^1 H D_1 A_1 L w_1 f_1(x) D - 2 \int_0^1 A_1 w_1 m(x) D f_1(x) D + 3 a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{V}{Z} A_1^2 \dot{A}_1 f_1^2(x) D + \\ &2 E^{I T_1 s_1} a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{V}{Z} A_2 \dot{A}_1^2 f_1^2(x) D + 2 a \int_0^1 f_2^e(x) D^2 \hat{a} x \frac{V}{Z} A_1 A_2 \dot{A}_2 f_1^2(x) D + \\ &E^{I T_1 s_1} a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{V}{Z} A_2 \dot{A}_1^2 f_2^2(x) D + 4 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{V}{Z} A_1 A_2 \dot{A}_2 f_2^2(x) D, \\ &-2 \int_0^1 H D_1 A_2 L w_2 f_2(x) D - 2 \int_0^1 A_2 w_2 m(x) D f_2(x) D + E^{-I T_1 s_1} a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{V}{Z} A_1^3 f_1^2(x) D + \\ &4 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{V}{Z} A_1 A_2 \dot{A}_1 f_1^2(x) D + 2 a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{V}{Z} A_1 A_2 \dot{A}_1 f_2^2(x) D + \\ &2 E^{I T_1 s_2} F \dot{A}_2 f_2^2(x) D + 3 a \int_0^1 f_2^e(x) D^2 \hat{a} x \frac{V}{Z} A_2^2 \dot{A}_2 f_2^2(x) D > \end{aligned}$$

As discussed in the case of principal parametric resonance of the first mode, the solvability conditions of **order1Eq** and **order1BC** demand that **ST** be orthogonal to $f_1(x)$ and $f_2(x)$, respectively. Imposing these conditions, we have

SCond =

Table@int@fj@xD ST@@jDD, 8x, 0, 1<D •. intRule1 •. intRule2 •. int -> Integrate, 8j, 2<D ==
0 •• Thread;

SCond •. displayRule

$$\begin{aligned}
 &:- 2 \int_0^1 f_1(x) D^2 \hat{a} x \frac{1}{\{}} HD_1 A_1 L w_1 - \\
 &2 \int_0^1 m(x) f_1(x) D \hat{a} x \frac{1}{\{}} A_1 w_1 + 3 a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{1}{\{}} \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x \frac{1}{\{}} A_1^2 \dot{A}_1 + \\
 &2 E^{I T_1 s_1} a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{1}{\{}} \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x \frac{1}{\{}} A_2 \dot{A}_1^2 + E^{I T_1 s_1} a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{1}{\{}} \\
 &\int_0^1 f_1(x) D f_2^2(x) D \hat{a} x \frac{1}{\{}} A_2 \dot{A}_1^2 + 2 a \int_0^1 f_2^e(x) D^2 \hat{a} x \frac{1}{\{}} \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x \frac{1}{\{}} A_1 A_2 \dot{A}_2 + \\
 &4 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{1}{\{}} \int_0^1 f_1(x) D f_2^2(x) D \hat{a} x \frac{1}{\{}} A_1 A_2 \dot{A}_2 == 0, \\
 &E^{-I T_1 s_1} a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{1}{\{}} \int_0^1 f_2(x) D f_1^2(x) D \hat{a} x \frac{1}{\{}} A_1^3 - 2 \int_0^1 f_2(x) D^2 \hat{a} x \frac{1}{\{}} HD_1 A_2 L w_2 - \\
 &2 \int_0^1 m(x) D f_2(x) D^2 \hat{a} x \frac{1}{\{}} A_2 w_2 + 4 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x \frac{1}{\{}} \int_0^1 f_2(x) D f_1^2(x) D \hat{a} x \frac{1}{\{}} A_1 A_2 \dot{A}_1 + \\
 &2 a \int_0^1 f_1^e(x) D^2 \hat{a} x \frac{1}{\{}} \int_0^1 f_2(x) D f_2^2(x) D \hat{a} x \frac{1}{\{}} A_1 A_2 \dot{A}_1 + 2 E^{I T_1 s_2} F \int_0^1 f_2(x) D f_2^2(x) D \hat{a} x \frac{1}{\{}} \dot{A}_2 + \\
 &3 a \int_0^1 f_2^e(x) D^2 \hat{a} x \frac{1}{\{}} \int_0^1 f_2(x) D f_2^2(x) D \hat{a} x \frac{1}{\{}} A_2^2 \dot{A}_2 == 0 >
 \end{aligned}$$

Using the notations introduced in the preceding section, we can express **SCond** in a more concise form as

eqMod = ExpandA- $\frac{1}{w_{\#}}$ HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<

$$\begin{aligned}
 &: 2 \int m_1 A_1 @ T_1 D + 8 g_{11} A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D + \\
 &8 E^{I T_1 s_1} d_1 A_2 @ T_1 D \dot{A}_1 @ T_1 D^2 + 8 g_{12} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + 2 \int A_1^e @ T_1 D == 0, \\
 &8 E^{-I T_1 s_1} d_2 A_1 @ T_1 D^3 + 2 \int m_2 A_2 @ T_1 D + 8 g_{21} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\
 &2 E^{I T_1 s_2} f_{s_2,2} \dot{A}_2 @ T_1 D + 8 g_{22} A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D + 2 \int A_2^e @ T_1 D == 0 >
 \end{aligned}$$

where

-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
TableForm

$$\begin{aligned}
 &8 g_{11} w_1 @ - 3 a \int_0^1 f_1^e(x) D^2 \hat{a} x M \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x \\
 &8 d_1 w_1 @ - 2 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x M \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x - a \int_0^1 f_1^e(x) D^2 \hat{a} x M \int_0^1 f_1(x) D f_2^2(x) D \hat{a} x \\
 &8 g_{12} w_1 @ - 2 a \int_0^1 f_2^e(x) D^2 \hat{a} x M \int_0^1 f_1(x) D f_1^2(x) D \hat{a} x - 4 a \int_0^1 f_1^e(x) D f_2^e(x) D \hat{a} x M \int_0^1 f_1(x) D f_2^2(x) D \hat{a} x
 \end{aligned}$$

-symbolList@2DD -> H-Coefficient@SCond@@2, 1DD, #D & .ž secularTerms@@2DDL •• Thread ••
TableForm

$$\begin{aligned} 8 d_2 w_2 & \otimes - a \int_0^1 f_1^e(xD^2) \hat{x} M \int_0^1 f_2(xD) f_1^2(xD) \hat{x} x \\ 8 g_{21} w_2 & \otimes - 4 a \int_0^1 f_1^e(xD) f_2^e(xD) \hat{x} M \int_0^1 f_2(xD) f_1^2(xD) \hat{x} x - 2 a \int_0^1 f_1^e(xD^2) \hat{x} M \int_0^1 f_2(xD) f_2^2(xD) \hat{x} x \\ 8 g_{22} w_2 & \otimes - 3 a \int_0^1 f_2^e(xD^2) \hat{x} M \int_0^1 f_2(xD) f_2^2(xD) \hat{x} x \end{aligned}$$

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm** defined in the preceding section, we obtain the modulation equations in polar form and the definitions for g_i as

PolarForm@eqModD

$$\begin{aligned} :: a_1 @ T_1 D & == -m_1 a_1 @ T_1 D - \sin@g_1 @ T_1 DD d_1 a_1 @ T_1 D^2 a_2 @ T_1 D, \\ a_2 @ T_1 D & == \sin@g_1 @ T_1 DD d_2 a_1 @ T_1 D^3 - m_2 a_2 @ T_1 D - f \sin@g_2 @ T_1 DD s_{2,2} a_2 @ T_1 D, \\ g_1 @ T_1 D & == s_1 + f \cos@g_2 @ T_1 DD s_{2,2} - 3 g_{11} a_1 @ T_1 D^2 + g_{21} a_1 @ T_1 D^2 + \\ & \quad \frac{\cos@g_1 @ T_1 DD d_2 a_1 @ T_1 D^3}{a_2 @ T_1 D} - 3 \cos@g_1 @ T_1 DD d_1 a_1 @ T_1 D a_2 @ T_1 D - 3 g_{12} a_2 @ T_1 D^2 + g_{22} a_2 @ T_1 D^2, \\ g_2 @ T_1 D & == s_2 - 2 f \cos@g_2 @ T_1 DD s_{2,2} - 2 g_{21} a_1 @ T_1 D^2 - \frac{2 \cos@g_1 @ T_1 DD d_2 a_1 @ T_1 D^3}{a_2 @ T_1 D} - 2 g_{22} a_2 @ T_1 D^2, \\ 8 g_1 @ T_1 D & == T_1 s_1 - 3 b_1 @ T_1 D + b_2 @ T_1 D, \quad g_2 @ T_1 D == T_1 s_2 - 2 b_2 @ T_1 D <> \end{aligned}$$

Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm** defined in the preceding section, we obtain the modulation equations in Cartesian form, the definitions for η_i , and the symmetry property as

CartesianForm@eqModD

```

:8p1@T1D == -m1 p1@T1D - n1 q1@T1D + g11 p1@T1D^2 q1@T1D - 2 d1 p1@T1D p2@T1D q1@T1D +
  g12 p2@T1D^2 q1@T1D + g11 q1@T1D^3 + d1 p1@T1D^2 q2@T1D - d1 q1@T1D^2 q2@T1D + g12 q1@T1D q2@T1D^2,
q1@T1D == n1 p1@T1D - g11 p1@T1D^3 - d1 p1@T1D^2 p2@T1D - g12 p1@T1D p2@T1D^2 - m1 q1@T1D -
  g11 p1@T1D q1@T1D^2 + d1 p2@T1D q1@T1D^2 - 2 d1 p1@T1D q1@T1D q2@T1D - g12 p1@T1D q2@T1D^2,
p2@T1D == -m2 p2@T1D + 3 d2 p1@T1D^2 q1@T1D - d2 q1@T1D^3 - n2 q2@T1D - f s2,2 q2@T1D +
  g21 p1@T1D^2 q2@T1D + g22 p2@T1D^2 q2@T1D + g21 q1@T1D^2 q2@T1D + g22 q2@T1D^3,
q2@T1D == -d2 p1@T1D^3 + n2 p2@T1D - f s2,2 p2@T1D - g21 p1@T1D^2 p2@T1D - g22 p2@T1D^3 +
  3 d2 p1@T1D q1@T1D^2 - g21 p2@T1D q1@T1D^2 - m2 q2@T1D - g22 p2@T1D q2@T1D^2<,
9n1 @ 1/6 H2 s1 + s2L, n2 @ s2/2 =, :8p1, q1, p2, q2<, : p1/2 - 3/2 q1, 3/2 p1 + q1/2, -p2, -q2>,
: - p1/2 - 3/2 q1, 3/2 p1 - q1/2, p2, q2>, 8-p1, -q1, -p2, -q2<,
: - p1/2 + 3/2 q1, -1/2 3 p1 - q1/2, p2, q2>, : p1/2 + 3/2 q1, -1/2 3 p1 + q1/2, -p2, -q2>>>

```

Combination Parametric Resonance of the Two Modes

In this case, $W \gg w_1 + w_2$. To describe the nearness of the internal and combination parametric resonances, we introduce the two detuning parameters s_1 and s_2 defined by

$$\text{ResonanceConds} = 8w_2 == 3w_1 + e s_1, W == w_1 + w_2 + e s_2<;$$

and define the following rules:

$$\text{OmgRule} = \text{Solve@ResonanceConds, Drop@omgList, 8\#<D~Join~8W<D@@1DD \& \cdot \checkmark 81, 2<$$

$$98w_2 @ e s_1 + 3w_1, W @ e s_1 + e s_2 + 4w_1<, 9w_1 @ \frac{1}{3} H - e s_1 + w_2 L, W @ \frac{1}{3} H - e s_1 + 3e s_2 + 4w_2 L ==$$

$$\text{expRule@i_D} := \text{Exp@arg_D} :> \text{Exp@Expand@arg} \cdot \text{OmgRule@@iDDD} \cdot e T_0 -> T_1 D$$

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

**ST = Table@Coefficient@order1Eq@@2DD •. expRule@iD, Exp@I w₁ T₀DD, 8i, 2<D;
ST •. displayRule**

$$\begin{aligned}
 &: -2 \int_{k_0}^1 \text{HD}_1 A_1 L w_1 f_1 @x D - 2 \int_{k_0}^1 A_1 w_1 m @x D f_1 @x D + 3 a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} A_1^2 \dot{A}_1 f_1^2 @x D + \\
 &2 E^{I T_1 s_1} a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} A_2 \dot{A}_1^2 f_1^2 @x D + 2 a \int_{k_0}^1 f_2^c @x D^2 \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_2 f_1^2 @x D + \\
 &E^{I T_1 s_1} a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} A_2 \dot{A}_1^2 f_2^2 @x D + 2 E^{I T_1 s_2} F \dot{A}_2 f_2^2 @x D + 4 a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_2 f_2^2 @x D, \\
 &-2 \int_{k_0}^1 \text{HD}_1 A_2 L w_2 f_2 @x D - 2 \int_{k_0}^1 A_2 w_2 m @x D f_2 @x D + E^{-I T_1 s_1} a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} A_1^3 f_1^2 @x D + \\
 &2 E^{I T_1 s_2} F \dot{A}_1 f_1^2 @x D + 4 a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_1 f_1^2 @x D + \\
 &2 a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_1 f_2^2 @x D + 3 a \int_{k_0}^1 f_2^c @x D^2 \hat{a} x \frac{V}{\{}} A_2^2 \dot{A}_2 f_2^2 @x D >
 \end{aligned}$$

The solvability conditions demand that **ST** be orthogonal to solutions of the adjoint homogeneous problem. Since the problem is self-adjoint, we have

SCond =

**Table@int@f_j@x D ST@@jDD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. int -> Integrate, 8j, 2<D ==
0 •• Thread;**

SCond •. displayRule

$$\begin{aligned}
 &: -2 \int_{k_0}^1 f_1 @x D^2 \hat{a} x \frac{V}{\{}} \text{HD}_1 A_1 L w_1 - 2 \int_{k_0}^1 m @x D f_1 @x D^2 \hat{a} x \frac{V}{\{}} A_1 w_1 + \\
 &3 a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_1 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} A_1^2 \dot{A}_1 + 2 E^{I T_1 s_1} a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} \\
 &\int_{k_0}^1 f_1 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} A_2 \dot{A}_1^2 + E^{I T_1 s_1} a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_1 @x D f_2^2 @x D \hat{a} x \frac{V}{\{}} A_2 \dot{A}_1^2 + \\
 &2 E^{I T_1 s_2} F \int_{k_0}^1 f_1 @x D f_2^2 @x D \hat{a} x \frac{V}{\{}} \dot{A}_2 + 2 a \int_{k_0}^1 f_2^c @x D^2 \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_1 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_2 + \\
 &4 a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_1 @x D f_2^2 @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_2 == 0, \\
 &E^{-I T_1 s_1} a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_2 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} A_1^3 - 2 \int_{k_0}^1 f_2 @x D^2 \hat{a} x \frac{V}{\{}} \text{HD}_1 A_2 L w_2 - \\
 &2 \int_{k_0}^1 m @x D f_2 @x D^2 \hat{a} x \frac{V}{\{}} A_2 w_2 + 2 E^{I T_1 s_2} F \int_{k_0}^1 f_2 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} \dot{A}_1 + \\
 &4 a \int_{k_0}^1 f_1^c @x D f_2^c @x D \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_2 @x D f_1^2 @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_1 + 2 a \int_{k_0}^1 f_1^c @x D^2 \hat{a} x \frac{V}{\{}} \\
 &\int_{k_0}^1 f_2 @x D f_2^2 @x D \hat{a} x \frac{V}{\{}} A_1 A_2 \dot{A}_1 + 3 a \int_{k_0}^1 f_2^c @x D^2 \hat{a} x \frac{V}{\{}} \int_{k_0}^1 f_2 @x D f_2^2 @x D \hat{a} x \frac{V}{\{}} A_2^2 \dot{A}_2 == 0 >
 \end{aligned}$$

Using the notations introduced in the preceding section, we can express **SCond** in a more concise form as

```
eqMod = ExpandA-  $\frac{1}{w\#}$  HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<
:2 I m1 A1@T1D + 8 g11 A1@T1D2 Ā1@T1D + 8 EI T1 s1 d1 A2@T1D Ā1@T1D2 +
2 EI T1 s2 f s1,2 Ā2@T1D + 8 g12 A1@T1D A2@T1D Ā2@T1D + 2 I A1ε@T1D == 0,
8 E-I T1 s1 d2 A1@T1D3 + 2 I m2 A2@T1D + 2 EI T1 s2 f s2,1 Ā1@T1D +
8 g21 A1@T1D A2@T1D Ā1@T1D + 8 g22 A2@T1D2 Ā2@T1D + 2 I A2ε@T1D == 0>
```

where

```
-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
TableForm
```

```
8 g11 w1 ® - 3 a I ũ01 f1ε@xD2 â xM ũ01 f1@xD f12@xD â x
8 d1 w1 ® - 2 a I ũ01 f1ε@xD f2ε@xD â xM ũ01 f1@xD f12@xD â x - a I ũ01 f1ε@xD2 â xM ũ01 f1@xD f22@xD â x
8 g12 w1 ® - 2 a I ũ01 f2ε@xD2 â xM ũ01 f1@xD f12@xD â x - 4 a I ũ01 f1ε@xD f2ε@xD â xM ũ01 f1@xD f22@xD â x
```

```
-symbolList@@2DD -> H-Coefficient@SCond@@2, 1DD, #D & •ž secularTerms@@2DDL •• Thread ••
TableForm
```

```
8 d2 w2 ® - a I ũ01 f1ε@xD2 â xM ũ01 f2@xD f12@xD â x
8 g21 w2 ® - 4 a I ũ01 f1ε@xD f2ε@xD â xM ũ01 f2@xD f12@xD â x - 2 a I ũ01 f1ε@xD2 â xM ũ01 f2@xD f22@xD â x
8 g22 w2 ® - 3 a I ũ01 f2ε@xD2 â xM ũ01 f2@xD f22@xD â x
```

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm**, we obtain the modulation equations in polar form and the definitions for g; as

```
PolarForm@eqModD
```

```
::a1ε@T1D == -m1 a1@T1D - f Sin@g2@T1DD s1,2 a2@T1D - Sin@g1@T1DD d1 a1@T1D2 a2@T1D,
a2ε@T1D == -f Sin@g2@T1DD s2,1 a1@T1D + Sin@g1@T1DD d2 a1@T1D3 - m2 a2@T1D,
g1ε@T1D == s1 - 3 g11 a1@T1D2 + g21 a1@T1D2 +  $\frac{f \cos@g_2@T_1DD s_{2,1} a_1@T_1D}{a_2@T_1D} + \frac{\cos@g_1@T_1DD d_2 a_1@T_1D^3}{a_2@T_1D}$  -
 $\frac{3 f \cos@g_2@T_1DD s_{1,2} a_2@T_1D}{a_1@T_1D} - 3 \cos@g_1@T_1DD d_1 a_1@T_1D a_2@T_1D - 3 g_{12} a_2@T_1D^2 + g_{22} a_2@T_1D^2,$ 
g2ε@T1D == s2 - g11 a1@T1D2 - g21 a1@T1D2 -  $\frac{f \cos@g_2@T_1DD s_{2,1} a_1@T_1D}{a_2@T_1D} - \frac{\cos@g_1@T_1DD d_2 a_1@T_1D^3}{a_2@T_1D}$  -
 $\frac{f \cos@g_2@T_1DD s_{1,2} a_2@T_1D}{a_1@T_1D} - \cos@g_1@T_1DD d_1 a_1@T_1D a_2@T_1D - g_{12} a_2@T_1D^2 - g_{22} a_2@T_1D^2>,
8g_1@T_1D == T_1 s_1 - 3 b_1@T_1D + b_2@T_1D, g_2@T_1D == T_1 s_2 - b_1@T_1D - b_2@T_1D<>$ 
```


Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm**, we obtain the modulation equations in Cartesian form, the definitions for n_i , and the symmetry property as

CartesianForm@eqModD

$$\begin{aligned}
 98p_1^c @T_1 D &== -m_1 p_1 @T_1 D - n_1 q_1 @T_1 D + g_{11} p_1 @T_1 D^2 q_1 @T_1 D - 2 d_1 p_1 @T_1 D p_2 @T_1 D q_1 @T_1 D + g_{12} p_2 @T_1 D^2 q_1 @T_1 D + \\
 &\quad g_{11} q_1 @T_1 D^3 - f s_{1,2} q_2 @T_1 D + d_1 p_1 @T_1 D^2 q_2 @T_1 D - d_1 q_1 @T_1 D^2 q_2 @T_1 D + g_{12} q_1 @T_1 D q_2 @T_1 D^2, \\
 q_1^c @T_1 D &== n_1 p_1 @T_1 D - g_{11} p_1 @T_1 D^3 - f s_{1,2} p_2 @T_1 D - d_1 p_1 @T_1 D^2 p_2 @T_1 D - g_{12} p_1 @T_1 D p_2 @T_1 D^2 - \\
 &\quad m_1 q_1 @T_1 D - g_{11} p_1 @T_1 D q_1 @T_1 D^2 + d_1 p_2 @T_1 D q_1 @T_1 D^2 - 2 d_1 p_1 @T_1 D q_1 @T_1 D q_2 @T_1 D - g_{12} p_1 @T_1 D q_2 @T_1 D^2, \\
 p_2^c @T_1 D &== -m_2 p_2 @T_1 D - f s_{2,1} q_1 @T_1 D + 3 d_2 p_1 @T_1 D^2 q_1 @T_1 D - d_2 q_1 @T_1 D^3 - n_2 q_2 @T_1 D + \\
 &\quad g_{21} p_1 @T_1 D^2 q_2 @T_1 D + g_{22} p_2 @T_1 D^2 q_2 @T_1 D + g_{21} q_1 @T_1 D^2 q_2 @T_1 D + g_{22} q_2 @T_1 D^3, \\
 q_2^c @T_1 D &== -f s_{2,1} p_1 @T_1 D - d_2 p_1 @T_1 D^3 + n_2 p_2 @T_1 D - g_{21} p_1 @T_1 D^2 p_2 @T_1 D - g_{22} p_2 @T_1 D^3 + \\
 &\quad 3 d_2 p_1 @T_1 D q_1 @T_1 D^2 - g_{21} p_2 @T_1 D q_1 @T_1 D^2 - m_2 q_2 @T_1 D - g_{22} p_2 @T_1 D q_2 @T_1 D^2, \\
 9n_1 &\otimes \frac{1}{4} H s_1 + s_2 L, \quad n_2 \otimes \frac{1}{4} H - s_1 + 3 s_2 L =, \quad 88p_1, \quad q_1, \quad p_2, \quad q_2 <, \quad 8 - q_1, \quad p_1, \quad q_2, \quad -p_2 <, \\
 &\quad 8 - p_1, \quad -q_1, \quad -p_2, \quad -q_2 <, \quad 8q_1, \quad -p_1, \quad -q_2, \quad p_2 < < =
 \end{aligned}$$

7.2.3 Discretization of the Continuous Problem

As an alternative, we apply the method of multiple scales to the discretized system of **EOM** and **BC**. To determine the discretized form, we expand $w(x, t)$ in terms of the linear mode shapes $f_m(x)$ as

$$wRule1 = w \rightarrow \text{Function}[8x, t], \quad \sum_{m=1}^{\infty} u_m(t) f_m(x);$$

To simplify the computation, we temporarily drop the $\sum$ sign in **wRule1**, as long as we know the repeated m represents a summation index, and rewrite it as

$$wRule2 = w \rightarrow \text{Function}[8x, t], \quad u_m(t) f_m(x);$$

where the $u_m(t)$ are the generalized coordinates. Substituting **wRule2** into **EOM**, multiplying the result with $f_n(x)$, and integrating the outcome from $x = 0$ to $x = 1$ using **intRule1** and **intRule2**, we obtain the following discretized form of the equations describing the response of the beam:

$$\begin{aligned}
 eq723a &= \text{int}[f_n(x) H] \cdot \text{Integrate} \rightarrow \text{int} \cdot wRule2L, \quad 8x, \quad 0, \quad 1 < D \cdot \text{intRule1} \cdot \text{intRule2} \cdot \\
 &\quad \text{int} \rightarrow \text{Integrate} \& \cdot \sum EOM \\
 -2 P \int_0^1 f_n(x) f_m^2(x) dx \sum_{\{m\}} u_m(t) D + \int_0^1 f_n(x) f_m^4(x) dx \sum_{\{m\}} u_m(t) D + \int_0^1 f_m(x) f_n(x) dx \sum_{\{m\}} u_m^2(t) D &== \\
 4 F e \cos \omega t W D \int_0^1 f_n(x) f_m^2(x) dx \sum_{\{m\}} u_m(t) D + \\
 a e \int_0^1 f_m^c(x) D^2 dx \sum_{\{m\}} \int_0^1 f_n(x) f_m^2(x) dx \sum_{\{m\}} u_m(t) D^3 - 2 e \int_0^1 m(x) f_m(x) f_n(x) dx \sum_{\{m\}} u_m^c(t) D
 \end{aligned}$$

where the damping is assumed to be modal. The nonlinear term, in general, should be rewritten as

$$\mathbf{NT} = \mathbf{I} \text{CoefficientAeq723a} @ @ 2 \mathbf{DD}, \mathbf{u}_m @ \mathbf{tD}^3 \mathbf{E} \cdot \mathbf{f}_m^e @ \mathbf{x} \mathbf{D}^2 \rightarrow \mathbf{f}_k^e @ \mathbf{x} \mathbf{D} \mathbf{f}_l^e @ \mathbf{x} \mathbf{D} \mathbf{u}_m @ \mathbf{tD} \mathbf{u}_k @ \mathbf{tD} \mathbf{u}_l @ \mathbf{tD}$$

$$a \, e \int_0^1 \mathbf{f}_k^e @ \mathbf{x} \mathbf{D} \mathbf{f}_l^e @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \, \mathbf{f}_n @ \mathbf{x} \mathbf{D} \mathbf{f}_m^2 @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \mathbf{u}_k @ \mathbf{tD} \mathbf{u}_m @ \mathbf{tD} \mathbf{u}_l @ \mathbf{tD}$$

Using the orthonormality of the mode shapes $\mathbf{f}_i @ \mathbf{x} \mathbf{D}$ and the results from the corresponding eigenvalue problem, we define the following rules:

$$\text{notationRule1} =$$

$$9 \int_0^1 \mathbf{f}_m @ \mathbf{x} \mathbf{D} \mathbf{f}_n @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \rightarrow \mathbf{d}_{m,n}, \int_0^1 \mathbf{f}_n @ \mathbf{x} \mathbf{D} \mathbf{f}_m^{\text{H4L}} @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \rightarrow 2 \mathbf{P} \int_0^1 \mathbf{f}_n @ \mathbf{x} \mathbf{D} \mathbf{f}_m^2 @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} + \mathbf{w}_m^2 \mathbf{d}_{m,n};$$

$$\text{notationRule2} = 9 \int_0^1 \mathbf{f}_m @ \mathbf{x} \mathbf{D} \mathbf{f}_m @ \mathbf{x} \mathbf{D} \mathbf{f}_n @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \rightarrow \mathbf{m}_m \mathbf{d}_{m,n}, \int_0^1 \mathbf{f}_n @ \mathbf{x} \mathbf{D} \mathbf{f}_m^2 @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \rightarrow -\mathbf{g}_{n,m},$$

$$\int_0^1 \mathbf{f}_k^e @ \mathbf{x} \mathbf{D} \mathbf{f}_l^e @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \int_0^1 \mathbf{f}_n @ \mathbf{x} \mathbf{D} \mathbf{f}_m^2 @ \mathbf{x} \mathbf{D} \hat{\mathbf{x}} \rightarrow -\mathbf{g}_{k,\{ \mathbf{g}_{n,m}, \mathbf{a}_{\mathbf{d}_{m,n}} : \mathbf{H} \mathbf{a} \cdot \mathbf{m} \rightarrow \mathbf{nL};$$

and then rewrite **eq723a** as

$$\mathbf{EOM1} =$$

$$\text{Expand@eq723a} @ @ 1 \mathbf{DD} \cdot \text{notationRule1D} == \text{Ieq723a} @ @ 2 \mathbf{DD} \cdot \mathbf{u}_m @ \mathbf{tD}^3 \rightarrow \mathbf{0M} + \mathbf{NT} \cdot \cdot \text{notationRule2}$$

$$\mathbf{w}_n^2 \mathbf{u}_n @ \mathbf{tD} + \mathbf{u}_n^2 @ \mathbf{tD} == -4 \mathbf{F} \mathbf{e} \cos @ \mathbf{t} \mathbf{WD} \mathbf{g}_{n,m} \mathbf{u}_m @ \mathbf{tD} - a \, e \mathbf{g}_{k,\{ \mathbf{g}_{n,m} \mathbf{u}_k @ \mathbf{tD} \mathbf{u}_m @ \mathbf{tD} \mathbf{u}_l @ \mathbf{tD} - 2 \, e \mathbf{m}_n \mathbf{u}_n^e @ \mathbf{tD}$$

where m , k , and $\{$ represent summation indicies.

Using the method of multiple scales, we seek a first-order uniform expansion in the form

$$\text{solRule} = \mathbf{u}_n \rightarrow \int_0^1 \hat{\mathbf{a}} \, e^j \mathbf{u}_{n,j} @ \#1, \#2 \mathbf{D} \, \mathbf{D} \, \mathbf{D};$$

Transforming the total time derivatives in **EOM1** into partial derivatives in terms of T_0 and T_1 , substituting the **solRule** into **EOM1**, expanding the result for small e , and discarding terms of order higher than e , we obtain

$$\text{eq723b} = \mathbf{HEOM1} \cdot \mathbf{8u}_s @ \mathbf{tD} \rightarrow \mathbf{u}_s @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D}, \text{Derivative@n_D@u}_s \mathbf{D} @ \mathbf{tD} \rightarrow \mathbf{dt@nD@u}_s @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{DD}, \mathbf{t} \rightarrow \mathbf{T}_0 < \cdot$$

$$\text{solRule} \cdot \cdot \text{ExpandAll} \cdot \cdot \mathbf{e}^{n-}; n > 1 \rightarrow \mathbf{0}$$

$$\mathbf{w}_n^2 \mathbf{u}_{n,0} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} + e \mathbf{w}_n^2 \mathbf{u}_{n,1} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} + 2 \, e \mathbf{u}_{n,0}^{\text{H1,1L}} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} + \mathbf{u}_{n,0}^{\text{H2,0L}} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} + e \mathbf{u}_{n,1}^{\text{H2,0L}} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} ==$$

$$-4 \mathbf{F} \mathbf{e} \cos @ \mathbf{T}_0 \mathbf{WD} \mathbf{g}_{n,m} \mathbf{u}_{m,0} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} -$$

$$a \, e \mathbf{g}_{k,\{ \mathbf{g}_{n,m} \mathbf{u}_{k,0} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} \mathbf{u}_{m,0} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} \mathbf{u}_{l,0} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D} - 2 \, e \mathbf{m}_n \mathbf{u}_{n,0}^{\text{H1,0L}} @ \mathbf{T}_0, \mathbf{T}_1 \mathbf{D}$$

Equating coefficients of like powers of e in **eq723b**, we obtain

eqEps = CoefficientList@#, eD & . ž eq723b . Thread

$$\begin{aligned} 9w_n^2 u_{n,0} @ T_0, T_1 D + u_{n,0}^{H2,0L} @ T_0, T_1 D == 0, \\ w_n^2 u_{n,1} @ T_0, T_1 D + 2 u_{n,0}^{H1,1L} @ T_0, T_1 D + u_{n,1}^{H2,0L} @ T_0, T_1 D == -4 F \cos @ T_0 W D g_{n,m} u_{m,0} @ T_0, T_1 D - \\ a g_{k,\{ g_{n,m} u_{k,0} @ T_0, T_1 D u_{m,0} @ T_0, T_1 D u_{\{,0} @ T_0, T_1 D - 2 m_n u_{n,0}^{H1,0L} @ T_0, T_1 D = \end{aligned}$$

Because in the presence of damping, all modes that are not directly or indirectly excited decay with time, all of the $u_{n,0}$ decay with time except $u_{1,0}$ and $u_{2,0}$. Then it follows from **eqEps[[1]]** that

$$\text{sol0} = u_{i,0} \rightarrow \text{FunctionA8T}_0, T_1 <, A_i @ T_1 D \text{Exp}@I w_i T_0 D + \dot{A}_i @ T_1 D \text{Exp}@-I w_i T_0 D E;$$

Taking account of the summation indicies, we define

$$\begin{aligned} \text{sumRule} = 9 g_{k,\{ g_{n,m} u_{k,0} @ a_D u_{m,0} @ a_D u_{\{,0} @ a_D \rightarrow \sum_{k=1}^2 \sum_{m=1}^2 \sum_{\{=1}^2 g_{k,\{ g_{n,m} u_{k,0} @ aD u_{m,0} @ aD u_{\{,0} @ aD, \\ g_{n,m} u_{m,0} @ a_D \rightarrow \sum_{m=1}^2 g_{n,m} u_{m,0} @ aD =; \end{aligned}$$

Hence, we rewrite **eqEps[[2]]** as

Clear@order1EqD

order1Eq@n_D = HeqEps@@1, 1DD . u_{n,0} -> u_{n,1}L ==

HeqEps@@1, 1DD . u_{n,0} -> u_{n,1}L - HSubtract žž eqEps@@2DDL . sumRule

$$\begin{aligned} w_n^2 u_{n,1} @ T_0, T_1 D + u_{n,1}^{H2,0L} @ T_0, T_1 D == -4 F \cos @ T_0 W D H g_{n,1} u_{1,0} @ T_0, T_1 D + g_{n,2} u_{2,0} @ T_0, T_1 D L - \\ a H g_{1,1} g_{n,1} u_{1,0} @ T_0, T_1 D^3 + g_{1,2} g_{n,1} u_{1,0} @ T_0, T_1 D^2 u_{2,0} @ T_0, T_1 D + \\ g_{2,1} g_{n,1} u_{1,0} @ T_0, T_1 D^2 u_{2,0} @ T_0, T_1 D + g_{1,1} g_{n,2} u_{1,0} @ T_0, T_1 D^2 u_{2,0} @ T_0, T_1 D + \\ g_{2,2} g_{n,1} u_{1,0} @ T_0, T_1 D u_{2,0} @ T_0, T_1 D^2 + g_{1,2} g_{n,2} u_{1,0} @ T_0, T_1 D u_{2,0} @ T_0, T_1 D^2 + \\ g_{2,1} g_{n,2} u_{1,0} @ T_0, T_1 D u_{2,0} @ T_0, T_1 D^2 + g_{2,2} g_{n,2} u_{2,0} @ T_0, T_1 D^3 L - 2 m_n u_{n,0}^{H1,0L} @ T_0, T_1 D - 2 u_{n,0}^{H1,1L} @ T_0, T_1 D \end{aligned}$$

Substituting **sol0** into **order1Eq[1]** and **order1Eq[2]** yields

eqOrder1 = Array@order1Eq, 2D . sol0 . TrigToExp . ExpandAll;

à Principal Parametric Resonance of the First Mode

ResonanceConds = 8w₂ == 3 w₁ + e s₁, W == 2 w₁ + e s₂<;

OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~Join~ 8W<D@@1DD & . ž 81, 2<

$$98w_2 @ e s_1 + 3 w_1, W @ e s_2 + 2 w_1 <, 9w_1 @ \frac{1}{3} H - e s_1 + w_2 L, W @ \frac{1}{3} H - 2 e s_1 + 3 e s_2 + 2 w_2 L ==$$

expRule@i_D := Exp@arg_D :> Exp@Expand@arg . OmgRule@@iDDD . e T₀ -> T₁D

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

```
SCond = Table@Coefficient@-eqOrder1@@i, 2DD . expRule@iD, Exp@I w1 T0DD == 0, 8i, 2<D;
SCond . displayRule
```

$$\begin{aligned} & :2 \text{ I HD}_1 \text{A}_1 \text{L w}_1 + 2 \text{ I A}_1 \text{ m}_1 \text{ w}_1 + 2 \text{ E}^{\text{I T}_1 \text{ s}_2} \text{ F } \dot{\text{A}}_1 \text{ g}_{1,1} + \\ & 3 \text{ a A}_1^2 \dot{\text{A}}_1 \text{ g}_{1,1}^2 + 2 \text{ E}^{\text{I T}_1 \text{ s}_1 - \text{I T}_1 \text{ s}_2} \text{ F A}_2 \text{ g}_{1,2} + 2 \text{ E}^{\text{I T}_1 \text{ s}_1} \text{ a A}_2 \dot{\text{A}}_1^2 \text{ g}_{1,1} \text{ g}_{1,2} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,2}^2 + \\ & \text{E}^{\text{I T}_1 \text{ s}_1} \text{ a A}_2 \dot{\text{A}}_1^2 \text{ g}_{1,1} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,2} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,1} \text{ g}_{2,2} == 0, \\ & 2 \text{ I HD}_1 \text{A}_2 \text{L w}_2 + 2 \text{ I A}_2 \text{ m}_2 \text{ w}_2 + 2 \text{ E}^{-\text{I T}_1 \text{ s}_1 + \text{I T}_1 \text{ s}_2} \text{ F A}_1 \text{ g}_{2,1} + \text{E}^{-\text{I T}_1 \text{ s}_1} \text{ a A}_1^3 \text{ g}_{1,1} \text{ g}_{2,1} + \\ & 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{1,2} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{2,1}^2 + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{1,1} \text{ g}_{2,2} + 3 \text{ a A}_2^2 \dot{\text{A}}_2 \text{ g}_{2,2}^2 == 0 > \end{aligned}$$

which is in agreement with that obtained by direct approach.

à Principal Parametric Resonance of the Second Mode

```
ResonanceConds = 8w2 == 3 w1 + e s1, W == 2 w2 + e s2<;
```

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D~Join~8W<D@@1DD & . ž 81, 2<
```

$$98w_2 \text{ ® } e s_1 + 3 w_1, W \text{ ® } 2 e s_1 + e s_2 + 6 w_1<, 9w_1 \text{ ® } \frac{1}{3} H - e s_1 + w_2 L, W \text{ ® } e s_2 + 2 w_2 ==$$

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg . . OmgRule@@iDDD . . e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

```
SCond = Table@Coefficient@-eqOrder1@@i, 2DD . expRule@iD, Exp@I w1 T0DD == 0, 8i, 2<D;
SCond . displayRule
```

$$\begin{aligned} & :2 \text{ I HD}_1 \text{A}_1 \text{L w}_1 + 2 \text{ I A}_1 \text{ m}_1 \text{ w}_1 + 3 \text{ a A}_1^2 \dot{\text{A}}_1 \text{ g}_{1,1}^2 + 2 \text{ E}^{\text{I T}_1 \text{ s}_1} \text{ a A}_2 \dot{\text{A}}_1^2 \text{ g}_{1,1} \text{ g}_{1,2} + \\ & 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,2}^2 + \text{E}^{\text{I T}_1 \text{ s}_1} \text{ a A}_2 \dot{\text{A}}_1^2 \text{ g}_{1,1} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,2} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_2 \text{ g}_{1,1} \text{ g}_{2,2} == 0, \\ & 2 \text{ I HD}_1 \text{A}_2 \text{L w}_2 + 2 \text{ I A}_2 \text{ m}_2 \text{ w}_2 + \text{E}^{-\text{I T}_1 \text{ s}_1} \text{ a A}_1^3 \text{ g}_{1,1} \text{ g}_{2,1} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{1,2} \text{ g}_{2,1} + \\ & 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{2,1}^2 + 2 \text{ E}^{\text{I T}_1 \text{ s}_2} \text{ F } \dot{\text{A}}_2 \text{ g}_{2,2} + 2 \text{ a A}_1 \text{ A}_2 \dot{\text{A}}_1 \text{ g}_{1,1} \text{ g}_{2,2} + 3 \text{ a A}_2^2 \dot{\text{A}}_2 \text{ g}_{2,2}^2 == 0 > \end{aligned}$$

which is in agreement with that obtained by direct approach.

à Combination Parametric Resonance of the Two Modes

```
ResonanceConds = 8w2 == 3 w1 + e s1, W == w1 + w2 + e s2<;
```

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D~Join~8W<D@@1DD & . ž 81, 2<
```

$$98w_2 \text{ ® } e s_1 + 3 w_1, W \text{ ® } e s_1 + e s_2 + 4 w_1<, 9w_1 \text{ ® } \frac{1}{3} H - e s_1 + w_2 L, W \text{ ® } \frac{1}{3} H - e s_1 + 3 e s_2 + 4 w_2 L ==$$

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg . . OmgRule@@iDDD . . e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

Principal Parametric Resonance of the First Mode

ResonanceConds = $8w_2 == 3w_1 + e s_1$, $W == 2w_1 + e s_2 < \bullet$. Equal -> Rule

$8w_2 @ e s_1 + 3w_1$, $W @ e s_2 + 2w_1 <$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

TAL = order1Lagr •. Exp@a_D := Exp@Expand@a •. ResonanceCondsD •. e T₀ -> T₁D •.

Exp@_ T₀ + _ .D -> 0 •. int -> Integrate

$$\begin{aligned}
 & -E^{-I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D^2 - 2 E^{I T_1 s_1 - I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_2 @ T_1 D \dot{A}_1 @ T_1 D - \\
 & E^{I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \dot{A}_1 @ T_1 D^2 - \frac{3}{2} a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D^2 - \\
 & E^{I T_1 s_1} a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_2 @ T_1 D \dot{A}_1 @ T_1 D^3 - \\
 & 2 E^{-I T_1 s_1 + I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D \dot{A}_2 @ T_1 D - \\
 & E^{-I T_1 s_1} a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D^3 \dot{A}_2 @ T_1 D - \\
 & 4 a \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\
 & 2 a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \int_0^1 \ddot{f}_2 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\
 & \frac{3}{2} a \int_0^1 \ddot{f}_2 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D^2 - \\
 & I \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } w_1 \dot{A}_1 @ T_1 D A_1 @ T_1 D - I \int_0^1 \ddot{f}_2 @ x D^2 \hat{a} x \frac{1}{\{ \} } w_2 \dot{A}_2 @ T_1 D A_2 @ T_1 D + \\
 & I \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } w_1 A_1 @ T_1 D \dot{A}_1 @ T_1 D + I \int_0^1 \ddot{f}_2 @ x D^2 \hat{a} x \frac{1}{\{ \} } w_2 A_2 @ T_1 D \dot{A}_2 @ T_1 D
 \end{aligned}$$

The Euler-Lagrange equations corresponding to the **TAL** can be written as

eqMod1 = DADATAL, $\dot{A}_1 @ T_1 D E$, $T_1 E - DATAL$, $\dot{A}_1 @ T_1 D E == 0$

$$\begin{aligned}
 & 2 E^{I T_1 s_1 - I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_2 @ T_1 D + 2 E^{I T_1 s_2} F \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \dot{A}_1 @ T_1 D + \\
 & 3 a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D + 3 E^{I T_1 s_1} a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_2 @ T_1 D \dot{A}_1 @ T_1 D^2 + \\
 & 4 a \int_0^1 \ddot{f}_1 @ x D \ddot{f}_2 @ x D \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + \\
 & 2 a \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } \int_0^1 \ddot{f}_2 @ x D^2 \hat{a} x \frac{1}{\{ \} } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + 2 I \int_0^1 \ddot{f}_1 @ x D^2 \hat{a} x \frac{1}{\{ \} } w_1 A_1 @ T_1 D == 0
 \end{aligned}$$

and

$$\text{eqMod2} = \text{DADATAL}, \dot{A}_2 @ T_1 D E, T_1 E - \text{DATAL}, \dot{A}_2 @ T_1 D E == 0$$

$$\begin{aligned} & 2 E^{-I T_1 s_1 + I T_1 s_2} F \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D + E^{-I T_1 s_1} a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D^3 + \\ & 4 a \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\ & 2 a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\ & 3 a \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D + 2 I \int_{k_0}^1 f_2 @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } w_2 A_2^c @ T_1 D == 0 \end{aligned}$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

Principal Parametric Resonance of the Second Mode

$$\text{ResonanceConds} = \text{Solve} @ 8 w_2 == 3 w_1 + e s_1, W == 2 w_2 + e s_2 <, 8 w_2, W < D @ @ 1 D D$$

$$8 w_2 @ e s_1 + 3 w_1, W @ 2 e s_1 + e s_2 + 6 w_1 <$$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

$$\text{TAL} = \text{order1Lagr} \bullet \bullet \text{Exp} @ a_D :> \text{Exp} @ \text{Expand} @ a \bullet \bullet \text{ResonanceCondsD} \bullet \bullet e T_0 -> T_1 D \bullet \bullet$$

$$\text{Exp} @ _ T_0 + _ . D -> 0 \bullet \bullet \text{int} -> \text{Integrate}$$

$$\begin{aligned} & - E^{-I T_1 s_2} F \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_2 @ T_1 D^2 - \frac{3}{2} a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D^2 - \\ & E^{I T_1 s_1} a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_2 @ T_1 D \dot{A}_1 @ T_1 D^3 - \\ & E^{-I T_1 s_1} a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D^3 \dot{A}_2 @ T_1 D - \\ & 4 a \int_{k_0}^1 f_1^c @ x D f_2^c @ x D \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\ & 2 a \int_{k_0}^1 f_1^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\ & E^{I T_1 s_2} F \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } \dot{A}_2 @ T_1 D^2 - \frac{3}{2} a \int_{k_0}^1 f_2^c @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D^2 - \\ & I \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } w_1 \dot{A}_1 @ T_1 D A_1^c @ T_1 D - I \int_{k_0}^1 f_2 @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } w_2 \dot{A}_2 @ T_1 D A_2^c @ T_1 D + \\ & I \int_{k_0}^1 f_1 @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } w_1 A_1 @ T_1 D \dot{A}_1 @ T_1 D + I \int_{k_0}^1 f_2 @ x D^2 \hat{a} x \frac{\mathcal{V}}{\{ } w_2 A_2 @ T_1 D \dot{A}_2 @ T_1 D \end{aligned}$$

The Euler-Lagrange equations corresponding to the **TAL** can be written as

$$\text{eqMod1} = \text{DADATL}, \dot{A}_1^e @ T_1 D E, T_1 E - \text{DATL}, \dot{A}_1 @ T_1 D E == 0$$

$$\begin{aligned} & 3 a \int_0^1 f_1^e @ x D^2 \hat{a} x \frac{y}{\{}} A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D + 3 E^{I T_1 s_1} a \int_0^1 f_1^e @ x D^2 \hat{a} x \frac{y}{\{}} \int_0^1 f_1^e @ x D f_2^e @ x D \hat{a} x \frac{y}{\{}} A_2 @ T_1 D \dot{A}_1 @ T_1 D^2 + \\ & 4 a \int_0^1 f_1^e @ x D f_2^e @ x D \hat{a} x \frac{y}{\{}} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + \\ & 2 a \int_0^1 f_1^e @ x D^2 \hat{a} x \frac{y}{\{}} \int_0^1 f_2^e @ x D^2 \hat{a} x \frac{y}{\{}} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + 2 I \int_0^1 f_1 @ x D^2 \hat{a} x \frac{y}{\{}} w_1 A_1^e @ T_1 D == 0 \end{aligned}$$

and

$$\text{eqMod2} = \text{DADATL}, \dot{A}_2^e @ T_1 D E, T_1 E - \text{DATL}, \dot{A}_2 @ T_1 D E == 0$$

$$\begin{aligned} & E^{-I T_1 s_1} a \int_0^1 f_1^e @ x D^2 \hat{a} x \frac{y}{\{}} \int_0^1 f_1^e @ x D f_2^e @ x D \hat{a} x \frac{y}{\{}} A_1 @ T_1 D^3 + 4 a \int_0^1 f_1^e @ x D f_2^e @ x D \hat{a} x \frac{y}{\{}} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\ & 2 a \int_0^1 f_1^e @ x D^2 \hat{a} x \frac{y}{\{}} \int_0^1 f_2^e @ x D^2 \hat{a} x \frac{y}{\{}} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + 2 E^{I T_1 s_2} F \int_0^1 f_2^e @ x D^2 \hat{a} x \frac{y}{\{}} \dot{A}_2 @ T_1 D + \\ & 3 a \int_0^1 f_2^e @ x D^2 \hat{a} x \frac{y}{\{}} A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D + 2 I \int_0^1 f_2 @ x D^2 \hat{a} x \frac{y}{\{}} w_2 A_2^e @ T_1 D == 0 \end{aligned}$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

Combination Parametric Resonance of the Two Modes

$$\text{ResonanceConds} = \text{Solve} @ 8 w_2 == 3 w_1 + e s_1, W == w_1 + w_2 + e s_2 \<, 8 w_2, W \< D @ 1 D D$$

$$8 w_2 @ e s_1 + 3 w_1, W @ e s_1 + e s_2 + 4 w_1 \<$$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

TAL = order1Lagr •. Exp@a_D := Exp@Expand@a •. ResonanceCondsD •. e T₀ -> T₁D •.
Exp@_ T₀ + _ .D -> 0 •. int -> Integrate

$$\begin{aligned}
 & -2 E^{-I T_1 s_2} F \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D - \frac{3}{2} a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D^2 - \\
 & E^{I T_1 s_1} a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_2 @ T_1 D \dot{A}_1 @ T_1 D^3 - \\
 & E^{-I T_1 s_1} a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D^3 \dot{A}_2 @ T_1 D - \\
 & 2 E^{I T_1 s_2} F \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - 4 a \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\
 & 2 a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2^e @ x D^2 \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D - \\
 & \frac{3}{2} a \int_{k_0}^1 f_2^e @ x D^2 \hat{x} \frac{V}{\{ } A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D^2 - \\
 & I \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } w_1 \dot{A}_1 @ T_1 D A_1^e @ T_1 D - I \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } w_2 \dot{A}_2 @ T_1 D A_2^e @ T_1 D + \\
 & I \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } w_1 A_1 @ T_1 D \dot{A}_1 @ T_1 D + I \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } w_2 A_2 @ T_1 D \dot{A}_2 @ T_1 D
 \end{aligned}$$

The Euler-Lagrange equations corresponding to the **TAL** can be written as

$$\mathbf{eqMod1} = \mathbf{DADATAL, \dot{A}_1 @ T_1 DE, T_1 E - DATAL, \dot{A}_1 @ T_1 DE == 0}$$

$$\begin{aligned}
 & 3 a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } A_1 @ T_1 D^2 \dot{A}_1 @ T_1 D + 3 E^{I T_1 s_1} a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_2 @ T_1 D \dot{A}_1 @ T_1 D^2 + \\
 & 2 E^{I T_1 s_2} F \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } \dot{A}_2 @ T_1 D + 4 a \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + \\
 & 2 a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2^e @ x D^2 \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_2 @ T_1 D + 2 I \int_{k_0}^1 f_1 @ x D^2 \hat{x} \frac{V}{\{ } w_1 A_1^e @ T_1 D == 0
 \end{aligned}$$

and

$$\mathbf{eqMod2} = \mathbf{DADATAL, \dot{A}_2 @ T_1 DE, T_1 E - DATAL, \dot{A}_2 @ T_1 DE == 0}$$

$$\begin{aligned}
 & E^{-I T_1 s_1} a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D^3 + \\
 & 2 E^{I T_1 s_2} F \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } \dot{A}_1 @ T_1 D + 4 a \int_{k_0}^1 f_1^e @ x D f_2^e @ x D \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\
 & 2 a \int_{k_0}^1 f_1^e @ x D^2 \hat{x} \frac{V}{\{ } \int_{k_0}^1 f_2^e @ x D^2 \hat{x} \frac{V}{\{ } A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\
 & 3 a \int_{k_0}^1 f_2^e @ x D^2 \hat{x} \frac{V}{\{ } A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D + 2 I \int_{k_0}^1 f_2 @ x D^2 \hat{x} \frac{V}{\{ } w_2 A_2^e @ T_1 D == 0
 \end{aligned}$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

7.3 Cantilever Beam

7.3.1 EOM and BC's

We consider the nonlinear nonplanar response of an inextensional cantilever beam to an external excitation of one of its flexural modes. The lowest torsional frequencies of the beam considered are much higher than the frequencies of the excited flexural modes so that the torsional inertia can be neglected. We assume that the beam is uniform and homogeneous. The transverse deflections $v(x, t)$ and $w(x, t)$ of the beam at the position x and time t are governed by the nondimensional integral-partial-differential equation (Crespo da Silva and Glynn, 1978)

$$\begin{aligned}
 \text{EOM1} = & \rho \frac{\partial^2 v}{\partial t^2} + b_y \frac{\partial^2 v}{\partial x^2} \frac{\partial v}{\partial x} = -2 e m \frac{\partial^2 v}{\partial t^2} + e H_1 - b_y L \\
 & \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} - \\
 & e \frac{H_1 - b_y L^2}{b_g} \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} - \\
 & e b_y \frac{\partial}{\partial x} \left[H \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[H \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} - \\
 & e \frac{1}{2} \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial v}{\partial x} - \\
 & \rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial w}{\partial x} = -2 e m \frac{\partial^2 w}{\partial t^2} - e H_1 - b_y L \\
 & \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \\
 & e \frac{H_1 - b_y L^2}{b_g} \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left[J \frac{\partial v}{\partial x} \right] \frac{\partial w}{\partial x} - \\
 & e \frac{\partial}{\partial x} \left[H \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[H \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial v}{\partial x} - \\
 & e \frac{1}{2} \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left[J \frac{\partial w}{\partial x} \right] \frac{\partial w}{\partial x} -
 \end{aligned}$$

along with the homogeneous boundary conditions

$$\begin{aligned}
 \text{BC1} = & 8v @ 0, tD == 0, \frac{\partial v}{\partial x} @ 0, tD == 0 \cdot x \rightarrow 0, \\
 & \frac{\partial v}{\partial x} @ 0, tD == 0 \cdot x \rightarrow 1, \frac{\partial v}{\partial x} @ 0, tD == 0 \cdot x \rightarrow 1 < \\
 & 8v @ 0, tD == 0, v^{H1, 0L} @ 0, tD == 0, v^{H2, 0L} @ 1, tD == 0, v^{H3, 0L} @ 1, tD == 0 < \\
 \text{BC} = & \text{Join} @ \text{BC1}, \text{BC1} \cdot v \rightarrow wD \\
 & 8v @ 0, tD == 0, v^{H1, 0L} @ 0, tD == 0, v^{H2, 0L} @ 1, tD == 0, v^{H3, 0L} @ 1, tD == 0, \\
 & w @ 0, tD == 0, w^{H1, 0L} @ 0, tD == 0, w^{H2, 0L} @ 1, tD == 0, w^{H3, 0L} @ 1, tD == 0 <
 \end{aligned}$$

The spatial derivatives outside the integrals on the right-hand sides of **EOM1** result in lengthy expressions. To simplify the symbolic computation, we first define some operators as

```

op1@v_, w_D =  $\int_{x,x} w @ x, tD \int_1^x \int_{x,x} v @ x, tD \int_{x,x} w @ x, tD \hat{a} x -$ 
 $\int_{x,x,x} w @ x, tD \int_0^x \int_{x,x} v @ x, tD \int_x w @ x, tD \hat{a} x \cdot \text{Integrate} \rightarrow \text{int@1D}$ 

int@1D@vH2,0L@x, tD wH2,0L@x, tD, 8x, 1, x<D wH2,0L@x, tD -
int@1D@wH1,0L@x, tD vH2,0L@x, tD, 8x, 0, x<D wH3,0L@x, tD

op2@v_, w_D =
HoldA  $\int_{x,x} w @ x, tD \int_0^x \int_1^x \int_{x,x} v @ x, tD \int_{x,x} w @ x, tD \hat{a} x \hat{a} x E \cdot \text{Integrate} \rightarrow \text{int@2D} \cdot \cdot \text{ReleaseHold}$ 

int@2D@vH2,0L@x, tD wH2,0L@x, tD, 8x, 0, x<, 8x, 1, x<D wH2,0L@x, tD

op3@v_, w_D =  $\int_{x,x} v @ x, tD \int_1^x \int_{t,t} J \int_0^x I H \int_x v @ x, tD L^2 + H \int_x w @ x, tD L^2 M \hat{a} x N \hat{a} x \cdot \text{Integrate} \rightarrow \text{int@1D}$ 

int@1DB
int@1DB2 vH1,1L@x, tD2 + 2 wH1,1L@x, tD2 + 2 vH1,0L@x, tD vH1,2L@x, tD + 2 wH1,0L@x, tD wH1,2L@x, tD,
8x, 0, x<F, 8x, 1, x<F vH1,0L@x, tD

```

where `int@iD` indicates the whole expression, including the integrand and the term multiplied by `int@iD@exprD`, will later be wrapped by the *i*th spatial derivative in the solvability condition.

Then we rewrite **EOM1** as

```

EOM = 9  $\int_{t,t} v @ x, tD + b_y \int_{x,x,x} v @ x, tD ==$ 
- 2 e m@xD  $\int_{t,t} v @ x, tD + e H1 - b_y L \text{op1@v, wD} - e \frac{H1 - b_y L^2}{b_g} \text{op2@v, wD} -$ 
e b_y  $\int_x H \int_x v @ x, tD \int_x H \int_x v @ x, tD \int_{x,x} v @ x, tD + \int_x w @ x, tD \int_{x,x} w @ x, tD L L -$ 
e  $\frac{1}{2} \text{op3@v, wD} - e F @ xD \text{Cos@W tD, } \int_{t,t} w @ x, tD + \int_{x,x,x} w @ x, tD ==$ 
- 2 e m@xD  $\int_{t,t} w @ x, tD - e H1 - b_y L \text{op1@w, vD} - e \frac{H1 - b_y L^2}{b_g} \text{op2@w, vD} -$ 
e  $\int_x H \int_x w @ x, tD \int_x H \int_x v @ x, tD \int_{x,x} v @ x, tD + \int_x w @ x, tD \int_{x,x} w @ x, tD L L - e \frac{1}{2} \text{op3@w, vD} =;$ 

```

It follows from **EOM1** and **BC** that the linear undamped natural frequencies and their corresponding mode shapes are given by

```

w1,m ==  $\sqrt{I_m^2 + b_y^2}$ , w2,n ==  $\sqrt{I_n^2}$ 
fi@xD ==  $\text{Cosh@ } l_i xD - \text{Cos@ } l_i xD + \frac{\text{Cos@ } l_i D + \text{Cosh@ } l_i D}{\text{Sin@ } l_i D + \text{Sinh@ } l_i D} H \text{Sin@ } l_i xD - \text{Sinh@ } l_i xD L H^* \quad i=m,n \quad *L=;$ 

```

where $w_{1,m}$ and $w_{2,n}$ are the natural frequencies in the *y* and *z* directions and the l_i are the roots of

$$1 + \text{Cos@ } l_i D \text{Cosh@ } l_i D == 0$$

We consider the case of one-to-one internal resonance between the m th mode in the y or v direction and the n th mode in the z or w direction; that is, $\omega_{1,m} \gg \omega_{2,n}$. To express the nearness of these frequencies quantitatively, we let

$$\text{betayRule} = 8b_y \rightarrow 1 + d_0 + \epsilon d_1;$$

so that $\omega_{1,m} = \sqrt{\frac{1}{1+d_0}} \omega_{2,n} = \omega_{2,n}$. The beam has a near-square cross-section when $d_0 = 0$. We assume neither of these two modes is involved in an internal resonance with any other mode. Moreover, we consider a primary resonance of the flexural mode in the y direction.

To manipulate some complicated integrals, we define the following rules:

```
intRule3 = 8int@i_D@fun_, arg2__D := int@iD@Expand@funD, arg2D<;
intRule4 = 8int@i_D@a_ + b_, arg2__D := int@iD@a, arg2D + int@iD@b, arg2D,
int@i_D@e^n_ fun_, arg2__D := e^n int@iD@fun, arg2D,
int@i_D@a_ fun_, a1_, b1__D := a int@iD@fun, a1, b1D *; FreeQ@a, First@a1DD,
int@i_D@int@i_D@a1_, a2_D fun_, a3__D := int@iD@a1, a2D int@iD@fun, a3D *;
FreeQ@Rest@a2D, First@a2DD, int@i_D@int@i_D@a1_, a2_D, a3__D := int@iD@a1, a3, a2D<;
```

§ 7.3.2 Direct Attack of the Continuous Problem

In this section, we directly attack the integral-partial-differential equation **EOM** and associated boundary conditions **BC** and seek a first-order uniform expansion of their solution in the form

```
solRule = 9v -> I EvaluateASumAe^j v_j@#1, #2, #3D, 8j, 0, 1<EE &M,
w -> I EvaluateASumAe^j w_j@#1, #2, #3D, 8j, 0, 1<EE &M=
8v @ Hv_0@#1, #2, #3D + e v_1@#1, #2, #3D &L, w @ Hw_0@#1, #2, #3D + e w_1@#1, #2, #3D &L<
```

where the first independent variable stands for x and the last two independent variables stand for the two time scales T_0 and T_1 . Transforming the total time derivatives in **EOM** into partial derivatives in terms of T_0 and T_1 , substituting the **solRule** and **betayRule** into **EOM** and **BC**, expanding the result for small ϵ , discarding terms of order higher than ϵ , and using **intRule3** and **intRule4** to simplify the expansions of the integrands, we obtain

```
leq732a =
HHJoin@EOM, BCD * . 8v@x_, tD -> v@x, T_0, T_1D, w@x_, tD -> w@x, T_0, T_1D, Derivative@m_, n_D@
w_D@x_, tD -> dt@nD@D@w@x, T_0, T_1D, 8x, m<DD, t -> T_0< * .
solRule * . betayRule * . ExpandAllL * . intRule3 * .
intRule4 * . ExpandAllL * . e^n_*; n>1 -> 0;M * . Timing
812.629 Second, Null<
```

Equating coefficients of like powers of ϵ in **eq732a**, we obtain

```
eqEps = Thread@CoefficientList@Subtract @@ #, eD == 0D & * . eq732a * . Transpose;
```

eqEps@@1DD •. displayRule

$$9v_0'''' + D_0^2 v_0 + H v_0'''' L d_0 == 0, w_0'''' + D_0^2 w_0 == 0, v_0 @ 0, T_0, T_1 D == 0, \\ v_0^{H1,0,0L} @ 0, T_0, T_1 D == 0, v_0^{H2,0,0L} @ 1, T_0, T_1 D == 0, v_0^{H3,0,0L} @ 1, T_0, T_1 D == 0, \\ w_0 @ 0, T_0, T_1 D == 0, w_0^{H1,0,0L} @ 0, T_0, T_1 D == 0, w_0^{H2,0,0L} @ 1, T_0, T_1 D == 0, w_0^{H3,0,0L} @ 1, T_0, T_1 D == 0 =$$

In the presence of damping, all modes that are not directly excited by the forcing or indirectly excited by the internal resonance will decay with time. Hence, the solution of **eqEps[[1]]** can be expressed in terms of the two excited linear free-vibration modes; that is,

$$\text{sol0} = 9v_0 \rightarrow \text{FunctionA8x}, T_0, T_1 \leq, A_1 @ T_1 D f_m @ x D \text{Exp} @ I w_{1,m} T_0 D + \dot{A}_1 @ T_1 D f_m @ x D \text{Exp} @ - I w_{1,m} T_0 D E, \\ w_0 \rightarrow \text{FunctionA8x}, T_0, T_1 \leq, A_2 @ T_1 D f_n @ x D \text{Exp} @ I w_{2,n} T_0 D + \dot{A}_2 @ T_1 D f_n @ x D \text{Exp} @ - I w_{2,n} T_0 D E =;$$

One can easily show that the mode shapes $f_i @ x D$ are orthogonal. We assume that these modes are normalized so that

$$\int_0^1 f_i @ x D f_j @ x D \hat{a} x \rightarrow d_{ij}$$

where d_{ij} is the Kronecker delta function.

Substituting **sol0** into the first-order equations, **eqEps[[2,1]]** and **eqEps[[2,2]]**, and using **intRule3** and **intRule4**, we obtain

Clear@order1EqD

$$\text{order1Eq@1D} = \text{HeqEps@@1, 1, 1DD} \cdot v_0 \rightarrow v_1 L == \\ \text{HHeqEps@@1, 1, 1DD} \cdot v_0 \rightarrow v_1 L - \text{HSubtract} \check{z} \check{z} \text{eqEps@@2, 1DD} \cdot \text{sol0} \cdot \text{TrigToExp} \cdot \\ \text{ExpandL} \cdot \text{intRule3} \cdot \text{intRule4} \cdot \text{ExpandL}; \\ \text{order1Eq@2D} = \text{HeqEps@@1, 2, 1DD} \cdot w_0 \rightarrow w_1 L == \\ \text{HHeqEps@@1, 2, 1DD} \cdot w_0 \rightarrow w_1 L - \text{HSubtract} \check{z} \check{z} \text{eqEps@@2, 2DD} \cdot \text{sol0} \cdot \text{TrigToExp} \cdot \\ \text{ExpandL} \cdot \text{intRule3} \cdot \text{intRule4} \cdot \text{ExpandL};$$

It follows from **eqEps[[2]]** that the first-order boundary conditions are

order1BC = Drop@eqEps@@2DD, 2D

$$9v_1 @ 0, T_0, T_1 D == 0, v_1^{H1,0,0L} @ 0, T_0, T_1 D == 0, v_1^{H2,0,0L} @ 1, T_0, T_1 D == 0, v_1^{H3,0,0L} @ 1, T_0, T_1 D == 0, \\ w_1 @ 0, T_0, T_1 D == 0, w_1^{H1,0,0L} @ 0, T_0, T_1 D == 0, w_1^{H2,0,0L} @ 1, T_0, T_1 D == 0, w_1^{H3,0,0L} @ 1, T_0, T_1 D == 0 =$$

We consider the case of primary resonance of the flexural mode in the y direction, $W \gg w_{1,m}$. To describe the nearness of the primary resonance, we introduce the detuning parameter s by

$$\text{omgList} = 8w_{1,m}, w_{2,n} \leq;$$

$$\text{ResonanceConds} = 8w_{2,n} == w_{1,m}, W == w_{1,m} + e s \leq;$$

and define the following rules:

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D~Join~8W<D@@1DD & . ž 81, 2<
```

```
88w2,n ® w1,m, W ® e s + w1,m<, 8w1,m ® w2,n, W ® e s + w2,n<<
```

```
expRule@i_D := Exp@arg_D :=> Exp@Expand@arg . . OmgRule@@iDDD . . e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand side of **order1Eq[1]** and **order1Eq[2]** and obtain the source of secular terms as

```
ST = Table@Coefficient@order1Eq@iD@@2DD . . expRule@iD, Exp@I omgList@@iDD T0DD, 8i, 2<D;
```

The solvability conditions of **order1Eq[i]** and **order1BC** demand that **ST[[i]]** be orthogonal to $f_i @ xD$. Imposing these conditions, we have

```
SCond1 = int@fm@xD ST@@1DD, 8x, 0, 1<D == 0 . . intRule1 . . . intRule2;
```

```
SCond2 = int@fn@xD ST@@2DD, 8x, 0, 1<D == 0 . . intRule1 . . . intRule2;
```

Recall that the i th spatial derivative should be recovered back from $\text{int} @ iD$ as stated in the preceding section. Hence, we define

```
intRule5@mode_D =  
  int@mode a_ int@i_D@b_, c__D, d_D :=> int@mode HoldForm@D@a int@b, cD, 8x, i<DD, dD;
```

and rewrite **SCond1** and **SCond2** as

```
SCond = 8SCond1 . . intRule5@fm@xDD, SCond2 . . intRule5@fn@xDD<;
```

To simplify the notation in the solvability conditions, we use the orthonormality of the mode shapes and define the following parameters:

```
notationRule = 9intAfi@xD2, 8x, 0, 1<E -> 1, intAfi@xD fiH4L@xD, 8x, 0, 1<E -> li4,  
  intAm@xD fi@xD2, 8x, 0, 1<E -> mi, int@F@xD fi@xD, 8x, 0, 1<D -> w1,i f=;
```

To identify the forms of the nonlinear terms in the solvability conditions, we first identify all of the possible forms of the nonhomogeneous terms in the first-order problem. To this end, we let

```
basicTerms = TableA9Ai@T1D EI omgList@@iDD T0, Ai@T1D E-I omgList@@iDD T0=, 8i, 2<E . . Flatten  
8EI T0 w1,m A1@T1D, E-I T0 w1,m A1@T1D, EI T0 w2,n A2@T1D, E-I T0 w2,n A2@T1D<
```

Then, all of the possible forms of the nonlinear terms in the first-order problem are given by

```
cubicTerms = Nest@Outer@Times, basicTerms, #D &, basicTerms, 2D . . Flatten . . Union;  
cubicTerms . . displayRule
```

```
: E3 I T0 w1,m A13, EI T0 H2 w1,m+w2,nL A12 A2, EI T0 H w1,m+2 w2,nL A1 A22, E3 I T0 w2,n A23, EI T0 w1,m A12 A1,  
  EI T0 w2,n A1 A2 A1, EI T0 H-w1,m+2 w2,nL A22 A1, E-I T0 w1,m A1 A12, EI T0 H-2 w1,m+w2,nL A2 A12, E-3 I T0 w1,m A13,  
  EI T0 H2 w1,m-w2,nL A12 A2, EI T0 w1,m A1 A2 A2, EI T0 w2,n A22 A2, E-I T0 w2,n A1 A1 A2, E-I T0 w1,m A2 A1 A2,  
  EI T0 H-2 w1,m-w2,nL A12 A2, EI T0 H w1,m-2 w2,nL A1 A22, E-I T0 w2,n A2 A22, EI T0 H-w1,m-2 w2,nL A1 A22, E-3 I T0 w2,n A23>
```


HsymbolList@1D -> coef@@1DD •• Thread •• Union •• RestL •. int -> Integrate •• Timing

: 29.683 Second, : 8 a_{1,1} w_{1,m} ®

$$\begin{aligned}
 & -3 \int_0^1 f_m @ x D f_m^2 @ x D^3 \hat{a} x - 12 \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_m^{H3L} @ x D \hat{a} x - 3 \int_0^1 f_m @ x D f_m^e @ x D^2 f_m^{H4L} @ x D \hat{a} x + \\
 & \int_k -3 \int_0^1 f_m @ x D f_m^2 @ x D^3 \hat{a} x - 12 \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_m^{H3L} @ x D \hat{a} x - 3 \int_0^1 f_m @ x D f_m^e @ x D^2 f_m^{H4L} @ x D \hat{a} x \frac{y}{z} d_0 + \\
 & 2 \int_k \int_0^1 \mathbb{I}_{8x,1<} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_m^e @ x D^2 \hat{a} x \hat{a} x N f_m @ x D \hat{a} x \frac{y}{z} w_{1,m}^2,
 \end{aligned}$$

$$\begin{aligned}
 & 8 a_{1,3} w_{1,m} \text{®} - \int_0^1 f_m @ x D f_m^2 @ x D f_n^2 @ x D^2 \hat{a} x - \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_n^{H3L} @ x D \hat{a} x - \\
 & 3 \int_0^1 f_m @ x D f_m^e @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a} x - \int_0^1 f_m @ x D f_m^e @ x D f_n^e @ x D f_n^{H4L} @ x D \hat{a} x + \\
 & \int_k \int_0^1 \mathbb{I}_{8x,1<} J f_n^{H3L} @ x D \hat{a}_0^x f_n^e @ x D f_m^2 @ x D \hat{a} x N f_m @ x D \hat{a} x - \int_0^1 \mathbb{I}_{8x,1<} J f_n^2 @ x D \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x N f_m @ x D \hat{a} x - \\
 & \int_0^1 f_m @ x D f_m^2 @ x D f_n^2 @ x D^2 \hat{a} x - \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_n^{H3L} @ x D \hat{a} x - \\
 & 3 \int_0^1 f_m @ x D f_m^e @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a} x - \int_0^1 f_m @ x D f_m^e @ x D f_n^e @ x D f_n^{H4L} @ x D \hat{a} x \frac{y}{z} d_0 - \\
 & \int_0^1 \mathbb{I}_{8x,2<} H f_n^2 @ x D \hat{a}_0^x \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x \hat{a} x L f_m @ x D \hat{a} x M d_0^2 \\
 & \frac{b_g}{b_g} + \\
 & 2 \int_k \int_0^1 \mathbb{I}_{8x,1<} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_n^e @ x D^2 \hat{a} x \hat{a} x N f_m @ x D \hat{a} x \frac{y}{z} w_{2,n}^2, \\
 & 8 a_{1,5} w_{1,m} \text{®} - 2 \int_0^1 f_m @ x D f_m^2 @ x D f_n^2 @ x D^2 \hat{a} x - 2 \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_n^{H3L} @ x D \hat{a} x - \\
 & 6 \int_0^1 f_m @ x D f_m^e @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a} x - 2 \int_0^1 f_m @ x D f_m^e @ x D f_n^e @ x D f_n^{H4L} @ x D \hat{a} x + \\
 & \int_k 2 \int_0^1 \mathbb{I}_{8x,1<} J f_n^{H3L} @ x D \hat{a}_0^x f_n^e @ x D f_m^2 @ x D \hat{a} x N f_m @ x D \hat{a} x - 2 \int_0^1 \mathbb{I}_{8x,1<} J f_n^2 @ x D \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x N f_m @ x D \\
 & \hat{a} x - 2 \int_0^1 f_m @ x D f_m^2 @ x D f_n^2 @ x D^2 \hat{a} x - 2 \int_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_n^{H3L} @ x D \hat{a} x - \\
 & 6 \int_0^1 f_m @ x D f_m^e @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a} x - 2 \int_0^1 f_m @ x D f_m^e @ x D f_n^e @ x D f_n^{H4L} @ x D \hat{a} x \frac{y}{z} d_0 - \\
 & 2 \int_0^1 \mathbb{I}_{8x,2<} H f_n^2 @ x D \hat{a}_0^x \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x \hat{a} x L f_m @ x D \hat{a} x M d_0^2 \\
 & \frac{b_g}{b_g} >>
 \end{aligned}$$

HsymbolList@2D -> coef@@2DD •• Thread •• Union •• RestL •. int -> Integrate •• Timing

```
:22.282 Second, :8 a2,2 w2,n ∫01 fn(x) fm2(x) fn2(x) dx -
6 ∫01 fn(x) fnε(x) fm2(x) fmH3L(x) dx - 2 ∫01 fn(x) fmε(x) fn2(x) fmH3L(x) dx -
2 ∫01 fn(x) fmε(x) fnε(x) fmH4L(x) dx + ∫k - 2 ∫01 ∫8x,1< J fmH3L(x) ∫0x fmε(x) fn2(x) dx xN fn(x) dx +
2 ∫01 ∫8x,1< J fm2(x) ∫1x fm2(x) fn2(x) dx xN fn(x) dx x ∫0x d0 -
2 ∫01 ∫8x,2< H fn2(x) ∫0x ∫1x fm2(x) fn2(x) dx xL fn(x) dx xM d02
bg, 8 a2,4 w2,n ∫01 fn(x) fm2(x) fn2(x) dx - 3 ∫01 fn(x) fnε(x) fm2(x) fmH3L(x) dx - ∫01 fn(x) fmε(x) fn2(x) fmH3L(x) dx -
∫01 fn(x) fmε(x) fnε(x) fmH4L(x) dx + ∫k - ∫01 ∫8x,1< J fmH3L(x) ∫0x fmε(x) fn2(x) dx xN fn(x) dx +
∫01 ∫8x,1< J fm2(x) ∫1x fm2(x) fn2(x) dx xN fn(x) dx x ∫0x d0 -
∫01 ∫8x,2< H fn2(x) ∫0x ∫1x fm2(x) fn2(x) dx xL fn(x) dx xM d02
bg +
2 ∫k ∫01 ∫8x,1< J fnε(x) ∫1x ∫0x fmε(x) fn2(x) dx xL fn(x) dx x ∫0x w1,m2 d0 -
8 a2,6 w2,n ∫01 fn(x) fn2(x) fn2(x) dx - 12 ∫01 fn(x) fnε(x) fn2(x) fnH3L(x) dx -
3 ∫01 fn(x) fnε(x) fnH4L(x) dx + 2 ∫k ∫01 ∫8x,1< J fnε(x) ∫1x ∫0x fnε(x) fn2(x) dx xL fn(x) dx x ∫0x w2,n2 d0>>
```

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm** defined in the preceding section, we obtain the modulation equations in polar form and the definitions for g_i as

PolarForm@eqModD

```
:: a1ε@T1D == 1/2 H - f Sin@g1@T1DD - 2 mm a1@T1D + 2 Sin@g2@T1DD a1,3 a1@T1D a2@T1D2L,
a2ε@T1D == -mn a2@T1D - Sin@g2@T1DD a2,4 a1@T1D2 a2@T1D,
g2ε@T1D == - d1 lm4 / (2 w1,m) - f Cos@g1@T1DD / (2 a1@T1D) + 2 a1,1 a1@T1D2 - 2 a2,2 a1@T1D2 -
2 Cos@g2@T1DD a2,4 a1@T1D2 + 2 Cos@g2@T1DD a1,3 a2@T1D2 + 2 a1,5 a2@T1D2 - 2 a2,6 a2@T1D2,
g1ε@T1D == s - d1 lm4 / (2 w1,m) - f Cos@g1@T1DD / (2 a1@T1D) + a1,1 a1@T1D2 + Cos@g2@T1DD a1,3 a2@T1D2 + a1,5 a2@T1D2>,
8 g1@T1D == T1 s - b1@T1D, g2@T1D == -2 b1@T1D + 2 b2@T1D<>
```

Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm** defined in the preceding section, we obtain the modulation equations in Cartesian form, the definitions for η_i , and the symmetry property as

CartesianForm@eqModD

$$\begin{aligned}
 &:: p_1 @ T_1 D == -m_m p_1 @ T_1 D - n_1 q_1 @ T_1 D + \frac{d_1 l_m^4 q_1 @ T_1 D}{2 w_{1,m}} - \\
 &\quad a_{1,1} p_1 @ T_1 D^2 q_1 @ T_1 D + a_{1,3} p_2 @ T_1 D^2 q_1 @ T_1 D - a_{1,5} p_2 @ T_1 D^2 q_1 @ T_1 D - a_{1,1} q_1 @ T_1 D^3 - \\
 &\quad 2 a_{1,3} p_1 @ T_1 D p_2 @ T_1 D q_2 @ T_1 D - a_{1,3} q_1 @ T_1 D q_2 @ T_1 D^2 - a_{1,5} q_1 @ T_1 D q_2 @ T_1 D^2, \quad q_1 @ T_1 D == \\
 &\quad -\frac{f}{2} + n_1 p_1 @ T_1 D - \frac{d_1 l_m^4 p_1 @ T_1 D}{2 w_{1,m}} + a_{1,1} p_1 @ T_1 D^3 + a_{1,3} p_1 @ T_1 D p_2 @ T_1 D^2 + a_{1,5} p_1 @ T_1 D p_2 @ T_1 D^2 - m_m q_1 @ T_1 D + \\
 &\quad a_{1,1} p_1 @ T_1 D q_1 @ T_1 D^2 + 2 a_{1,3} p_2 @ T_1 D q_1 @ T_1 D q_2 @ T_1 D - a_{1,3} p_1 @ T_1 D q_2 @ T_1 D^2 + a_{1,5} p_1 @ T_1 D q_2 @ T_1 D^2, \\
 & p_2 @ T_1 D == -m_n p_2 @ T_1 D - 2 a_{2,4} p_1 @ T_1 D p_2 @ T_1 D q_1 @ T_1 D - n_2 q_2 @ T_1 D - a_{2,2} p_1 @ T_1 D^2 q_2 @ T_1 D + \\
 &\quad a_{2,4} p_1 @ T_1 D^2 q_2 @ T_1 D - a_{2,6} p_2 @ T_1 D^2 q_2 @ T_1 D - a_{2,2} q_1 @ T_1 D^2 q_2 @ T_1 D - a_{2,4} q_1 @ T_1 D^2 q_2 @ T_1 D - a_{2,6} q_2 @ T_1 D^3, \\
 & q_2 @ T_1 D == n_2 p_2 @ T_1 D + a_{2,2} p_1 @ T_1 D^2 p_2 @ T_1 D + a_{2,4} p_1 @ T_1 D^2 p_2 @ T_1 D + a_{2,6} p_2 @ T_1 D^3 + a_{2,2} p_2 @ T_1 D q_1 @ T_1 D^2 - \\
 &\quad a_{2,4} p_2 @ T_1 D q_1 @ T_1 D^2 - m_n q_2 @ T_1 D + 2 a_{2,4} p_1 @ T_1 D q_1 @ T_1 D q_2 @ T_1 D + a_{2,6} p_2 @ T_1 D q_2 @ T_1 D^2, \\
 & 8 n_1 \otimes s, \quad n_2 \otimes s <, \quad 88 p_1, \quad q_1, \quad p_2, \quad q_2 <, \quad 8 p_1, \quad q_1, \quad -p_2, \quad -q_2 <<>
 \end{aligned}$$

7.3.3 Discretization of the Continuous Problem

As an alternative, we apply the method of multiple scales to the discretized system of **EOM** and **BC**. To determine the discretized form, we express $v @ x, t$ and $w @ x, t$ in terms of the linear mode shapes $f_m @ x$ and $f_n @ x$, respectively, as

```

discretRule = 8v -> Function@8x, t<, q_m@tD f_m@xDD, w -> Function@8x, t<, q_n@tD f_n@xDD<;

modes = 8f_m@xD, f_n@xD<;

```

where the $q_i @ t$ are the generalized coordinates. We consider only $q_m @ t$ and $q_n @ t$ because in the presence of damping, all other modes that are not directly or indirectly excited decay with time. Substituting **discretRule** into each of **EOM**, multiplying the results with $f_m @ x$ and $f_n @ x$, respectively, and integrating the outcome from $x = 0$ to $x = 1$ using **intRule3**, **intRule4**, **intRule1**, **intRule2**, and **intRule5**, we obtain the following discretized form of the equations describing the response of the beam:

$$\begin{aligned}
 &\text{eq733a} = \\
 &\quad \text{Table@int@modes@kDD \# \cdot discretRule, 8x, 0, 1<D \cdot intRule3 \cdot \cdot \cdot intRule4 \cdot \cdot \cdot intRule1 \cdot \cdot \cdot} \\
 &\quad \quad \text{intRule2 \cdot \cdot \cdot intRule5@modes@kDDD \cdot \cdot \cdot} \\
 &\quad \text{int@a_, b_D -> HoldForm@Integrate@a, bDD \& \cdot \checkmark \text{EOM@kDD, 8k, 2<D} \\
 &:: \int_0^1 f_m @ x D f_m^{H4L} @ x D \hat{a} x \checkmark \{ b_y q_m @ t D + \int_0^1 f_m @ x D^2 \hat{a} x \checkmark \{ q_m^2 @ t D == \\
 &\quad - e \text{Cos@t WD} \int_0^1 F @ x D f_m @ x D \hat{a} x - e \int_0^1 f_m @ x D f_m^2 @ x D^3 \hat{a} x \checkmark \{ b_y q_m @ t D^3 -
 \end{aligned}$$

$$\begin{aligned}
& 4 e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_m^e @ x D f_m^2 @ x D f_m^{H3L} @ x D \hat{a}_0^x \{ b_y q_m @ t D^3 - e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_m^e @ x D^2 f_m^{H4L} @ x D \hat{a}_0^x \{ b_y q_m @ t D^3 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_n^{H3L} @ x D \hat{a}_0^x f_n^e @ x D f_m^2 @ x D \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D q_n @ t D^2 + \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_n^2 @ x D \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D q_n @ t D^2 + \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_n^{H3L} @ x D \hat{a}_0^x f_n^e @ x D f_m^2 @ x D \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ b_y q_m @ t D q_n @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_n^2 @ x D \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ b_y q_m @ t D q_n @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_m^2 @ x D f_n^2 @ x D^2 \hat{a}_0^x \{ b_y q_m @ t D q_n @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_n^e @ x D f_m^2 @ x D f_n^{H3L} @ x D \hat{a}_0^x \{ b_y q_m @ t D q_n @ t D^2 - 3 e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_m^e @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a}_0^x \{ \\
& b_y q_m @ t D q_n @ t D^2 - e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_m @ x D f_m^e @ x D f_n^e @ x D f_n^{H4L} @ x D \hat{a}_0^x \{ b_y q_m @ t D q_n @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 2 <} H f_n^2 @ x D \hat{a}_0^x \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x \hat{a}_0^x L f_m @ x D \hat{a}_0^x M q_m @ t D q_n @ t D^2 \\
& + \\
& b_g \\
& 2 e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 2 <} H f_n^2 @ x D \hat{a}_0^x \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x \hat{a}_0^x L f_m @ x D \hat{a}_0^x M b_y q_m @ t D q_n @ t D^2 \\
& - \\
& b_g \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 2 <} H f_n^2 @ x D \hat{a}_0^x \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x \hat{a}_0^x L f_m @ x D \hat{a}_0^x M b_y^2 q_m @ t D q_n @ t D^2 \\
& - \\
& b_g \\
& 2 e \int_{\mathbf{k}}^1 \hat{a}_0^1 m @ x D f_m @ x D^2 \hat{a}_0^x \{ q_m^e @ t D - e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_m^e @ x D^2 \hat{a}_0^x \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D q_m^e @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_n^e @ x D^2 \hat{a}_0^x \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D q_n^e @ t D^2 - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_m^e @ x D^2 \hat{a}_0^x \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D^2 q_m^2 @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^e @ x D \hat{a}_1^x \hat{a}_0^x f_n^e @ x D^2 \hat{a}_0^x \hat{a}_0^x N f_m @ x D \hat{a}_0^x \{ q_m @ t D q_n @ t D q_n^2 @ t D, \\
& \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D f_n^{H4L} @ x D \hat{a}_0^x \{ q_n @ t D + \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D^2 \hat{a}_0^x \{ q_n^2 @ t D == \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^{H3L} @ x D \hat{a}_0^x f_m^e @ x D f_n^2 @ x D \hat{a}_0^x N f_n @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^2 @ x D \hat{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a}_0^x N f_n @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D f_m^2 @ x D^2 f_n^2 @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - 3 e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D f_n^e @ x D f_m^2 @ x D f_m^{H3L} @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D f_m^e @ x D f_n^2 @ x D f_m^{H3L} @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 f_n @ x D f_m^e @ x D f_n^e @ x D f_m^{H4L} @ x D \hat{a}_0^x \{ q_m @ t D^2 q_n @ t D - \\
& e \int_{\mathbf{k}}^1 \hat{a}_0^1 \mathbb{I}_{8x, 1 <} J f_m^{H3L} @ x D \hat{a}_0^x f_m^e @ x D f_n^2 @ x D \hat{a}_0^x N f_n @ x D \hat{a}_0^x \{ b_y q_m @ t D^2 q_n @ t D +
\end{aligned}$$

$$\begin{aligned}
& e \int_0^1 \ddot{a}_{8x,1} J f_m^2 @ x D \ddot{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x N f_n @ x D \hat{a} x \sum_y b_y q_m @ t D^2 q_n @ t D - \\
& e \int_0^1 \ddot{a}_{8x,2} H f_m^2 @ x D \ddot{a}_1^x \ddot{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x \hat{a} x L f_n @ x D \hat{a} x M q_m @ t D^2 q_n @ t D \\
& \quad + b_g \\
& 2 e \int_0^1 \ddot{a}_{8x,2} H f_m^2 @ x D \ddot{a}_1^x \ddot{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x \hat{a} x L f_n @ x D \hat{a} x M b_y q_m @ t D^2 q_n @ t D \\
& \quad - b_g \\
& e \int_0^1 \ddot{a}_{8x,2} H f_m^2 @ x D \ddot{a}_1^x \ddot{a}_1^x f_m^2 @ x D f_n^2 @ x D \hat{a} x \hat{a} x L f_n @ x D \hat{a} x M b_y^2 q_m @ t D^2 q_n @ t D \\
& \quad - b_g \\
& e \int_0^1 f_n @ x D f_n^2 @ x D^3 \hat{a} x \sum_y q_n @ t D^3 - 4 e \int_0^1 f_n @ x D f_n^c @ x D f_n^2 @ x D f_n^{H3L} @ x D \hat{a} x \sum_y q_n @ t D^3 - \\
& e \int_0^1 f_n @ x D f_n^c @ x D^2 f_n^{H4L} @ x D \hat{a} x \sum_y q_n @ t D^3 - \\
& e \int_0^1 \ddot{a}_{8x,1} J f_n^c @ x D \ddot{a}_1^x \ddot{a}_0^x f_m^c @ x D^2 \hat{a} x \hat{a} x N f_n @ x D \hat{a} x \sum_y q_n @ t D q_m^c @ t D^2 - \\
& 2 e \int_0^1 m @ x D f_n @ x D^2 \hat{a} x \sum_y q_n^c @ t D - e \int_0^1 \ddot{a}_{8x,1} J f_n^c @ x D \ddot{a}_1^x \ddot{a}_0^x f_n^c @ x D^2 \hat{a} x \hat{a} x N f_n @ x D \hat{a} x \sum_y q_n @ t D q_n^c @ t D^2 - \\
& e \int_0^1 \ddot{a}_{8x,1} J f_n^c @ x D \ddot{a}_1^x \ddot{a}_0^x f_m^c @ x D^2 \hat{a} x \hat{a} x N f_n @ x D \hat{a} x \sum_y q_m @ t D q_n @ t D q_m^2 @ t D - \\
& e \int_0^1 \ddot{a}_{8x,1} J f_n^c @ x D \ddot{a}_1^x \ddot{a}_0^x f_n^c @ x D^2 \hat{a} x \hat{a} x N f_n @ x D \hat{a} x \sum_y q_n @ t D^2 q_n^2 @ t D >
\end{aligned}$$

where the damping is assumed to be modal. Using the orthonormality of the mode shapes $f_i @ x D$, we define

$$\begin{aligned}
& \text{notationRule} = 9 \text{intAf}_i @ x D^2, 8x, 0, 1 < E \rightarrow 1, \text{intAf}_i @ x D f_i^{H4L} @ x D, 8x, 0, 1 < E \rightarrow l_i^4, \\
& \text{intAm} @ x D f_i @ x D^2, 8x, 0, 1 < E \rightarrow m_i, \text{intF} @ x D f_i @ x D, 8x, 0, 1 < D \rightarrow w_{1,i} f = . \\
& \text{int@a_}, b_D \rightarrow \text{HoldForm@Integrate@a, bDD} \\
& : \ddot{a}_0^1 f_i @ x D^2 \hat{a} x \otimes 1, \ddot{a}_0^1 f_i @ x D f_i^{H4L} @ x D \hat{a} x \otimes l_i^4, \ddot{a}_0^1 m @ x D f_i @ x D^2 \hat{a} x \otimes m_i, \ddot{a}_0^1 F @ x D f_i @ x D \hat{a} x \otimes f_{w_{1,i}} >
\end{aligned}$$

To simplify the expression for **eq733a**, we first find possible nonlinear terms as follows:

$$\begin{aligned}
& \text{basicTerms} = \text{Table}@D@q_m @ t D, 8t, i < D, 8i, 0, 2 < D \\
& 8q_m @ t D, q_m^c @ t D, q_m^2 @ t D < \\
& \text{linearTerms} = \text{Join}@ \text{basicTerms}, \text{basicTerms} \cdot . m \rightarrow n D \\
& 8q_m @ t D, q_m^c @ t D, q_m^2 @ t D, q_n @ t D, q_n^c @ t D, q_n^2 @ t D < \\
& \text{cubicTerms} = e \text{Nest}@ \text{Outer}@ \text{Times}, \text{linearTerms}, \# D \&, \text{linearTerms}, 2 D \cdot \cdot \text{Flatten} \cdot \cdot \text{Union};
\end{aligned}$$

The coefficients corresponding to **cubicTerms** in **eq733a** are obtained as

$$\text{coefList} = \text{Coefficient}@ \text{eq733a} @ \#, 2 D D, \text{cubicTerms} D \& \cdot \checkmark 81, 2 <;$$

We replace these coefficients by $a_{i,j}$ as

```
symbolList = MapIndexed@If@#1 != 0, aSequence && #2, 0D &, coefList, 82<D;
```

Using **notationRule**, **symbolList**, and **cubicTerms**, we can rewrite **eq733a** as

```
EOM1 = Table@eq733a@@k, 1DD == Heq733a@@k, 2DD •. Thread@cubicTerms -> 0DL +
symbolList@@kDD.cubicTerms •. notationRule, 8k, 2<D

8by l_m^4 q_m@tD + q_m^2@tD == -f e Cos@t WD w_{1,m} + e a_{1,1} q_m@tD^3 + e a_{1,3} q_m@tD q_n@tD^2 - 2 e m_m q_m^e@tD +
e a_{1,8} q_m@tD q_m^e@tD^2 + e a_{1,17} q_m@tD q_n^e@tD^2 + e a_{1,21} q_m@tD^2 q_m^2@tD + e a_{1,37} q_m@tD q_n@tD q_n^2@tD,
l_n^4 q_n@tD + q_n^2@tD == e a_{2,2} q_m@tD^2 q_n@tD + e a_{2,4} q_n@tD^3 + e a_{2,9} q_n@tD q_m^e@tD^2 -
2 e m_n q_n^e@tD + e a_{2,18} q_n@tD q_n^e@tD^2 + e a_{2,22} q_m@tD q_n@tD q_m^2@tD + e a_{2,38} q_n@tD^2 q_n^2@tD<
```

Using the method of multiple scales, we seek a first-order uniform expansion in the form

$$\text{solRule} = q_n \rightarrow \left\{ \begin{array}{l} 1 \\ \hat{a} \end{array} \right\} e^j q_{n,j} @ \#1, \#2D \& \frac{Y}{X};$$

Transforming the total time derivatives in **EOM1** into partial derivatives in terms of T_0 and T_1 , substituting the **solRule** and **betayRule** into **EOM1**, expanding the result for small ϵ , and discarding terms of order higher than ϵ , we obtain

```
eq733b = HEOM1 •. 8q_s_@tD -> q_s@T_0, T_1D, Derivative@n_D@q_s_D@tD -> dt@nD@q_s@T_0, T_1DD, t -> T_0< •.
solRule •. betayRule •• ExpandAll •. e^{n_•}; n>1 -> 0;
```

Equating coefficients of like powers of ϵ in **eq733b**, we obtain

```
eqEps = Thread@CoefficientList@Subtract && #, eD == 0D & •& eq733b •• Transpose

: 9 l_m^4 q_{m,0}@T_0, T_1D + d_0 l_m^4 q_{m,0}@T_0, T_1D + q_{m,0}^{H2,0L}@T_0, T_1D == 0, l_n^4 q_{n,0}@T_0, T_1D + q_{n,0}^{H2,0L}@T_0, T_1D == 0=,
: f Cos@T_0 WD w_{1,m} + d_1 l_m^4 q_{m,0}@T_0, T_1D - a_{1,1} q_{m,0}@T_0, T_1D^3 + l_m^4 q_{m,1}@T_0, T_1D + d_0 l_m^4 q_{m,1}@T_0, T_1D -
a_{1,3} q_{m,0}@T_0, T_1D q_{n,0}@T_0, T_1D^2 + 2 m_m q_{m,0}^{H1,0L}@T_0, T_1D - a_{1,8} q_{m,0}@T_0, T_1D q_{m,0}^{H1,0L}@T_0, T_1D^2 -
a_{1,17} q_{m,0}@T_0, T_1D q_{n,0}^{H1,0L}@T_0, T_1D^2 + 2 q_{m,0}^{H1,1L}@T_0, T_1D - a_{1,21} q_{m,0}@T_0, T_1D^2 q_{m,0}^{H2,0L}@T_0, T_1D +
q_{m,1}^{H2,0L}@T_0, T_1D - a_{1,37} q_{m,0}@T_0, T_1D q_{n,0}@T_0, T_1D q_{n,0}^{H2,0L}@T_0, T_1D == 0,
- a_{2,2} q_{m,0}@T_0, T_1D^2 q_{n,0}@T_0, T_1D - a_{2,4} q_{n,0}@T_0, T_1D^3 + l_n^4 q_{n,1}@T_0, T_1D -
a_{2,9} q_{n,0}@T_0, T_1D q_{m,0}^{H1,0L}@T_0, T_1D^2 + 2 m_n q_{n,0}^{H1,0L}@T_0, T_1D - a_{2,18} q_{n,0}@T_0, T_1D q_{n,0}^{H1,0L}@T_0, T_1D^2 +
2 q_{n,0}^{H1,1L}@T_0, T_1D - a_{2,22} q_{m,0}@T_0, T_1D q_{n,0}@T_0, T_1D q_{m,0}^{H2,0L}@T_0, T_1D -
a_{2,38} q_{n,0}@T_0, T_1D^2 q_{n,0}^{H2,0L}@T_0, T_1D + q_{n,1}^{H2,0L}@T_0, T_1D == 0>>
```

Then it follows from **eqEps[[1]]** that

```
sol0 = 9q_{m,0} -> FunctionA8T_0, T_1<, A_1@T_1D Exp@I w_{1,m} T_0D + A_1@T_1D Exp@-I w_{1,m} T_0DE,
q_{n,0} -> FunctionA8T_0, T_1<, A_2@T_1D Exp@I w_{2,n} T_0D + A_2@T_1D Exp@-I w_{2,n} T_0DE=;
```

Substituting **sol0** into **eqEps[[2]]** yields

```
order1Eq = H#@@1DD & •& eqEps@@1DD •. q_{k,0} -> q_{k,1}L == H#@@1DD & •& eqEps@@1DD •. q_{k,0} -> q_{k,1}L -
H#@@1DD & •& eqEps@@2DDL •. sol0 •• TrigToExp •• ExpandAll •• Thread;
```

We consider the case of primary resonance of the flexural mode in the y direction, $W \gg w_{1,m}$. To describe the nearness of the primary resonance, we introduce the detuning parameter S and define

$$\text{omgList} = \{w_{1,m}, w_{2,n}\};$$

$$\text{ResonanceConds} = \{w_{2,n} == w_{1,m}, W == w_{1,m} + \epsilon s\};$$

$$\text{OmgRule} = \text{Solve}[\text{ResonanceConds}, \text{Drop}[\text{omgList}, \#] \sim W \ll 1 \text{ \& } \epsilon \ll 1, 2];$$

$$88w_{2,n} \gg w_{1,m}, W \gg \epsilon s + w_{1,m}, 8w_{1,m} \gg w_{2,n}, W \gg \epsilon s + w_{2,n} \ll$$

$$\text{expRule}[i_D] := \text{Exp}[\arg_D] \rightarrow \text{Exp}[\text{Expand}[\arg] \cdot \text{OmgRule}[[i]] \cdot \epsilon T_0] \rightarrow T_1 D$$

We substitute **ResonanceConds** into the right-hand sides of **order1Eq** and obtain the solvability condition as

$$S\text{Cond} =$$

$$\text{Table}[\text{Coefficient}[\text{order1Eq}[[i], 2] \cdot \text{expRule}[i], \text{Exp}[\epsilon T_0]] == 0, \{i, 2\}];$$

$$S\text{Cond} \cdot \text{displayRule}$$

$$9A_1 d_1 l_m^4 - 3A_1^2 \dot{A}_1 a_{1,1} - A_2^2 \dot{A}_1 a_{1,3} - 2A_1 A_2 \dot{A}_2 a_{1,3} +$$

$$\frac{1}{2} E^I T_1 s f w_{1,m} + 2 I_H D_1 A_1 L w_{1,m} + 2 I A_1 m w_{1,m} - A_1^2 \dot{A}_1 a_{1,8} w_{1,m}^2 + 3 A_1^2 \dot{A}_1 a_{1,21} w_{1,m}^2 +$$

$$A_2^2 \dot{A}_1 a_{1,17} w_{2,n}^2 - 2 A_1 A_2 \dot{A}_2 a_{1,17} w_{2,n}^2 + A_2^2 \dot{A}_1 a_{1,37} w_{2,n}^2 + 2 A_1 A_2 \dot{A}_2 a_{1,37} w_{2,n}^2 == 0,$$

$$- 2 A_1 A_2 \dot{A}_1 a_{2,2} - A_1^2 \dot{A}_2 a_{2,2} - 3 A_2^2 \dot{A}_2 a_{2,4} - 2 A_1 A_2 \dot{A}_1 a_{2,9} w_{1,m}^2 + A_1^2 \dot{A}_2 a_{2,9} w_{1,m}^2 + 2 A_1 A_2 \dot{A}_1 a_{2,22} w_{1,m}^2 +$$

$$A_1^2 \dot{A}_2 a_{2,22} w_{1,m}^2 + 2 I_H D_1 A_2 L w_{2,n} + 2 I A_2 m w_{2,n} - A_2^2 \dot{A}_2 a_{2,18} w_{2,n}^2 + 3 A_2^2 \dot{A}_2 a_{2,38} w_{2,n}^2 == 0 =$$

where

```
# -> H# . . ai,j . :> coefList@@i, jDD . . betayRule . . e -> 0 . . Expand . . Collect@#, d0D &L & . ž
HsymbolList . . Flatten . . Union . . RestL
```

$$\begin{aligned}
& :a_{1,1} \textcircled{\text{R}} -\hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D^3 \hat{a} x - 4 \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_m^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D^2 f_m^{H4L} \textcircled{\text{X}} D \hat{a} x + \\
& \quad \int_k -\hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D^3 \hat{a} x - 4 \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_m^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D^2 f_m^{H4L} \textcircled{\text{X}} D \hat{a} x \frac{y}{\{}} d_0, \\
& a_{1,3} \textcircled{\text{R}} -\hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D^2 \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_n^{H3L} \textcircled{\text{X}} D \hat{a} x - \\
& \quad 3 \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D f_n^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_n^{H4L} \textcircled{\text{X}} D \hat{a} x + \\
& \quad \int_k \hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^{H3L} \textcircled{\text{X}} D \hat{a}_0^x f_n^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^2 \textcircled{\text{X}} D \hat{a}_1^x f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x - \\
& \quad \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D^2 \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_n^{H3L} \textcircled{\text{X}} D \hat{a} x - \\
& \quad 3 \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D f_n^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_m \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_n^{H4L} \textcircled{\text{X}} D \hat{a} x \frac{y}{\{}} d_0 - \\
& \quad \mathbb{I}_{8x,2<}^1 H f_m^2 \textcircled{\text{X}} D \hat{a}_0^x \hat{a}_1^x f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D \hat{a} x \hat{a} x L f_m \textcircled{\text{X}} D \hat{a} x M d_0^2, \\
& \quad \text{b}_g \\
& a_{1,8} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_m^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x, \\
& a_{1,17} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_n^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x, \\
& a_{1,21} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_m^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x, \\
& a_{1,37} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_n^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_m \textcircled{\text{X}} D \hat{a} x, \\
& a_{2,2} \textcircled{\text{R}} -\hat{a}_0^1 f_n \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D^2 f_n^2 \textcircled{\text{X}} D \hat{a} x - 3 \hat{a}_0^1 f_n \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_m^2 \textcircled{\text{X}} D f_m^{H3L} \textcircled{\text{X}} D \hat{a} x - \\
& \quad \hat{a}_0^1 f_n \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D f_m^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_n \textcircled{\text{X}} D f_m^e \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_m^{H4L} \textcircled{\text{X}} D \hat{a} x + \\
& \quad \int_k -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^{H3L} \textcircled{\text{X}} D \hat{a}_0^x f_m^e \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x + \hat{a}_0^1 \mathbb{I}_{8x,1<} J f_m^2 \textcircled{\text{X}} D \hat{a}_1^x f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x \frac{y}{\{}} \\
& \quad d_0 - \mathbb{I}_{8x,2<}^1 H f_m^2 \textcircled{\text{X}} D \hat{a}_0^x \hat{a}_1^x f_m^2 \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D \hat{a} x \hat{a} x L f_n \textcircled{\text{X}} D \hat{a} x M d_0^2, \\
& \quad \text{b}_g \\
& a_{2,4} \textcircled{\text{R}} -\hat{a}_0^1 f_n \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D^3 \hat{a} x - 4 \hat{a}_0^1 f_n \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D f_n^2 \textcircled{\text{X}} D f_n^{H3L} \textcircled{\text{X}} D \hat{a} x - \hat{a}_0^1 f_n \textcircled{\text{X}} D f_n^e \textcircled{\text{X}} D^2 f_n^{H4L} \textcircled{\text{X}} D \hat{a} x, \\
& a_{2,9} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_m^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x, \\
& a_{2,18} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_n^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x, \\
& a_{2,22} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_m^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x, \\
& a_{2,38} \textcircled{\text{R}} -\hat{a}_0^1 \mathbb{I}_{8x,1<} J f_n^e \textcircled{\text{X}} D \hat{a}_1^x \hat{a}_0^x f_n^e \textcircled{\text{X}} D^2 \hat{a} x \hat{a} x N f_n \textcircled{\text{X}} D \hat{a} x >
\end{aligned}$$

which is in agreement with that obtained by direct approach.

7.3.4 Method of Time-Averaged Lagrangian

As a second alternative, we derive the modulation equations by using the method of time-averaged Lagrangian. The nondimensional Lagrangian of the beam can be expressed as (Crespo da Silva and Glynn, 1978; Arafat, Nayfeh, and Chin, 1998)

$$\begin{aligned} \text{Lagr1} = & \text{HoldA} \int_0^1 \frac{1}{2} \int_0^1 e \int_0^1 \frac{1}{2} \int_0^1 \left[H \eta_{x,x} v @ x, t D L^2 + H \eta_{x,x} w @ x, t D L^2 M \hat{a} x^2 + H \eta_{t,x} v @ x, t D L^2 + H \eta_{t,x} w @ x, t D L^2 \right. \\ & - e H_1 - b_y L J \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \hat{a}_0 \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \hat{a} x N - \\ & \frac{1}{2} H \eta_{x,x} w @ x, t D L^2 + e H \eta_{x,x} w @ x, t D L^2 H \eta_{x,x} w @ x, t D L^2 M - \frac{1}{2} b_y H \eta_{x,x} v @ x, t D L^2 + \\ & e H \eta_{x,x} v @ x, t D L^2 H \eta_{x,x} v @ x, t D L^2 + e^2 \eta_{x,x} v @ x, t D \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \eta_{x,x} w @ x, t D M - \\ & e \frac{H_1 - b_y L^2}{2 b_g} \int_0^1 J \hat{a}_1 \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \hat{a} x N^2 + 2 \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \\ & \left. \hat{a}_0 \hat{a}_1 \eta_{x,x} v @ x, t D \eta_{x,x} w @ x, t D \hat{a} x \hat{a} x^2 - e F @ x D \cos @ W t D v @ x, t D \hat{a} x E \right] \\ & \text{Integrate} \rightarrow \text{int} \cdot \text{HoldPattern} @ \eta_{t,x} \text{int} @ a_, b_{DD} := \text{int} @ \eta_{t,x} a, b_D \cdot \text{ReleaseHold}; \end{aligned}$$

where **Hold** and **int** are used to keep **Integrate** from being evaluated. Transforming the total time derivatives into partial derivatives in terms of T_0 and T_1 , we modify **Lagr1** as

$$\begin{aligned} \text{Lagr2} = & \text{Lagr1} \cdot 8 v @ x_, t D \rightarrow v @ x, T_0, T_1 D, w @ x_, t D \rightarrow w @ x, T_0, T_1 D, \\ & \text{Derivative} @ m_, n_D @ w_D @ x_, t D \rightarrow dt @ n_D @ D @ w @ x, T_0, T_1 D, 8 x, m < D D, t \rightarrow T_0 <; \end{aligned}$$

To this end, we let

$$\begin{aligned} \text{solRule} = & 9 v \rightarrow \text{FunctionA} 8 x, T_0, T_1 <, A_1 @ T_1 D f_m @ x D \text{Exp} @ I w_{1,m} T_0 D + \dot{A}_1 @ T_1 D f_m @ x D \text{Exp} @ - I w_{1,m} T_0 D E, \\ w \rightarrow & \text{FunctionA} 8 x, T_0, T_1 <, A_2 @ T_1 D f_n @ x D \text{Exp} @ I w_{2,n} T_0 D + \dot{A}_2 @ T_1 D f_n @ x D \text{Exp} @ - I w_{2,n} T_0 D E =; \end{aligned}$$

and define

$$\begin{aligned} \text{intRule6} = & 8 \text{int} @ c1_. \text{int} @ a1_ D^n _, b1_ D := \\ & c1 \text{int} @ \text{int} @ a1 D^n \cdot \text{intRule1} \cdot \text{intRule2} \cdot e \rightarrow 0, b1 D \cdot; \text{FreeQ} @ c1, \text{First} @ b1 D D, \\ & \text{int} @ c1_ \text{int} @ a1_ D^n _, b1_ D := \text{int} @ c1 \text{int} @ a1 D^n \cdot \text{intRule1} \cdot \text{intRule2} \cdot e \rightarrow 0, b1 D <; \end{aligned}$$

Substituting **solRule** and **betayRule** into **Lagr2**, using **intRule1**, **intRule2**, and **intRule6** to simplify the expansions of the integrands, and collecting the coefficients of e , we obtain


```

Horder1Lagr =
  Lagr2 •. solRule •. betayRule •. intRule1 •. intRule2 •. intRule6 •. intRule1 •.
    intRule2 •. intRule6 •. intRule1 •. intRule2 •.
    TrigToExp •. Expand •. Coefficient@#, eD &L •. Timing

814.951 Second, Null<

ResonanceConds = 8w2,n == w1,m, W == w1,m + e s < •. Equal -> Rule

8w2,n @ w1,m, W @ e s + w1,m <

notationRule1 = 9intAfi@xD2, 8x, 0, 1<E -> 1,
  int@F@xD fi@xD, 8x, 0, 1<D -> f w1,i, intAfi2@xD2, 8x, 0, 1<E -> li4=;

```

Using **ResonanceConds** and **notationRule1**, we obtain the slowly varying terms from **order1Lagr** as

```

TAL = order1Lagr •. Exp@a_D :=> Exp@Expand@a •. ResonanceCondsD •. e T0 -> T1D •.
  Exp@_ T0 + _ .D -> 0 •. notationRule1;

```

The Euler-Lagrange equations corresponding to the **TAL** can be written as

eqMod1 =

DADATAL, $\dot{A}_1@T_1DE$, $T_1E - DATAL$, $\dot{A}_1@T_1DE == 0 \dots \text{int@a_}$, $b_D \rightarrow \text{HoldForm@Integrate@a, bDD}$

$$\begin{aligned}
 & \frac{1}{2} E^{T_1 s} f_{w_{1,m}} + d_1 l_m^4 A_1@T_1D + 6 \int_{k_0}^1 f_m^e @xD^2 f_m^2 @xD^2 \hat{a} x \frac{y}{z} A_1@T_1D^2 \dot{A}_1@T_1D + \\
 & 6 \int_{k_0}^1 f_m^e @xD^2 f_m^2 @xD^2 \hat{a} x \frac{y}{z} d_0 A_1@T_1D^2 \dot{A}_1@T_1D - 2 \int_{k_0}^1 J \hat{a}_0^x f_m^e @xD^2 \hat{a} x N^2 \hat{a} x \frac{y}{z} w_{1,m}^2 A_1@T_1D^2 \dot{A}_1@T_1D + \\
 & 2 \int_{k_0}^1 f_m^e @xD f_n^e @xD f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} A_2@T_1D^2 \dot{A}_1@T_1D - \\
 & 2 \int_{k_0}^1 J \hat{a}_0^x f_n^e @xD f_m^2 @xD \hat{a} x N f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} d_0 A_2@T_1D^2 \dot{A}_1@T_1D + \\
 & 2 \int_{k_0}^1 f_m^e @xD f_n^e @xD f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} d_0 A_2@T_1D^2 \dot{A}_1@T_1D + \\
 & l_0^1 H \hat{u}_0^x f_m^2 @xD f_n^2 @xD \hat{a} x L^2 \hat{a} x M d_0^2 A_2@T_1D^2 \dot{A}_1@T_1D \\
 & \frac{2}{b_g} l_0^1 H \hat{u}_0^x f_m^2 @xD f_n^2 @xD \hat{a} x \hat{a} x L f_m^2 @xD f_n^2 @xD \hat{a} x M d_0^2 A_2@T_1D^2 \dot{A}_1@T_1D \\
 & - \frac{2}{b_g} l_0^1 H \hat{u}_0^x f_m^2 @xD f_n^2 @xD \hat{a} x \hat{a} x L f_m^2 @xD f_n^2 @xD \hat{a} x M d_0^2 A_2@T_1D^2 \dot{A}_1@T_1D \\
 & 2 \int_{k_0}^1 J \hat{a}_0^x f_m^e @xD^2 \hat{a} x N \hat{a}_0^x f_n^e @xD^2 \hat{a} x \hat{a} x \frac{y}{z} w_{1,m} w_{2,n} A_2@T_1D^2 \dot{A}_1@T_1D + \\
 & 4 \int_{k_0}^1 f_m^e @xD f_n^e @xD f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} A_1@T_1D A_2@T_1D \dot{A}_2@T_1D - \\
 & 4 \int_{k_0}^1 J \hat{a}_0^x f_n^e @xD f_m^2 @xD \hat{a} x N f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} d_0 A_1@T_1D A_2@T_1D \dot{A}_2@T_1D + \\
 & 4 \int_{k_0}^1 f_m^e @xD f_n^e @xD f_m^2 @xD f_n^2 @xD \hat{a} x \frac{y}{z} d_0 A_1@T_1D A_2@T_1D \dot{A}_2@T_1D + \\
 & \frac{2}{b_g} l_0^1 H \hat{u}_0^x f_m^2 @xD f_n^2 @xD \hat{a} x L^2 \hat{a} x M d_0^2 A_1@T_1D A_2@T_1D \dot{A}_2@T_1D \\
 & + \frac{4}{b_g} l_0^1 H \hat{u}_0^x f_m^2 @xD f_n^2 @xD \hat{a} x \hat{a} x L f_m^2 @xD f_n^2 @xD \hat{a} x M d_0^2 A_1@T_1D A_2@T_1D \dot{A}_2@T_1D \\
 & + 2 I w_{1,m} A_1^e @T_1D == 0
 \end{aligned}$$

and

eqMod2 =

DADATL, $\ddot{A}_2@T_1DE$, $T_1E - DATL$, $\dot{A}_2@T_1DE == 0 \dots \text{int@a_}$, $b_D \rightarrow \text{HoldForm@Integrate@a, bDD}$

$$\begin{aligned}
 & 4 \int_0^1 \ddot{f}_m @ x D \ddot{f}_n @ x D \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D - \\
 & 4 \int_0^1 J \ddot{a}_0^x \ddot{f}_n @ x D \ddot{f}_m @ x D \hat{x} N \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} d_0 A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\
 & 4 \int_0^1 \ddot{f}_m @ x D \ddot{f}_n @ x D \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} d_0 A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D + \\
 & \frac{2 \int_0^1 H \ddot{u}_0^x \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} L^2 \hat{x} M d_0^2 A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D}{b_g} + \\
 & \frac{4 \int_0^1 H \ddot{u}_0^x \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} L^2 \hat{x} M d_0^2 A_1 @ T_1 D A_2 @ T_1 D \dot{A}_1 @ T_1 D}{b_g} + \\
 & 2 \int_0^1 \ddot{f}_m @ x D \ddot{f}_n @ x D \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D - \\
 & 2 \int_0^1 J \ddot{a}_0^x \ddot{f}_n @ x D \ddot{f}_m @ x D \hat{x} N \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} d_0 A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D + \\
 & 2 \int_0^1 \ddot{f}_m @ x D \ddot{f}_n @ x D \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} \frac{y}{z} d_0 A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D + \\
 & \frac{2 \int_0^1 H \ddot{u}_0^x \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} L^2 \hat{x} M d_0^2 A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D}{b_g} + \\
 & \frac{2 \int_0^1 H \ddot{u}_0^x \ddot{f}_m @ x D \ddot{f}_n @ x D \hat{x} L^2 \hat{x} M d_0^2 A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D}{b_g} - \\
 & 2 \int_0^1 J \ddot{a}_0^x \ddot{f}_m @ x D^2 \hat{x} N \ddot{a}_0^x \ddot{f}_n @ x D^2 \hat{x} \frac{y}{z} w_{1,m} w_{2,n} A_1 @ T_1 D^2 \dot{A}_2 @ T_1 D + \\
 & 6 \int_0^1 \ddot{f}_n @ x D^2 \ddot{f}_n @ x D^2 \hat{x} \frac{y}{z} A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D - \\
 & 2 \int_0^1 J \ddot{a}_0^x \ddot{f}_n @ x D^2 \hat{x} N^2 \hat{x} \frac{y}{z} w_{2,n}^2 A_2 @ T_1 D^2 \dot{A}_2 @ T_1 D + 2 \int_0^1 w_{2,n} A_2 @ T_1 D == 0
 \end{aligned}$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

Chapter 8

Continuous Systems with Quadratic and Cubic Nonlinearities

In this chapter, we use the method of multiple scales to determine second-order uniform asymptotic solutions of continuous or distributed-parameter systems with quadratic and cubic nonlinearities. We consider some of the internal resonances arising from the cubic nonlinearities, namely, [one-to-one](#) and [three-to-one internal resonances](#). [Two-to-one](#) internal resonances are treated in Chapter 9. We directly attack these continuous systems because treatment of reduced-order models obtained with the Galerkin procedure might lead to erroneous results. We consider buckled beams in Section 8.1, circular cylindrical shells in Section 8.2, and near-square plates in Section 8.3.

à Preliminaries

```
Off@General::spell1, Integrate::generD
```

```
Needs@"Utilities`Notation`"D
```

To determine second-order uniform asymptotic expansions of the solutions of buckled beams, circular cylindrical shells, and near-square plates by using the method of multiple scales, we introduce the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, where ϵ is a small nondimensional parameter and the order of the amplitude of oscillations. Moreover, we symbolize these scales according to

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = {T0, T1, T2};
```

In terms of the time scales T_0 , T_1 , and T_2 , the time derivatives can be expressed as

```
dt@0D@expr_D := expr; dt@1D@expr_D := Sum[Ae^i D@expr, timeScales@@i + 1DDD, {i, 0, 2}E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD // ExpandL // . e^i_.; i>3 -> 0;
```

In the course of the analysis, we need the complex conjugates of A and other variables. We define them by using the following rule:

```
conjugateRule = {A -> A-bar, A-bar -> A, Complex@m_, n_D -> Complex@m, -nD};
```

To speed up manipulations of the integrals involved in the governing equations with *Mathematica*, we introduce the rules

```

intRule1 = 8int@fun_, arg2__D := int@Expand@funD, arg2D<;
intRule2 = 8int@a_ + b_, arg2__D := int@a, arg2D + int@b, arg2D,
int@e^n_ fun_, arg2__D := e^n int@fun, arg2D,
int@a_ fun_, a1_, b1__D := a int@fun, a1, b1D •; FreeQ@a, First@a1DD,
int@int@a1_, a2_D fun_, a3__D := int@a1, a2D int@fun, a3D •; FreeQ@Rest@a2D, First@a2DD<;

```

To display the outputs in easily read expressions, we introduce the rules

```

displayRule = 9Derivative@a_, b__DAw_i_ E@_, T_0, __D :=>
SequenceFormAIfAarg1 = Times žž MapIndexedAD^#1_#2@@1DD-1 &, 8b<E; arg1 != 1, arg1, ""E,
w_i_ SequenceForm žž Table@""", 8a<D E,
Derivative@a__D@A_i_ D@__D :=> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD &, 8a<E, A_i E,
Derivative@a__D@A_i_ D@__D :=> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD &, 8a<E, A_i E,
w_i_ @_, T_0, __D -> w_i, A_i_ @__D -> A_i, A_i_ @__D -> A_i,
Exp@a_. + b_. Complex@0, m_D T_0 + c_. Complex@0, n_D T_0D -> Exp@a + Hm * b + n * cL I T_0D=;

```

à 8.1 Buckled Beams

In this section, we consider the nonlinear transverse response of a buckled beam possessing either a one-to-one or a three-to-one internal resonance to a principal parametric resonance of the higher mode. The analysis assumes a unimodal buckled deflection.

The equation governing nonlinear transverse vibrations of a parametrically excited homogeneous beam is (Nayfeh and Mook, 1979)

eq81a =

$$m w_{t,t} + c w_t + E_s I_s w_{x,x,x,x} + \int_k P - f_p \cos \omega t D - \frac{E_s A}{2} \int_0^L H w_x L^2 \hat{a} x \frac{y}{\{ } w_{x,x} = 0 \bullet . w_m _ :> \P_m w @ x, t D ;$$

where P is a static axial load, $f_p \cos \omega t$ is an axial harmonic excitation, $w(x, t)$ is the transverse deflection of the beam, x is the distance along the undeflected beam, and t is time. Other dimensional parameters are the damping coefficient c and the beam length $\{$, mass per unit length m , cross-sectional area A , modulus of elasticity E_s , and cross-sectional area moment of inertia I_s . We note that the subscript s is used with I and E so that *Mathematica* will not confuse them with $\bullet$ and the exponential function. Equation eq81a is supplemented with the four linear homogeneous boundary conditions

$$bc81a = 8B_1@w@0, tDD == 0, B_2@w@0, tDD == 0, B_3@w@L, tDD == 0, B_4@w@L, tDD == 0<;$$

where the B_i are linear homogeneous operators.

It is good practice to cast the governing equations and boundary conditions in nondimensional form. Thus, we introduce the following nondimensional variables:

$$T = \frac{m L^4}{E_s I_s} ;$$

```
nondimRule = 9w -> H{W@#1 • {, #2 • TD &L, x -> {x̃, t -> T t̃, W -> W̃ • T,
P -> P̃ Es Is • {2, fp -> f̃ Es Is • {2, c -> c̃ • {Es Is m̃3 , {2, A -> 2 a Is • {2 =;
```

where the tilde indicates a nondimensional variable. Substituting **nondimRule** into **eq81a** and dropping the tildes, we obtain

```
eq81b = ∫k Es Is f3 eq81a@@1DD •. Integrate -> int •. nondimRule •. s̃ -> s •.
int@a_, 8x {, __<D -> {int@a, 8x, 0, 1<D •• Expand •• PowerExpandy == 0;
{
eq81b •.
int ->
Integrate
c wH0,1L@x, tD + wH0,2L@x, tD + P wH2,0L@x, tD -
f Cos@t WD wH2,0L@x, tD - a ∫k=01 wH1,0L@x, tD2 â x ∫{ wH2,0L@x, tD + wH4,0L@x, tD == 0
```

8.1.1 Postbuckling Deflection

The first step in analyzing the nonlinear vibrations of a buckled beam is the determination of the critical Euler buckling loads P_k and their corresponding mode shapes $f_k(x)$. We symbolize the P_k as

```
Symbolize@PkD;
```

Dropping the time derivative and neglecting the forcing and nonlinear terms in **eq81b**, we define the linear buckling problem as

```
buckEq = eq81b •. int -> H0 &L •. 8P -> Pk, f -> 0, w -> Hfk@#1D &L<
Pk fk2@xD + fkH4L@xD == 0
```

Solving **buckEq** for $f_k^{H4L}(x)$ yields

```
buckRule = SolveAbuckEq, fkH4L@xDE@@1DD
9 fkH4L@xD ® -Pk fk2@xD =
```

Next, we increase the axial load P beyond P_k and assume that the postbuckling displacement is $b f_k$, where b is a nondimensional measure of the buckling level. Substituting this displacement into **eq81b**, dropping the forcing term and the time derivative, and using **intRule2** and **buckRule**, we obtain

```
bEq = Factor@eq81b@@1DD •. 8f -> 0, w -> Hb fk@#1D &L< •. intRule2 •. buckRuleD == 0
-b H - P + Pk + b2 a int@fk4@xD2, 8x, 0, 1<D fk2@xD == 0
```

It follows from **bEq** that either $b = 0$ (unbuckled case) or b is given by

l bRule = b² -> l b² •. Solve@bEq, bD@@2DDMM •. int -> Integrate

$$b^2 \otimes \frac{P - P_k}{a \int_0^1 f_k^e @ x D^2 \hat{a} x}$$

And solving **bEq** for P yields

HPRule = Solve@bEq, PD@@1DDL •. int -> Integrate

$$: P \otimes P_k + b^2 a \int_0^1 f_k^e @ x D^2 \hat{a} x >$$

Because P is greater than P_k , **PRule** yields the amplitude of the k th buckling mode.

Having solved the postbuckling problem, we assume that the beam deflection is the sum of the static buckled displacement $b f_k @ x D$, corresponding to the k th buckled mode, and a time-dependent relative deflection $u @ x, t D$; that is,

wRule = 8w -> Hb f_k @ #1D + u @ #1, #2D &L<;

Substituting **wRule** into **eq81b** and using **intRule1**, **intRule2**, **buckRule**, and **PRule**, we find that the nonlinear transverse vibrations of the beam around its buckled configuration are given by

EOM = eq81b •. wRule •. intRule1 •. intRule2 •. buckRule •. PRule •. ExpandAll;

EOM •. int -> Integrate

$$\begin{aligned} & -b f \cos @ t W D f_k^2 @ x D - 2 b^2 a \int_0^1 f_k^e @ x D u^{H1, 0L} @ x, t D \hat{a} x \int_0^1 f_k^2 @ x D - b a \int_0^1 u^{H1, 0L} @ x, t D^2 \hat{a} x \int_0^1 f_k^2 @ x D + \\ & c u^{H0, 1L} @ x, t D + u^{H0, 2L} @ x, t D + P_k u^{H2, 0L} @ x, t D - f \cos @ t W D u^{H2, 0L} @ x, t D - \\ & 2 b a \int_0^1 f_k^e @ x D u^{H1, 0L} @ x, t D \hat{a} x \int_0^1 u^{H2, 0L} @ x, t D - a \int_0^1 u^{H1, 0L} @ x, t D^2 \hat{a} x \int_0^1 u^{H2, 0L} @ x, t D + u^{H4, 0L} @ x, t D == 0 \end{aligned}$$

BC = bc81a •. 8w -> u, { -> 1<

$$8B_1 @ u @ 0, t D D == 0, B_2 @ u @ 0, t D D == 0, B_3 @ u @ 1, t D D == 0, B_4 @ u @ 1, t D D == 0 <$$

In the following analysis, we focus on **hinged-hinged, first mode-buckled beams**. Then, the boundary conditions for the static deflection $f_k @ x D$ can be written as

$$\mathbf{bBC} = 8f_k @ 0D == 0, f_k^2 @ 0D == 0, f_k @ 1D == 0, f_k^2 @ 1D == 0 <;$$

and the boundary conditions for the dynamic deflection $u @ x, t D$ can be written as

$$\mathbf{BC1} = 8u @ 0, t D == 0, \mathbb{1}_{x, x} u @ x, t D == 0 \cdot x -> 0, u @ 1, t D == 0, \mathbb{1}_{x, x} u @ x, t D == 0 \cdot x -> 1 <$$

$$8u @ 0, t D == 0, u^{H2, 0L} @ 0, t D == 0, u @ 1, t D == 0, u^{H2, 0L} @ 1, t D == 0 <$$

For the case of first buckled mode, it follows from **buckEq** and **bBC** that the first critical buckling load and associated mode-shape are

```
bucklingLoad = Pk -> p2;
modeshape = 8fk -> Hsin@p #D &L<;
```

where the buckling modeshape was normalized so that $f_k(1) \cdot 2D = 1$.

Linear Vibrations

The next step in analyzing the nonlinear vibrations of a buckled beam is the calculation of its linear natural frequencies and corresponding modeshapes. To this end, we linearize **EOM** in $u(x, t)$, drop the forcing, damping, and nonlinear terms, and obtain

```
linearEq = EOM . 8c -> 0, f -> 0 < . u -> He u@#1, #2D &L . intRule2 . en-*; n>1 -> 0 . e -> 1;
linearEq . int -> Integrate

- 2 b2 a ∫01 fkε@xD uH1, 0L@x, tD â x ∑{ fk2@xD + uH0, 2L@x, tD + Pk uH2, 0L@x, tD + uH4, 0L@x, tD == 0
```

The boundary conditions are the same as **BC1**, which we list below as

```
linearBC = BC1

8u@0, tD == 0, uH2, 0L@0, tD == 0, u@1, tD == 0, uH2, 0L@1, tD == 0 <
```

Then, using separation of variables, we seek the solution of **linearEq** and **linearBC** in the form

```
uSol = u -> Function@8x, t<, F@xD Exp@I w tDD;
```

where w is the nondimensional linear natural frequency and $F(x)$ is the corresponding modeshape. Substituting **uSol** into **linearEq** and **linearBC** and using **intRule2** and **Thread**, we obtain the equation governing $F(x)$ as

```
FEq = linearEq . uSol . intRule2 . ThreadAE-I w t #, EqualE & . ExpandAll;
FEq . int -> Integrate

- w2 F@xD + Pk F2@xD - 2 b2 a ∫01 Fε@xD fkε@xD â x ∑{ fk2@xD + FH4L@xD == 0
```

and the boundary conditions as

```
FBC = linearBC . u -> HF@#1D &L

8F@0D == 0, F2@0D == 0, F@1D == 0, F2@1D == 0 <
```

To find the general solution of **FEq** and **FBC**, we follow Nayfeh, Kreider, and Anderson (1995) and treat the definite integral as a constant B ; thus, we have a nonhomogeneous linear ordinary-differential equation. Hence, its general solution can be expressed as a linear combination of homogeneous and particular solutions. Because $f_k(x)$ is a solution of **buckEq**, the particular solution is proportional to $f_k^2(x)$. Consequently, we express the general solution of **FEq** and **FBC** as

```
cList = Table@ci, 8i, 5<D;

shapeFunc = 8Cos@ l1 xD, Sin@ l1 xD, Cosh@ l2 xD, Sinh@ l2 xD, fk2@xD<;
```


FSol = F -> Function@x, cList.shapeFunc •. modeshape •• Evaluated

F @ Function@x, Cos@x l₁D c₁ + Sin@x l₁D c₂ + Cosh@x l₂D c₃ + Sinh@x l₂D c₄ - p² Sin@p xD c₅D

where the l_i satisfy the characteristic equation

charEq =

FEq •. b -> 0 •. bucklingLoad •. F -> HExp@l #D &L •• ThreadAE⁻¹x #, EqualE & •• ExpandAll

p² l² + l⁴ - w² == 0

lRule = 9- l₁², l₂² -> l l² •. Solve@charEq, lD •• UnionM •• Thread

: - l₁² @ $\frac{1}{2} \int_k -p^2 - \frac{p^4 + 4w^2}{4} \frac{y}{\{}$, l₂² @ $\frac{1}{2} \int_k -p^2 + \frac{p^4 + 4w^2}{4} \frac{y}{\{}$

Substituting **bucklingLoad**, **modeshape**, and **FSol** into the left-hand sides of **FBC** and **FEq** and using **intRule1** and **intRule2**, we obtain

expr1 = Append@#@@1DD & •ž FBC •. FSol,

FEq@@1DD •. bucklingLoad •. modeshape •. FSol •. intRule1 ••. intRule2 •. int -> Integrated

**: c₁ + c₃, - c₁ l₁² + c₃ l₂², Cos@l₁D c₁ + Sin@l₁D c₂ + Cosh@l₂D c₃ + Sinh@l₂D c₄,
 - Cos@l₁D c₁ l₁² - Sin@l₁D c₂ l₁² + Cosh@l₂D c₃ l₂² + Sinh@l₂D c₄ l₂²,
 - p⁶ Sin@p xD c₅ - w² HCos@x l₁D c₁ + Sin@x l₁D c₂ + Cosh@x l₂D c₃ + Sinh@x l₂D c₄ - p² Sin@p xD c₅L +
 Cos@x l₁D c₁ l₁⁴ + Sin@x l₁D c₂ l₁⁴ + Cosh@x l₂D c₃ l₂⁴ + Sinh@x l₂D c₄ l₂⁴ +
 p² Hp⁴ Sin@p xD c₅ - Cos@x l₁D c₁ l₁² - Sin@x l₁D c₂ l₁² + Cosh@x l₂D c₃ l₂² + Sinh@x l₂D c₄ l₂² L +
 2 b² p² a Sin@p xD $\int_k -\frac{1}{2} p^4 c_5 + \frac{p \sin@l_1 D c_2 l_1^2}{H_p - l_1 L H_p + l_1 L} - p c_1 l_1 \int_k \frac{l_1}{H_p - l_1 L H_p + l_1 L} - \frac{\cos@l_1 D l_1}{H_p - l_1 L H_p + l_1 L} \frac{y}{\{}$ -
 $\frac{p \sinh@l_2 D c_4 l_2^2}{H_p - l_2 L H_p + l_2 L} + p c_3 l_2 \int_k \frac{l_2}{H_p - l_2 L H_p + l_2 L} - \frac{\cosh@l_2 D l_2}{H_p - l_2 L H_p + l_2 L} \frac{y}{\{}$**

The coefficient matrix of **cList** in **expr1** can be obtained as

coefMat = Outer@Coefficient, expr1, cListD

```
:81, 0, 1, 0, 0<, 8-l12, 0, l22, 0, 0<, 8Cos@l1D, Sin@l1D, Cosh@l2D, Sinh@l2D, 0<,
8-Cos@l1D l12, -Sin@l1D l12, Cosh@l2D l22, Sinh@l2D l22, 0<, :-w2 Cos@x l1D -
p2 Cos@x l1D l12 + Cos@x l1D l14 +  $\frac{2 b^2 p^3 a \sin@p x D l_1^2}{H p - l_1 L H p + l_1 L} + \frac{2 b^2 p^3 a \cos@l_1 D \sin@p x D l_1^2}{H p - l_1 L H p + l_1 L}$ ,
-w2 Sin@x l1D - p2 Sin@x l1D l12 + Sin@x l1D l14 +  $\frac{2 b^2 p^3 a \sin@p x D \sin@l_1 D l_1^2}{H p - l_1 L H p + l_1 L}$ , -w2 Cosh@x l2D +
p2 Cosh@x l2D l22 -  $\frac{2 b^2 p^3 a \sin@p x D l_2^2}{H p - l_2 L H p + l_2 L} - \frac{2 b^2 p^3 a \cosh@l_2 D \sin@p x D l_2^2}{H p - l_2 L H p + l_2 L} + \cosh@x l_2 D l_2^4$ ,
-w2 Sinh@x l2D + p2 Sinh@x l2D l22 -  $\frac{2 b^2 p^3 a \sin@p x D \sinh@l_2 D l_2^2}{H p - l_2 L H p + l_2 L} + \sinh@x l_2 D l_2^4$ ,
-b2 p6 a Sin@p xD + p2 w2 Sin@p xD>>
```

Substituting **charEq** into **coefMat[[5]]** and eliminating the common factor **Sin@p xD**, we have

coefMat@@5DD = Coefficient@coefMat@@5DD, Sin@p xDD

```
:  $\frac{2 b^2 p^3 a l_1^2}{H p - l_1 L H p + l_1 L} + \frac{2 b^2 p^3 a \cos@l_1 D l_1^2}{H p - l_1 L H p + l_1 L}$ ,  $\frac{2 b^2 p^3 a \sin@l_1 D l_1^2}{H p - l_1 L H p + l_1 L}$ ,
-  $\frac{2 b^2 p^3 a l_2^2}{H p - l_2 L H p + l_2 L} - \frac{2 b^2 p^3 a \cosh@l_2 D l_2^2}{H p - l_2 L H p + l_2 L}$ ,  $-\frac{2 b^2 p^3 a \sinh@l_2 D l_2^2}{H p - l_2 L H p + l_2 L}$ , -b2 p6 a + p2 w2>
```

Setting the determinant of **coefMat** equal to zero yields the following characteristic equation for the natural frequencies:

Factor@Det@coefMatDD == 0

$p^2 H b^2 p^4 a - w^2 L \sin@l_1 D \sinh@l_2 D H l_1^2 + l_2^2 L^2 == 0$

There are two possibilities: $b^2 p^4 a - w^2 = 0$ and $\sin@l_1 D = 0$. Hence, either

wSol1 = SolveAb² p⁴ a - w² == 0, wE@@2DD

$9 w \otimes b p^2 \cdot \frac{1}{a} =$

or

wSol2 = SolveA- l₁² + Hn pL² == 0 •. lRule, wE@@2DD

$: w \otimes n \cdot \frac{1}{-1 + n^2} p^2 >$

for $n > 1$. The lowest five natural frequencies are

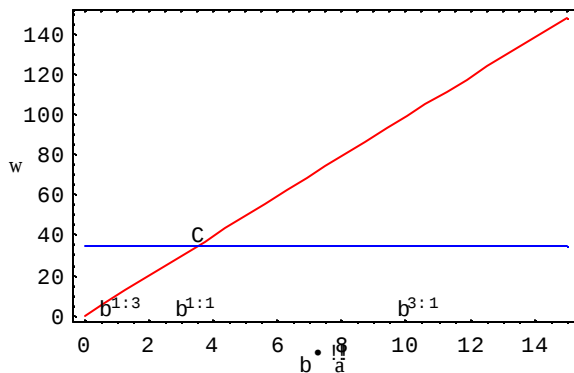
frequencies = 8w₁ -> Hw •. wSol1L, w₂ -> Hw •. wSol2 •. n -> 2L,

w₃ -> Hw •. wSol2 •. n -> 3L, w₄ -> Hw •. wSol2 •. n -> 4L, w₅ -> Hw •. wSol2 •. n -> 5L<

$9 w_1 \otimes b p^2 \cdot \frac{1}{a}, w_2 \otimes 2 \cdot \frac{1}{3} p^2, w_3 \otimes 6 \cdot \frac{1}{2} p^2, w_4 \otimes 4 \cdot \frac{1}{15} p^2, w_5 \otimes 10 \cdot \frac{1}{6} p^2 =$

The first two frequencies vs. the buckling level are plotted as follows:

```
PlotAw = 9wSol1 . b .  $\frac{1}{a^3}$  -> s, wSol2 . n -> 2 = Evaluate,
8s, 0, 15 <, PlotStyle -> 8RGBColor@1, 0, 0D, RGBColor@0, 0, 1D <,
Frame -> True, FrameLabel -> 9"b .  $\frac{1}{a^3}$ ", "w" =, RotateLabel -> False,
Epilog -> 9TextA"C", 92 .  $\frac{1}{3}$ , 2 .  $\frac{1}{3}$  p2 + 5 = E, TextA"b1:3", 92 , .  $\frac{1}{3}$ , 3 = E,
TextA"b1:1", 92 .  $\frac{1}{3}$ , 3 = E, TextA"b3:1", 96 .  $\frac{1}{3}$ , 3 = E = E;
```



where the low-frequency mode (red line) is symmetric and the second mode (blue line) is antisymmetric. As the buckling level increases from zero, the frequency w_1 of the first mode increases from zero and crosses the frequency w_2 of the second mode at point C. Hence, there are two possible three-to-one internal resonances: $w_2 = 3w_1$ when $b \cdot \frac{1}{a^3} = b^{1:3}$ and $w_1 = 3w_2$ when $b^{3:1}$. There is also a possible one-to-one internal resonance $w_2 = w_1$ when $b \cdot \frac{1}{a^3} = b^{1:1}$. Whether or not these candidates for internal resonances will be activated depends on the corresponding modeshapes. In this section, we consider the case of one-to-one internal resonance between the first and second modes. In Section 8.1.4, we consider three-to-one internal resonances between the first two modes. In Sections 8.1.5 and 8.1.6, we consider one-to-one and three-to-one internal resonances between the first and third modes and the first and fourth modes, respectively.

In the next section, we consider a combination of a one-to-one internal resonance between the lowest two modes and a principal parametric resonance of the second mode; that is,

```
omgList = 8w1, w2 <;
ResonanceConds = 9w2 == w1 + e2 s1, W == 2 w2 + e2 s2 =;
```

where the s_i are detuning parameters that describe the nearness of the resonances. The buckling level for which the natural frequencies of the first and second modes are equal (i.e., $w_2 = w_1$) is given by

```
values1 = Solve@w1 == w2 . frequencies, bD@@1DD
```

```
: b @  $\frac{2}{3} \cdot \frac{1}{a^3}$  >
```

The corresponding l_m are

```
IRuleN = Solve@IRule . w -> w1 . frequencies . values1 . Rule -> Equal, 8 l1, l2 <D@@4DD
9 l1 @ 2 p, l2 @  $\frac{1}{3}$  p =
```

In order that the influence of the damping and nonlinearity balance the influence of the forcing, we scale the damping and forcing terms as

$$\text{scaling} = 9c \rightarrow 2e^2 m, \quad f \rightarrow e^2 f;$$

Substituting for the natural frequency w_1 back into **coefMat**, we determine the modeshape as

$$\text{NullSpace@coefMat} \cdot w \rightarrow w_1 \cdot \text{frequenciesD} @ 1 \text{DD} . \text{shapeFunc} \cdot \text{modeshape} \\ - p^2 \text{Sin}@p \text{ xD}$$

which we normalize as

$$\text{shape1} = c \text{Sin}@p \text{ xD} \cdot \text{SolveA} \hat{a}_0^1 c^2 \text{Sin}@p \text{ xD}^2 \hat{a} x == 1, \quad cE @ 2 \text{DD} \\ \cdot \frac{1}{2} \text{Sin}@p \text{ xD}$$

Substituting w_2 into **coefMat**, we determine the modeshape as

$$\text{NullSpace@coefMat} \cdot \text{IRuleN} \cdot w \rightarrow w_2 \cdot \text{frequenciesD} @ 1 \text{DD} . \text{shapeFunc} \cdot \text{IRuleN} \\ \text{Sin}@2 p \text{ xD}$$

which we also normalize as

$$\text{shape2} = c \text{Sin}@2 p \text{ xD} \cdot \text{SolveA} \hat{a}_0^1 c^2 \text{Sin}@2 p \text{ xD}^2 \hat{a} x == 1, \quad cE @ 2 \text{DD} \\ \cdot \frac{1}{2} \text{Sin}@2 p \text{ xD}$$

In a similar fashion, we find that the next three normalized modeshapes are

$$\begin{aligned} \text{shape3} &= \frac{1}{2} \text{Sin}@3 p \text{ xD}; \\ \text{shape4} &= \frac{1}{2} \text{Sin}@4 p \text{ xD}; \\ \text{shape5} &= \frac{1}{2} \text{Sin}@5 p \text{ xD}; \end{aligned}$$

To treat a one-to-one internal resonance between the first and second modes, we define f_k and the first two eigenmodes of the buckled beam as

$$\text{modeshapes} = \text{Join}@ \text{modeshape}, \\ 8F_1 \rightarrow \text{Function}@x, \text{shape1} \cdot \text{EvaluateD}, F_2 \rightarrow \text{Function}@x, \text{shape2} \cdot \text{EvaluateD} < \text{D};$$

8.1.2 Perturbation Analysis

We use the method of multiple scales to directly attack **EOM** and **BC1**. To transform the time derivatives in **EOM** in terms of the scales T_0 , T_1 , and T_2 , we let

```
multiScales = 8u@x_, tD -> u@x, T0, T1, T2D,
Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;
```

Then, we seek a second-order approximate solution in the form

```
solRule = u -> I EvaluateASum Ae^j u_j@#1, #2, #3, #4D, 8j, 3<EE &M
u @ He u_1@#1, #2, #3, #4D + e^2 u_2@#1, #2, #3, #4D + e^3 u_3@#1, #2, #3, #4D &L
```

where #1 stands for x and #2, #3, and #4 stand for T_0 , T_1 , and T_2 , respectively.

Substituting **multiScales**, **solRule**, and **scaling** into **EOM** and **BC1**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```
eq81c = HJoin@8EOM<, BC1D •. multiScales •. solRule •. scaling •. intRule1 ••. intRule2 ••
TrigToExp •• ExpandAllL •. e^n-; n>3 -> 0;
```

Equating coefficients of like powers of ϵ , we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq81c •• Transpose •• Rest;
eqEps •. displayRule

98P_k Hu_1''L + u_1'''' + D_0^2 u_1 - 2 b^2 a int@Hu_1' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0,
u_1 == 0, u_1' == 0, u_1 == 0, u_1' == 0<,
9- 2 b a int@Hu_1' L f_k^e @xD, 8x, 0, 1<D Hu_1''L + P_k Hu_2''L + u_2'''' + D_0^2 u_2 + 2 HD_0 D_1 u_1L - 1/2 b E^{-I T_0 W} f f_k^2 @xD -
1/2 b E^{I T_0 W} f f_k^2 @xD - b a int@Hu_1' L^2, 8x, 0, 1<F f_k^2 @xD -
2 b^2 a int@Hu_2' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0, u_2 == 0, u_2' == 0, u_2 == 0, u_2' == 0=,
9- 1/2 E^{-I T_0 W} f Hu_1' L - 1/2 E^{I T_0 W} f Hu_1' L - a int@Hu_1' L^2, 8x, 0, 1<F Hu_1' L -
2 b a int@Hu_2' L f_k^e @xD, 8x, 0, 1<D Hu_1''L - 2 b a int@Hu_1' L f_k^e @xD, 8x, 0, 1<D Hu_2''L + P_k Hu_3''L +
u_3'''' + 2 m HD_0 u_1L + D_0^2 u_3 + 2 HD_0 D_1 u_2L + D_1^2 u_1 + 2 HD_0 D_2 u_1L - 2 b a int@Hu_1' L Hu_2' L, 8x, 0, 1<D f_k^2 @xD -
2 b^2 a int@Hu_3' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0, u_3 == 0, u_3' == 0, u_3 == 0, u_3' == 0==
```

§ First-Order Solution

Because in the presence of damping all of the modes that are not directly or indirectly excited decay with time, the solution of **eqEps[[1]]** is taken to consist of the two modes that might be involved in the internal resonance; that is,

```
sol1 = 9u_1 -> FunctionA8x, T0, T1, T2<,
SumAA_i@T1, T2D F_i@xD Exp@I w_i T0D + A_i@T1, T2D F_i@xD Exp@-I w_i T0D, 8i, 2<E •• EvaluateE=;
```

where the F_i are the eigenmodes, which satisfy the orthonormality condition $\int_0^1 F_n(x) F_m(x) dx = \delta_{n,m}$.

Second-Order Solution

Substituting **sol1** into **eqEps[[2,1]]** and using **intRule1** and **intRule2** yields

order2Eq =

HeqEps@@1, 1, 1DD . u₁ -> u₂L == HeqEps@@1, 1, 1DD . u₁ -> u₂L - eqEps@@2, 1, 1DD . sol1 .

intRule1 . . . intRule2 . . ExpandAll; order2Eq . . displayRule

$$\begin{aligned}
 & P_k H u_2'' L + u_2'''' + D_0^2 u_2 - 2 b^2 a \int @ H u_2' L f_k^\epsilon @ x D, 8 x, 0, 1 < D f_k^2 @ x D == \\
 & - 2 I E^{I T_0 w_1} H D_1 A_1 L w_1 F_1 @ x D + 2 I E^{-I T_0 w_1} H D_1 \dot{A}_1 L w_1 F_1 @ x D - \\
 & 2 I E^{I T_0 w_2} H D_1 A_2 L w_2 F_2 @ x D + 2 I E^{-I T_0 w_2} H D_1 \dot{A}_2 L w_2 F_2 @ x D + \frac{1}{2} b E^{-I T_0 w} f f_k^2 @ x D + \\
 & \frac{1}{2} b E^{I T_0 w} f f_k^2 @ x D + b E^{2 I T_0 w_1} a \int @ F_1^\epsilon @ x D^2, 8 x, 0, 1 < D A_1^2 f_k^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 + w_2 L} a \int @ F_1^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_1 A_2 f_k^2 @ x D + \\
 & b E^{2 I T_0 w_2} a \int @ F_2^\epsilon @ x D^2, 8 x, 0, 1 < D A_2^2 f_k^2 @ x D + 2 b a \int @ F_1^\epsilon @ x D^2, 8 x, 0, 1 < D A_1 \dot{A}_1 f_k^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 + w_2 L} a \int @ F_1^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_2 \dot{A}_1 f_k^2 @ x D + \\
 & b E^{-2 I T_0 w_1} a \int @ F_1^\epsilon @ x D^2, 8 x, 0, 1 < D \dot{A}_1^2 f_k^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 - w_2 L} a \int @ F_1^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_1 \dot{A}_2 f_k^2 @ x D + 2 b a \int @ F_2^\epsilon @ x D^2, 8 x, 0, 1 < D A_2 \dot{A}_2 f_k^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 - w_2 L} a \int @ F_1^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D \dot{A}_1 \dot{A}_2 f_k^2 @ x D + \\
 & b E^{-2 I T_0 w_2} a \int @ F_2^\epsilon @ x D^2, 8 x, 0, 1 < D \dot{A}_2^2 f_k^2 @ x D + 2 b E^{2 I T_0 w_1} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D A_1^2 F_1^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 + w_2 L} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_1 A_2 F_1^2 @ x D + \\
 & 4 b a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D A_1 \dot{A}_1 F_1^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 + w_2 L} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_2 \dot{A}_1 F_1^2 @ x D + \\
 & 2 b E^{-2 I T_0 w_1} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D \dot{A}_1^2 F_1^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 - w_2 L} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_1 \dot{A}_2 F_1^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 - w_2 L} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D \dot{A}_1 \dot{A}_2 F_1^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 + w_2 L} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D A_1 A_2 F_2^2 @ x D + \\
 & 2 b E^{2 I T_0 w_2} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_2^2 F_2^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 + w_2 L} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D A_2 \dot{A}_1 F_2^2 @ x D + \\
 & 2 b E^{I T_0 H w_1 - w_2 L} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D A_1 \dot{A}_2 F_2^2 @ x D + \\
 & 4 b a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D A_2 \dot{A}_2 F_2^2 @ x D + \\
 & 2 b E^{I T_0 H - w_1 - w_2 L} a \int @ f_k^\epsilon @ x D F_1^\epsilon @ x D, 8 x, 0, 1 < D \dot{A}_1 \dot{A}_2 F_2^2 @ x D + \\
 & 2 b E^{-2 I T_0 w_2} a \int @ f_k^\epsilon @ x D F_2^\epsilon @ x D, 8 x, 0, 1 < D \dot{A}_2^2 F_2^2 @ x D
 \end{aligned}$$

To collect the terms that might lead to secular terms from the right-hand side of **order2Eq**, we define the rules:

OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D~Join~8W<D@@1DD & . # omgList

$$88 w_2 \otimes e^2 s_1 + w_1, W \otimes 2 e^2 s_1 + e^2 s_2 + 2 w_1 <, 8 w_1 \otimes - e^2 s_1 + w_2, W \otimes e^2 s_2 + 2 w_2 <<$$

expRule1@i_D := Exp@arg_D :> ExpAExpand@arg . . OmgRule@@iDDD . . e² T₀ -> T₂E

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_1 T_0}$, we have

```
ST1 = CoefficientAorder2Eq@@2DD •. expRule1@#D, EI w# T0 E & •ž 81, 2<;
ST1 •. displayRule
```

$$8 - 2 \int \text{HD}_1 A_1 L w_1 F_1 @x D - 2 \int E^{I T_2 s_1} \text{HD}_1 A_2 L w_2 F_2 @x D, - 2 \int E^{-I T_2 s_1} \text{HD}_1 A_1 L w_1 F_1 @x D - 2 \int \text{HD}_1 A_2 L w_2 F_2 @x D <$$

Because the homogeneous parts of **order2Eq** and corresponding boundary conditions have a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. These conditions demand that **ST1** be orthogonal to every solution of the adjoint homogeneous problem. In this case, the problem is self-adjoint and hence solutions of the adjoint problem are given by the $F_i @x D$. Demanding that **ST1[[i]]** be orthogonal to the $F_i @x D$, we obtain the solvability conditions

```
SCond1 = Table[A01 ST1@@kDD Fk@x D â x == 0, {k, 2}<E •. modeshapes
```

$$9 - 2 \int w_1 A_1^{H1, OL} @T_1, T_2 D == 0, - 2 \int w_2 A_2^{H1, OL} @T_1, T_2 D == 0 =$$

```
SCond1Rule = SolveASCond1, 9A1H1, OL @T1, T2D, A2H1, OL @T1, T2D=E@@1DD
```

$$9A_1^{H1, OL} @T_1, T_2 D @ 0, A_2^{H1, OL} @T_1, T_2 D @ 0 =$$

whose complexconjugate is

```
ccSCond1Rule = SCond1Rule •. conjugateRule
```

$$:A_1^{H1, OL} @T_1, T_2 D @ 0, A_2^{H1, OL} @T_1, T_2 D @ 0 >$$

Substituting **SCond1Rule** and **ccSCond1Rule** into **order2Eq** and using **modeshapes**, we have

```
order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule •. modeshapes •. int -> Integrate •.
```

```
Integrate -> int;
```

```
order2Eqm •. displayRule
```

$$\begin{aligned} & P_k H u_2'' L + u_2'''' + D_0^2 u_2 + 2 b^2 p^3 a \int \cos @p x D H u_2' L, 8x, 0, 1 < D \sin @p x D == \\ & - \frac{1}{2} b E^{-I T_0 w} f p^2 \sin @p x D - \frac{1}{2} b E^{I T_0 w} f p^2 \sin @p x D - 3 b E^{2 I T_0 w_1} p^4 a \sin @p x D A_1^2 - \\ & 8 b E^{I T_0 H w_1 + w_2 L} p^4 a \sin @2 p x D A_1 A_2 - 4 b E^{2 I T_0 w_2} p^4 a \sin @p x D A_2^2 - \\ & 6 b p^4 a \sin @p x D A_1 \dot{A}_1 - 8 b E^{I T_0 H - w_1 + w_2 L} p^4 a \sin @2 p x D A_2 \dot{A}_1 - \\ & 3 b E^{-2 I T_0 w_1} p^4 a \sin @p x D \dot{A}_1^2 - 8 b E^{I T_0 H w_1 - w_2 L} p^4 a \sin @2 p x D A_1 \dot{A}_2 - \\ & 8 b p^4 a \sin @p x D A_2 \dot{A}_2 - 8 b E^{I T_0 H - w_1 - w_2 L} p^4 a \sin @2 p x D \dot{A}_1 \dot{A}_2 - 4 b E^{-2 I T_0 w_2} p^4 a \sin @p x D \dot{A}_2^2 \end{aligned}$$

The associated boundary conditions are

```
order2BC = eqEps@@2DD •• Rest
```

$$\begin{aligned} & 9 u_2 @0, T_0, T_1, T_2 D == 0, u_2^{H2, 0, 0, OL} @0, T_0, T_1, T_2 D == 0, \\ & u_2 @1, T_0, T_1, T_2 D == 0, u_2^{H2, 0, 0, OL} @1, T_0, T_1, T_2 D == 0 = \end{aligned}$$

We use the method of undetermined coefficients to determine a particular solution of **order2Eqm** and **order2BC**. To accomplish this, we first determine the forms of the nonhomogeneous terms (i.e., the terms on the right-hand side of **order2Eqm**) as

```
rhsTerms = Cases@order2Eqm@@2DD, #D & . ž 9a_ f_@b_ xD :=> f@b xD .; FreeQ@a, T0D,
_ E^a_ T0+b_ . f_@c_ xD -> E^a_ T0+b_ f@c xD== .. Flatten .. Union
```

```
8Sin@p xD, E^-I T0 W Sin@p xD, E^I T0 W Sin@p xD, E^-2 I T0 w1 Sin@p xD,
E^2 I T0 w1 Sin@p xD, E^-2 I T0 w2 Sin@p xD, E^2 I T0 w2 Sin@p xD, E^-I T0 w1 -I T0 w2 Sin@2 p xD,
E^I T0 w1 -I T0 w2 Sin@2 p xD, E^-I T0 w1+I T0 w2 Sin@2 p xD, E^I T0 w1+I T0 w2 Sin@2 p xD<
```

which all satisfy the **order2BC**. Since we have only even spatial derivatives on the left-hand side of **order2Eqm**, we seek a particular solution as a linear combination of the **rhsTerms** as follows:

```
symbolList = Table@a_i, 8i, Length@rhsTermsD<D
```

```
8a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11<
```

```
sol2Form = u2 -> Function@8x, T0, T1, T2<, symbolList.rhsTerms .. Evaluated;
```

Substituting **sol2Form** into **order2Eqm**, equating coefficients of like terms, and solving for the **symbolList**, we have

```
symbolRule =
Solve@Coefficient@Subtract žž order2Eqm . sol2Form . intRule1 .. intRule2 . int ->
Integrate, rhsTermsD == 0 . Exp@_ T0 + _ .D -> 0 .. Thread, symbolListD@@1DD

: a1 ® - 
$$\frac{2 H^3 b p^2 a A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D + 4 b p^2 a A_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D L}{p^2 - P_k + b^2 p^2 a},$$

a2 ® - 
$$\frac{b f p^2}{2 H p^4 - p^2 P_k + b^2 p^4 a - W^2 L}, a3 ® - \frac{b f p^2}{2 H p^4 - p^2 P_k + b^2 p^4 a - W^2 L},$$

a4 ® - 
$$\frac{3 b p^4 a \dot{A}_1 @ T_1, T_2 D^2}{p^4 - p^2 P_k + b^2 p^4 a - 4 w_1^2}, a5 ® - \frac{3 b p^4 a A_1 @ T_1, T_2 D^2}{p^4 - p^2 P_k + b^2 p^4 a - 4 w_1^2},$$

a6 ® - 
$$\frac{4 b p^4 a \dot{A}_2 @ T_1, T_2 D^2}{p^4 - p^2 P_k + b^2 p^4 a - 4 w_2^2}, a7 ® - \frac{4 b p^4 a A_2 @ T_1, T_2 D^2}{p^4 - p^2 P_k + b^2 p^4 a - 4 w_2^2},$$

a8 ® - 
$$\frac{8 b p^4 a \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{16 p^4 - 4 p^2 P_k - w_1^2 - 2 w_1 w_2 - w_2^2}, a9 ® - \frac{8 b p^4 a A_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{16 p^4 - 4 p^2 P_k - w_1^2 + 2 w_1 w_2 - w_2^2},$$

a10 ® - 
$$\frac{8 b p^4 a A_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{16 p^4 - 4 p^2 P_k - w_1^2 + 2 w_1 w_2 - w_2^2}, a11 ® - \frac{8 b p^4 a \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{16 p^4 - 4 p^2 P_k - w_1^2 - 2 w_1 w_2 - w_2^2} >$$

```

Substituting for the parameter values in **symbolRule** yields


```
symbolRuleN = symbolRule /. W -> 2 w2 . frequencies . bucklingLoad . values1
```

$$\begin{aligned}
 a_1 &\otimes - \frac{6 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D + 8 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{6 p^2}, a_2 \otimes \frac{f}{12 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}}, \\
 a_3 &\otimes \frac{f}{12 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}}, a_4 \otimes \frac{\ddot{a}_1 @ T_1, T_2 D^2}{2 \cdot \frac{1}{3}}, a_5 \otimes \frac{\ddot{a}_1 @ T_1, T_2 D^2}{2 \cdot \frac{1}{3}}, a_6 \otimes \frac{2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D^2}{3 \cdot \frac{1}{3}}, \\
 a_7 &\otimes \frac{2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D^2}{3 \cdot \frac{1}{3}}, a_8 \otimes \frac{4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot \frac{1}{3}}, a_9 \otimes - \frac{4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot \frac{1}{3}}, \\
 a_{10} &\otimes - \frac{4 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3 \cdot \frac{1}{3}}, a_{11} \otimes \frac{4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot \frac{1}{3}} >
 \end{aligned}$$

Substituting **symbolRuleN** into **sol2Form** yields

```
sol2 = sol2Form . symbolRuleN
```

```
u2 @ FunctionB8x, T0, T1, T2<,
```

$$\begin{aligned}
 &\frac{\sin @ p \times D \mid - \mid 6 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D + 8 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{6 p^2} + \\
 &\frac{E^{-I T_0 W} \sin @ p \times D f}{12 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}} + \frac{E^{I T_0 W} \sin @ p \times D f}{12 \cdot \frac{1}{3} p^2 \cdot \frac{1}{3} \ddot{a}} + \frac{E^{-2 I T_0 w_1} \sin @ p \times D J \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D^2 N}{2 \cdot \frac{1}{3}} + \\
 &\frac{E^{2 I T_0 w_1} \sin @ p \times D \mid \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D^2 M}{2 \cdot \frac{1}{3}} + \frac{E^{-2 I T_0 w_2} \sin @ p \times D J 2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D^2 N}{3 \cdot \frac{1}{3}} + \\
 &\frac{E^{2 I T_0 w_2} \sin @ p \times D \mid 2 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D^2 M}{3 \cdot \frac{1}{3}} + \frac{E^{-I T_0 w_1 - I T_0 w_2} \sin @ 2 p \times D \mid 4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D M}{3 \cdot \frac{1}{3}} + \\
 &\frac{E^{I T_0 w_1 - I T_0 w_2} \sin @ 2 p \times D \mid - 4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D M}{3 \cdot \frac{1}{3}} + \\
 &\frac{E^{-I T_0 w_1 + I T_0 w_2} \sin @ 2 p \times D \mid - 4 \cdot \frac{1}{3} \ddot{a}_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D M}{3 \cdot \frac{1}{3}} + \\
 &\frac{E^{I T_0 w_1 + I T_0 w_2} \sin @ 2 p \times D \mid 4 \cdot \frac{1}{3} \ddot{a}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D M}{3 \cdot \frac{1}{3}} F
 \end{aligned}$$

Ÿ Solvability Conditions

Substituting **sol1** and **sol2** into **eqEps[[3,1]]** and using **intRule1** and **intRule2** yields

```
order3Eq = HeqEps@@1, 1, 1DD . u1 -> u3L ==
```

```

  HHeqEps@@1, 1, 1DD . u1 -> u3L - eqEps@@3, 1, 1DD . sol1 . sol2 . intRule1 . . . intRule2 . .
  modeshapes . int -> Integrate . . ExpandL; . . Timing

```

```
810.475 Second, Null<
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{i\omega_i T_0}$, we have

ST2 = CoefficientAorder3Eq@@2DD . expRule1@#D, E^{I w# T0}E & .ž 81, 2<;

ST2 . displayRule

$$\begin{aligned}
 &:- \frac{1}{2} \text{HD}_1^2 A_1 L \sin @ p \, x D - \frac{1}{2} E^{I T_2 s_1} \text{HD}_1^2 A_2 L \sin @ 2 p \, x D - \\
 &2 I \frac{1}{2} \text{HD}_2 A_1 L \sin @ p \, x D w_1 - 2 I \frac{1}{2} m \sin @ p \, x D A_1 w_1 - 2 I \frac{1}{2} E^{I T_2 s_1} \text{HD}_2 A_2 L \sin @ 2 p \, x D w_2 - \\
 &2 I \frac{1}{2} E^{I T_2 s_1} m \sin @ 2 p \, x D A_2 w_2 - \frac{E^{2 I T_2 s_1 + I T_2 s_2} f p^2 \sin @ p \, x D \dot{A}_1}{2} - \\
 &\frac{b E^{2 I T_2 s_1 + I T_2 s_2} f p^2 \sin @ p \, x D \dot{A}_1}{2} - 3 \frac{1}{2} p^4 a \sin @ p \, x D A_1^2 \dot{A}_1 + 5 \frac{3}{2} b p^4 a^{3 \cdot 2} \sin @ p \, x D A_1^2 \dot{A}_1 - \\
 &8 \frac{1}{2} E^{I T_2 s_1} p^4 a \sin @ 2 p \, x D A_1 A_2 \dot{A}_1 + \frac{68}{3} \frac{2}{3} b E^{I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_1 A_2 \dot{A}_1 - \\
 &4 \frac{1}{2} E^{2 I T_2 s_1} p^4 a \sin @ p \, x D A_2^2 \dot{A}_1 + 14 \frac{2}{3} b E^{2 I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ p \, x D A_2^2 \dot{A}_1 - \\
 &2 \frac{1}{2} E^{I T_2 s_1 + I T_2 s_2} f p^2 \sin @ 2 p \, x D \dot{A}_2 - \frac{1}{3} \frac{2}{3} b E^{I T_2 s_1 + I T_2 s_2} f p^2 \sin @ 2 p \, x D \dot{A}_2 - \\
 &4 \frac{1}{2} E^{-I T_2 s_1} p^4 a \sin @ 2 p \, x D A_1^2 \dot{A}_2 + 14 \frac{2}{3} b E^{-I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_1^2 \dot{A}_2 - \\
 &8 \frac{1}{2} p^4 a \sin @ p \, x D A_1 A_2 \dot{A}_2 + \frac{68}{3} \frac{2}{3} b p^4 a^{3 \cdot 2} \sin @ p \, x D A_1 A_2 \dot{A}_2 - \\
 &48 \frac{1}{2} E^{I T_2 s_1} p^4 a \sin @ 2 p \, x D A_2^2 \dot{A}_2 + \frac{40}{3} \frac{2}{3} b E^{I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_2^2 \dot{A}_2, \\
 &- \frac{1}{2} E^{-I T_2 s_1} \text{HD}_1^2 A_1 L \sin @ p \, x D - \frac{1}{2} \text{HD}_1^2 A_2 L \sin @ 2 p \, x D - 2 I \frac{1}{2} E^{-I T_2 s_1} \text{HD}_2 A_1 L \sin @ p \, x D w_1 - \\
 &2 I \frac{1}{2} E^{-I T_2 s_1} m \sin @ p \, x D A_1 w_1 - 2 I \frac{1}{2} \text{HD}_2 A_2 L \sin @ 2 p \, x D w_2 - 2 I \frac{1}{2} m \sin @ 2 p \, x D A_2 w_2 - \\
 &\frac{E^{I T_2 s_1 + I T_2 s_2} f p^2 \sin @ p \, x D \dot{A}_1}{2} - \frac{b E^{I T_2 s_1 + I T_2 s_2} f p^2 \sin @ p \, x D \dot{A}_1}{2} - \\
 &3 \frac{1}{2} E^{-I T_2 s_1} p^4 a \sin @ p \, x D A_1^2 \dot{A}_1 + 5 \frac{3}{2} b E^{-I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ p \, x D A_1^2 \dot{A}_1 - \\
 &8 \frac{1}{2} p^4 a \sin @ 2 p \, x D A_1 A_2 \dot{A}_1 + \frac{68}{3} \frac{2}{3} b p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_1 A_2 \dot{A}_1 - \\
 &4 \frac{1}{2} E^{I T_2 s_1} p^4 a \sin @ p \, x D A_2^2 \dot{A}_1 + 14 \frac{2}{3} b E^{I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ p \, x D A_2^2 \dot{A}_1 - \\
 &2 \frac{1}{2} E^{I T_2 s_2} f p^2 \sin @ 2 p \, x D \dot{A}_2 - \frac{1}{3} \frac{2}{3} b E^{I T_2 s_2} f p^2 \sin @ 2 p \, x D \dot{A}_2 - \\
 &4 \frac{1}{2} E^{-2 I T_2 s_1} p^4 a \sin @ 2 p \, x D A_1^2 \dot{A}_2 + 14 \frac{2}{3} b E^{-2 I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_1^2 \dot{A}_2 - \\
 &8 \frac{1}{2} E^{-I T_2 s_1} p^4 a \sin @ p \, x D A_1 A_2 \dot{A}_2 + \frac{68}{3} \frac{2}{3} b E^{-I T_2 s_1} p^4 a^{3 \cdot 2} \sin @ p \, x D A_1 A_2 \dot{A}_2 - \\
 &48 \frac{1}{2} p^4 a \sin @ 2 p \, x D A_2^2 \dot{A}_2 + \frac{40}{3} \frac{2}{3} b p^4 a^{3 \cdot 2} \sin @ 2 p \, x D A_2^2 \dot{A}_2 >
 \end{aligned}$$

Demanding that $\mathbf{F}_i(\mathbf{x})$ be orthogonal to the $\mathbf{F}_i(\mathbf{x})$, we obtain the solvability conditions

SCond2 =

```
Table@int@ST2@@kDD F_k@x, 8x, 0, 1<D == 0, 8k, 2<D . intRule1 . . intRule2 . . modeshapes . .
  values1 . . int -> Integrate . . D@SCond1Rule, T1D;
```

SCond2 . . displayRule

$$9 - 2 \text{I} \text{HD}_2 \text{A}_1 \text{L} \text{w}_1 - 2 \text{I} \text{m} \text{A}_1 \text{w}_1 - \text{E}^{2 \text{I} \text{T}_2 \text{s}_1 + \text{I} \text{T}_2 \text{s}_2} \text{f} \text{p}^2 \dot{\text{A}}_1 + 12 \text{p}^4 \text{a} \text{A}_1^2 \dot{\text{A}}_1 + \\ 24 \text{E}^{2 \text{I} \text{T}_2 \text{s}_1} \text{p}^4 \text{a} \text{A}_2^2 \dot{\text{A}}_1 + \frac{112}{3} \text{p}^4 \text{a} \text{A}_1 \text{A}_2 \dot{\text{A}}_2 == 0, -2 \text{I} \text{HD}_2 \text{A}_2 \text{L} \text{w}_2 - 2 \text{I} \text{m} \text{A}_2 \text{w}_2 + \\ \frac{112}{3} \text{p}^4 \text{a} \text{A}_1 \text{A}_2 \dot{\text{A}}_1 - \frac{8}{3} \text{E}^{\text{I} \text{T}_2 \text{s}_2} \text{f} \text{p}^2 \dot{\text{A}}_2 + 24 \text{E}^{-2 \text{I} \text{T}_2 \text{s}_1} \text{p}^4 \text{a} \text{A}_1^2 \dot{\text{A}}_2 - \frac{64}{3} \text{p}^4 \text{a} \text{A}_2^2 \dot{\text{A}}_2 == 0 =$$

§ 8.1.3 The Function MMSDirect11

Collecting the commands in the preceding section, we build a function **MMSDirect11**, specifically for the nonlinear transverse vibrations of a hinged-hinged beam around its first buckled mode (**EOM** and **BC1**), to determine the solvability conditions for different combinations of **frequencies**, **modeshapes**, and **ResonanceConds** in which the first mode might be involved in either a one-to-one or a three-to-one internal resonance with one of the other modes.

```
MMSDirect11@frequencies_, modeshapes_, ResonanceConds_D :=
  ModuleA8<,
    multiScales = 8u@x_, tD -> u@x, T0, T1, T2D,
    Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;
    solRule = u -> IEvaluateASumAe^j u_j@#1, #2, #3, #4D, 8j, 3<EE &M;
    scaling = 9c -> 2 e^2 m, f -> e^2 f=;
    eqs = HJoin@8EOM<, BC1D . multiScales . solRule . scaling . intRule1 . . intRule2 . .
      TrigToExp . . ExpandAllL . . e^N_ ; n>3 -> 0;
    eqEps = Thread@CoefficientList@Subtract z z #, eD == 0D & . z eqs . Transpose . . Rest;
    sol1 = 9u1 -> FunctionA8x, T0, T1, T2<,
      SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ai@T1, T2D Fi@xD Exp@-I wi T0D, 8i, 2<E . . EvaluateE=;
    order2Eq = HeqEps@@1, 1, 1DD . u1 -> u2L == HeqEps@@1, 1, 1DD . u1 -> u2L - eqEps@@2, 1, 1DD .
      sol1 . intRule1 . . intRule2 . . ExpandAll;
    omgList = 8w1, w2<;
  OmgRule =
    Solve@ResonanceConds, Complement@omgList, 8#<D ~Join~ 8W<D@@1DD & . z omgList;
    expRule1@i_D := Exp@arg_D :> ExpAExpand@arg . OmgRule@@iDDD . e^2 T0 -> T2E;
    ST1 = CoefficientAorder2Eq@@2DD . expRule1@#D, E^I w# T0 E & . z 81, 2<;
    SCond1 = TableAa^1 ST1@@kDD F_k@xD a x == 0, 8k, 2<E . . modeshapes;
    SCond1Rule = SolveASCond1, 9A1^H1, 0L@T1, T2D, A2^H1, 0L@T1, T2D=E@@1DD;
    ccSCond1Rule = SCond1Rule . conjugateRule;
    order2Eqm = order2Eq . SCond1Rule . ccSCond1Rule . modeshapes . int -> Integrate . .
      Integrate -> int;
```

```

rhsTerms = Cases@order2Eqm@@2DD, #D & . ž 9a_ f_@b_ xD := f@b xD .; FreeQ@a, T0D,
  _ E^a_ T0+b_ . f_@c_ xD -> E^a_ T0+b_ f@c xD == .. Flatten .. Union;
symbolList = Table@a_i, 8i, Length@rhsTermsD<D;
sol2Form = u2 -> Function@8x, T0, T1, T2<, symbolList.rhsTerms .. Evaluated;
symbolRule =
  Solve@Coefficient@Subtract žž order2Eqm . sol2Form . intRule1 .. intRule2 .
    int -> Integrate, rhsTermsD == 0 . Exp@_ T0 + _ .D -> 0 .. Thread, symbolListD@@1DD;
  values1 = Solve@Select@ResonanceConds, FreeQ@#, WD &D . e -> 0 . frequencies, bD ..
  Flatten;
symbolRuleN = symbolRule . W -> 2 w2 . frequencies . bucklingLoad . values1;
sol2 = sol2Form . symbolRuleN;
order3Eq = HeqEps@@1, 1, 1DD . u1 -> u3L ==
  HHeqEps@@1, 1, 1DD . u1 -> u3L - eqEps@@3, 1, 1DD . sol1 . sol2 . intRule1 .. intRule2 .
    modeshapes . int -> Integrate .. ExpandL;
ST2 = CoefficientAorder3Eq@@2DD . expRule1@#D, E^I w# T0 E & . ž 81, 2<;
SCond2 =
  Table@int@ST2@@kDD F_k@xD, 8x, 0, 1<D == 0, 8k, 2<D . intRule1 .. intRule2 . modeshapes .
    values1 . int -> Integrate . D@SCond1Rule, T1D;
  Return@SCond2 . A_i_ -> HA_i@#2D &L . f_@T1, T2D -> f@T2DD
E

```

§ 8.1.4 Three-to-One Internal Resonances Between the First Two Modes

In this section, we consider the case $w_2 \gg 3w_1$ and $W \gg 2w_2$. To describe the nearness of the resonances, we introduce the detuning parameters s_i defined by

$$\text{ResonanceConds1} = 9w_2 == 3w_1 + e^2 s_1, \quad W == 2w_2 + e^2 s_2;$$

Using **MMSDirect11**, we obtain the solvability conditions

MMSDirect11@frequencies, modeshapes, ResonanceConds1D .. Timing

921.501 Second,

$$\begin{aligned}
 & 9 - 2 I m w_1 A_1 @ T_2 D + 12 p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D + \frac{624}{35} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 I w_1 A_1^\epsilon @ T_2 D == 0, \\
 & - 2 I m w_2 A_2 @ T_2 D + \frac{624}{35} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - \\
 & \frac{72}{35} E^{I T_2 s_2} f p^2 \dot{A}_2 @ T_2 D - \frac{576}{35} p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 I w_2 A_2^\epsilon @ T_2 D == 0 ==
 \end{aligned}$$

which indicates that this three-to-one internal resonance is not activated.

Next, we consider the case in which the frequency of the first mode is three times that of the second mode; that is, the buckling load is above the first crossover value. To accomplish this, we redefine the frequencies and denote the frequency of the first mode by w_2 and that of the second mode by w_1 as follows:

```
frequencies2 = 8w2 -> Hw . wSol1L, w1 -> Hw . wSol2 . n -> 2L<
```

```
9w2 @ b p^2 . a^111, w1 @ 2 . a^111 p^2=
```

Moreover, we redefine the modeshapes according to

```
modeshapes2 = Join@modeshape,
8F1 -> Function@x, shape2 . . Evaluated, F2 -> Function@x, shape1 . . Evaluated<D;
```

To describe the nearness of the resonances, we introduce the two detuning parameters S_i defined by

```
ResonanceConds2 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies2, modeshapes2, ResonanceConds2D . . Timing
```

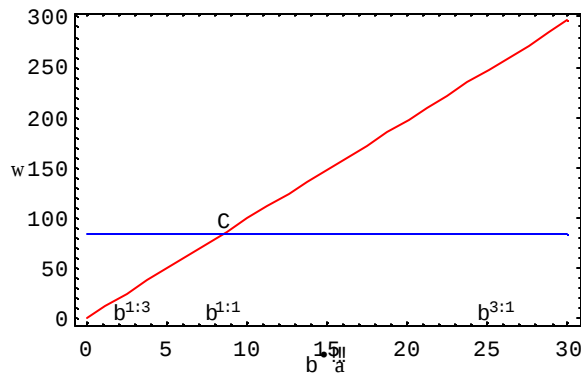
```
921.721 Second,
```

```
9 - 2 I m w1 A1@T2D + 64 p^4 a A1@T2D^2 A1@T2D - 496 p^4 a A1@T2D A2@T2D A2@T2D A2@T2D - 2 I w1 A1@T2D == 0,
- 2 I m w2 A2@T2D - 496 p^4 a A1@T2D A2@T2D A1@T2D -
E^I T2 s2 f p^2 A2@T2D + 12 p^4 a A2@T2D^2 A2@T2D - 2 I w2 A2@T2D == 0==
```

Again, this three-to-one internal resonance is not activated.

8.1.5 One-to-One and Three-to-One Internal Resonances Between the First and Third Modes

Variations of the first and third natural frequencies with the buckling level are plotted as follows:



It follows from this figure that the frequencies of the first and third modes are equal when $b \cdot a^111 = b^{1:1}$, the frequency of the third mode is three times that of the first mode when $b \cdot a^111 = b^{1:3}$, and the frequency of the first mode is three times that of the third mode when $b \cdot a^111 = b^{3:1}$. Next, we use the function **MMSDirect11** to treat these three cases.

¶ One-to-One Internal Resonance

We denote the frequency of the third mode by w_2 and that of the first mode by w_1 as follows:

$$\text{frequencies3} = \text{8w}_1 \rightarrow \text{Hw} \cdot \text{wSol1L}, \text{w}_2 \rightarrow \text{Hw} \cdot \text{wSol2} \cdot \text{n} \rightarrow \text{3L} < \\ 9w_1 \otimes b p^2 \cdot \frac{!!}{a}, w_2 \otimes 6 \cdot \frac{!!}{2} p^2 =$$

Moreover, we define f_k and the first and third eigenmodes of the buckled beam as

$$\text{modeshapes3} = \text{Join@modeshape}, \\ \text{8F}_1 \rightarrow \text{Function@x}, \text{shape1} \cdot \cdot \text{Evaluated}, \text{F}_2 \rightarrow \text{Function@x}, \text{shape3} \cdot \cdot \text{Evaluated} < \text{D};$$

We consider a combination of a one-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

$$\text{ResonanceConds3} = 9w_2 == w_1 + e^2 s_1, W == 2 w_2 + e^2 s_2 =;$$

Using **MMSDirect11**, we obtain the solvability conditions

$$\text{MMSDirect11@frequencies3, modeshapes3, ResonanceConds3D} \cdot \cdot \text{Timing} \\ 826.809 \text{ Second}, \\ 8 - 2 \text{ I m } w_1 A_1 @ T_2 D - E^{2 \text{ I } T_2 s_1 + \text{I } T_2 s_2} f p^2 \dot{A}_1 @ T_2 D + 12 p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D + 144 E^{2 \text{ I } T_2 s_1} p^4 a A_2 @ T_2 D^2 \dot{A}_1 @ T_2 D + \\ 144 p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 \text{ I } w_1 A_1^c @ T_2 D == 0, \\ - 2 \text{ I m } w_2 A_2 @ T_2 D + 144 p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - 6 E^{\text{I } T_2 s_2} f p^2 \dot{A}_2 @ T_2 D + \\ 144 E^{-2 \text{ I } T_2 s_1} p^4 a A_1 @ T_2 D^2 \dot{A}_2 @ T_2 D - 108 p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 \text{ I } w_2 A_2^c @ T_2 D == 0 <<$$

which indicates that the one-to-one internal resonance between the first and third modes is activated.

¶ Three-to-One Internal Resonances

To treat the case in which the natural frequency of the third mode is approximately three times that of the first mode using the function **MMSDirect11**, we denote these frequencies by w_2 and w_1 , respectively, according to

$$\text{frequencies4} = \text{8w}_1 \rightarrow \text{Hw} \cdot \text{wSol1L}, \text{w}_2 \rightarrow \text{Hw} \cdot \text{wSol2} \cdot \text{n} \rightarrow \text{3L} < \\ 9w_1 \otimes b p^2 \cdot \frac{!!}{a}, w_2 \otimes 6 \cdot \frac{!!}{2} p^2 =$$

Moreover, we define f_k and the first and third eigenmodes of the buckled beam as

$$\text{modeshapes4} = \text{Join@modeshape}, \\ \text{8F}_1 \rightarrow \text{Function@x}, \text{shape1} \cdot \cdot \text{Evaluated}, \text{F}_2 \rightarrow \text{Function@x}, \text{shape3} \cdot \cdot \text{Evaluated} < \text{D};$$

We consider a combination of a three-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

$$\text{ResonanceConds4} = 9w_2 == 3 w_1 + e^2 s_1, W == 2 w_2 + e^2 s_2 =;$$

Using **MMSDirect11**, we obtain the solvability conditions

MMSDirect11@frequencies4, modeshapes4, ResonanceConds4D •• Timing

921.651 Second,

$$\begin{aligned} 9 - 2 I m w_1 A_1 @ T_2 D + 12 p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D + \frac{1584}{35} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 I w_1 A_1 @ T_2 D & == 0, \\ - 2 I m w_2 A_2 @ T_2 D + \frac{1584}{35} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - \\ \frac{162}{35} E^{I T_2 s_2} f p^2 \dot{A}_2 @ T_2 D - \frac{2916}{35} p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 I w_2 A_2 @ T_2 D & == 0 == \end{aligned}$$

which indicates that this three-to-one internal resonance between the first and third modes is not activated.

When the frequency of first mode is approximately three times that of the third mode, we define the frequencies as

frequencies5 = 8w₂ -> Hw •. wSol1L, w₁ -> Hw •. wSol2 •. n -> 3L<

$$9w_2 @ b p^2 \cdot \frac{1}{a}, w_1 @ 6 \cdot \frac{1}{2} p^2 =$$

Moreover, we define f_k and the third and first eigenmodes of the buckled beam as

modeshapes5 = Join@modeshape,

8F₁ -> Function@x, shape3 •• EvaluateD, F₂ -> Function@x, shape1 •• EvaluateD<D;

We consider a combination of a three-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

$$\text{ResonanceConds5} = 9w_2 == 3w_1 + e^2 s_1, W == 2w_2 + e^2 s_2;$$

Using **MMSDirect11**, we obtain the solvability conditions

MMSDirect11@frequencies5, modeshapes5, ResonanceConds5D •• Timing

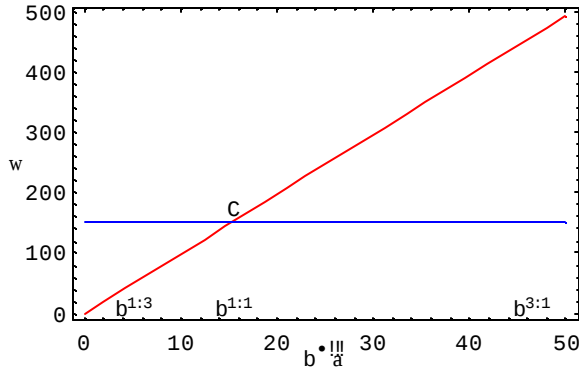
921.712 Second,

$$\begin{aligned} 9 - 2 I m w_1 A_1 @ T_2 D + \frac{324}{5} p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D - \frac{2736}{5} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 I w_1 A_1 @ T_2 D & == 0, \\ - 2 I m w_2 A_2 @ T_2 D - \frac{2736}{5} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - \\ E^{I T_2 s_2} f p^2 \dot{A}_2 @ T_2 D + 12 p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 I w_2 A_2 @ T_2 D & == 0 == \end{aligned}$$

which indicates that this three-to-one internal resonance between the first and third modes is not activated.

§ 8.1.6 One-to-One and Three-to-One Internal Resonances Between the First and Fourth Modes

Variations of the first and fourth natural frequencies with the buckling level are plotted as follows:



Again, there exist buckling levels for which the frequencies of the first and fourth modes are in the ratio of 1:1, 3:1, and 1:3. We use the function **MMSDirect11** to treat these cases.

One-to-One Internal Resonance

When the frequencies of the first and fourth modes are approximately equal, we denote them by w_1 and w_2 , respectively, according to

frequencies6 = 8w₁ -> Hw • wSol1L, w₂ -> Hw • wSol2 • n -> 4L<

$$9w_1 \otimes b p^2 \cdot \frac{1}{a}, w_2 \otimes 4 \cdot \frac{1}{15} p^2 =$$

Moreover, we define f_k and the first and fourth eigenmodes of the buckled beam as

modeshapes6 = Join@modeshape,
8F₁ -> Function@x, shape1 • EvaluateD, F₂ -> Function@x, shape4 • EvaluateD<D;

We consider a combination of a one-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

$$\text{ResonanceConds6} = 9w_2 == w_1 + \epsilon^2 s_1, W == 2w_2 + \epsilon^2 s_2;$$

Using **MMSDirect11**, we obtain the solvability conditions

MMSDirect11@frequencies6, modeshapes6, ResonanceConds6D • Timing

929.522 Second,

$$\begin{aligned} & 9 - 2 \operatorname{Im} w_1 A_1 @ T_2 D - E^{2 \operatorname{Im} T_2 s_1 + \operatorname{Im} T_2 s_2} f p^2 \dot{A}_1 @ T_2 D + 12 p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D + 480 E^{2 \operatorname{Im} T_2 s_1} p^4 a A_2 @ T_2 D^2 \dot{A}_1 @ T_2 D + \\ & \frac{1216}{3} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 \operatorname{Im} w_1 A_1 @ T_2 D == 0, \\ & - 2 \operatorname{Im} w_2 A_2 @ T_2 D + \frac{1216}{3} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - \frac{32}{3} E^{\operatorname{Im} T_2 s_2} f p^2 \dot{A}_2 @ T_2 D + \\ & 480 E^{-2 \operatorname{Im} T_2 s_1} p^4 a A_1 @ T_2 D^2 \dot{A}_2 @ T_2 D - \frac{1024}{3} p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 \operatorname{Im} w_2 A_2 @ T_2 D == 0 == \end{aligned}$$

which indicates that this one-to-one internal resonance between the first and fourth modes is activated.

§ Three-to-One Internal Resonances

When the frequency of the fourth mode is approximately three times that of the first mode, we denote these frequencies by ω_2 and ω_1 , respectively, as

$$\text{frequencies7} = 8\omega_1 \rightarrow \text{Hw} \cdot \text{wSol1L}, \omega_2 \rightarrow \text{Hw} \cdot \text{wSol2} \cdot n \rightarrow 4L<$$

$$9\omega_1 \otimes b p^2 \cdot \frac{!!!}{a}, \omega_2 \otimes 4 \cdot \frac{!!!!}{15} p^2 =$$

Moreover, we define f_k and the first and fourth eigenmodes of the buckled beam as

$$\text{modeshapes7} = \text{Join}@\text{modeshape},$$

$$8F_1 \rightarrow \text{Function}@x, \text{shape1} \cdot \cdot \text{EvaluatedD}, F_2 \rightarrow \text{Function}@x, \text{shape4} \cdot \cdot \text{EvaluatedD}<D;$$

We consider a combination of a three-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

$$\text{ResonanceConds7} = 9\omega_2 == 3\omega_1 + e^2 s_1, W == 2\omega_2 + e^2 s_2=;$$

Using **MMSDirect11**, we obtain the solvability conditions

$$\text{MMSDirect11}@\text{frequencies7}, \text{modeshapes7}, \text{ResonanceConds7D} \cdot \cdot \text{Timing}$$

922.653 Second,

$$\begin{aligned} & 9 - 2 \operatorname{Im} \omega_1 A_1 @T_2 D + 12 p^4 a A_1 @T_2 D^2 \dot{A}_1 @T_2 D + \frac{3264}{35} p^4 a A_1 @T_2 D A_2 @T_2 D \dot{A}_2 @T_2 D - 2 \operatorname{Im} \omega_1 A_1^c @T_2 D == 0, \\ & - 2 \operatorname{Im} \omega_2 A_2 @T_2 D + \frac{3264}{35} p^4 a A_1 @T_2 D A_2 @T_2 D \dot{A}_1 @T_2 D - \\ & \frac{288}{35} E^I T_2 s_2 f p^2 \dot{A}_2 @T_2 D - \frac{9216}{35} p^4 a A_2 @T_2 D^2 \dot{A}_2 @T_2 D - 2 \operatorname{Im} \omega_2 A_2^c @T_2 D == 0 == \end{aligned}$$

which indicates that this three-to-one internal resonance between the first and fourth modes is not activated.

When the frequency of the first mode is approximately three times that of the fourth mode, we denote these frequencies by ω_2 and ω_1 , respectively, as

$$\text{frequencies8} = 8\omega_2 \rightarrow \text{Hw} \cdot \text{wSol1L}, \omega_1 \rightarrow \text{Hw} \cdot \text{wSol2} \cdot n \rightarrow 4L<$$

$$9\omega_2 \otimes b p^2 \cdot \frac{!!!}{a}, \omega_1 \otimes 4 \cdot \frac{!!!!}{15} p^2 =$$

Moreover, we define f_k and the fourth and first eigenmodes of the buckled beam as

$$\text{modeshapes8} = \text{Join}@\text{modeshape},$$

$$8F_1 \rightarrow \text{Function}@x, \text{shape4} \cdot \cdot \text{EvaluatedD}, F_2 \rightarrow \text{Function}@x, \text{shape1} \cdot \cdot \text{EvaluatedD}<D;$$

We consider a combination of a three-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

$$\text{ResonanceConds8} = 9\omega_2 == 3\omega_1 + e^2 s_1, W == 2\omega_2 + e^2 s_2=;$$

Using **MMSDirect11**, we obtain the solvability conditions

MMSDirect11@frequencies8, modeshapes8, ResonanceConds8D •• Timing

922.072 Second,

$$\begin{aligned} & 9 - 2 \operatorname{Im} w_1 A_1 @ T_2 D + \frac{1024}{5} p^4 a A_1 @ T_2 D^2 \dot{A}_1 @ T_2 D - \frac{8896}{5} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_2 @ T_2 D - 2 \operatorname{Im} w_1 A_1 @ T_2 D == 0, \\ & - 2 \operatorname{Im} w_2 A_2 @ T_2 D - \frac{8896}{5} p^4 a A_1 @ T_2 D A_2 @ T_2 D \dot{A}_1 @ T_2 D - \\ & E^{T_2 s_2} f p^2 \dot{A}_2 @ T_2 D + 12 p^4 a A_2 @ T_2 D^2 \dot{A}_2 @ T_2 D - 2 \operatorname{Im} w_2 A_2 @ T_2 D == 0 == \end{aligned}$$

which indicates again that this three-to-one internal resonance between the first and fourth modes is not activated.

à 8.2 Circular Cylindrical Shells

Following McIvor (1966) and Raouf and Nayfeh (1990), we write the equations of motion governing the dynamic response of an infinitely long cylindrical shell to a primary-resonance excitation of one of its two orthogonal flexural modes as

$$\begin{aligned} \text{EOM} = & 9w - y_q + w_{t,t} + a^2 Hw + 2w_{q,q} + w_{q,q,q} L = \\ & - \frac{w_q^2}{2} - w w_q^2 - y_t^2 + w y_t^2 - 2w y_q + \frac{1}{2} w_q^2 y_q + y_q^2 - w y_q^2 + f e^3 \cos n q D \cos W t D H_1 - w + y_q L - \\ & w w_{q,q} - w^2 w_{q,q} + \frac{3}{2} w_q^2 w_{q,q} + y_q w_{q,q} + w y_q w_{q,q} - y_q^2 w_{q,q} + w_q y_{q,q} + w w_q y_{q,q} - \\ & 2w_q y_q y_{q,q} + a^2 \left\{ \begin{array}{l} - 3w^2 - \frac{11 w_q^2}{2} - 11 w w_{q,q} + 4 y_q w_{q,q} - 6 w_{q,q}^2 + 4 w_q y_{q,q} - 8 w_q w_{q,q,q} + \\ 8 y_{q,q} w_{q,q,q} + w y_{q,q,q} + 5 w_{q,q} y_{q,q,q} - 4 w w_{q,q,q,q} + 4 y_q w_{q,q,q,q} + w_q y_{q,q,q,q} \end{array} \right\} \\ & w_q + y_{t,t} - y_{q,q} = 2w w_q + \frac{w^3}{2} + 2w_t y_t - 2w w_t y_t - 2w_q y_q + 2w w_q y_q + w_q w_{q,q} + \\ & w w_q w_{q,q} - 2w_q y_q w_{q,q} + 2w y_{t,t} - w^2 y_{t,t} - 2w y_{q,q} + w^2 y_{q,q} - w_q^2 y_{q,q} + \\ & a^2 Hw_q w_{q,q} - w w_{q,q,q} - w_{q,q} w_{q,q,q} + w_q w_{q,q,q,q} L = \bullet. \quad 8w \rightarrow w @ q, t D, Hs : w \in y L_m : > q_m s @ q, t D <; \end{aligned}$$

We seek a second-order uniform asymptotic expansion of the solution of **EOM** in the form

$$\begin{aligned} \text{solRule} = & 9w \rightarrow \text{IEvaluateASumAe}^j w_j @ \#1, \#2, \#3, \#4 D, 8j, 3 < EE \& M, \\ y \rightarrow & \text{IEvaluateASumAe}^j y_j @ \#1, \#2, \#3, \#4 D, 8j, 3 < EE \& M = \\ 8w \& He & w_1 @ \#1, \#2, \#3, \#4 D + e^2 w_2 @ \#1, \#2, \#3, \#4 D + e^3 w_3 @ \#1, \#2, \#3, \#4 D \& L, \\ y \& He & y_1 @ \#1, \#2, \#3, \#4 D + e^2 y_2 @ \#1, \#2, \#3, \#4 D + e^3 y_3 @ \#1, \#2, \#3, \#4 D \& L < \end{aligned}$$

where #1 stands for q and #2, #3, and #4 stand for T_0 , T_1 , and T_2 , respectively. Transforming the time derivatives in **EOM** in terms of the scales T_0 , T_1 , and T_2 , substituting **solRule** into **EOM**, expanding the result for small e , and discarding terms of order higher than e^3 , we obtain

```
eq82a = HEOM •. 8w@x_, tD -> w@x, T0, T1, T2D, y@x_, tD -> y@x, T0, T1, T2D,
Derivative@m_, n_D@w_D@x_, tD -> dt@nD@D@w@x, T0, T1, T2D, 8x, m<DD,
t -> T0< •. solRule •• ExpandAllL •. e^n_*; n>3 -> 0;
```

Equating coefficients of like powers of t , we have

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq82a •• Transpose;
eqEps •. displayRule
```

```
982 a^2 Hw1'' L + a^2 Hw1'''' L - y1' + D0^2 w1 + w1 + a^2 w1 == 0, w1' - y1'' + D0^2 y1 == 0<,
9  $\frac{1}{2}$  Hw1' L^2 +  $\frac{11}{2}$  a^2 Hw1' L^2 + 6 a^2 Hw1' L^2 + 8 a^2 Hw1' L Hw1'' L + 2 a^2 Hw2' L + a^2 Hw2'' L -
Hw1' L Hy1' L - 4 a^2 Hw1' L Hy1' L - 4 a^2 Hw1'' L Hy1' L - Hy1' L^2 - Hw1' L Hy1'' L - 4 a^2 Hw1' L Hy1'' L -
8 a^2 Hw1'' L Hy1'' L - 5 a^2 Hw1' L Hy1'' L - a^2 Hw1' L Hy1'''' L - y2' + HD0 y1 L^2 + D0^2 w2 + 2 HD0 D1 w1 L +
Hw1' L w1 + 11 a^2 Hw1' L w1 + 4 a^2 Hw1'' L w1 + 2 Hy1' L w1 - a^2 Hy1'' L w1 + 3 a^2 w1^2 + w2 + a^2 w2 == 0,
-Hw1' L Hw1' L - a^2 Hw1' L Hw1' L + a^2 Hw1' L Hw1'' L - a^2 Hw1' L Hw1'''' L + w2' + 2 Hw1' L Hy1' L - y2'' -
2 HD0 w1 L HD0 y1 L + D0^2 y2 + 2 HD0 D1 y1 L - 2 Hw1' L w1 + a^2 Hw1'' L w1 + 2 Hy1' L w1 - 2 HD0^2 y1 L w1 == 0=,
9 - f Cos@n qD Cos@T0 WD -  $\frac{3}{2}$  Hw1' L^2 Hw1' L + Hw1' L Hw2' L + 11 a^2 Hw1' L Hw2' L + 8 a^2 Hw1'' L Hw2' L +
12 a^2 Hw1' L Hw2' L + 8 a^2 Hw1' L Hw2'' L + 2 a^2 Hw3' L + a^2 Hw3'' L -  $\frac{1}{2}$  Hw1' L^2 Hy1' L - Hw2' L Hy1' L -
4 a^2 Hw2' L Hy1' L - 4 a^2 Hw2'' L Hy1' L + Hw1' L Hy1' L^2 - Hw2' L Hy1'' L - 4 a^2 Hw2' L Hy1'' L - 8 a^2 Hw2'' L Hy1'' L +
2 Hw1' L Hy1' L Hy1' L - 5 a^2 Hw2' L Hy1'' L - a^2 Hw2' L Hy1'''' L - Hw1' L Hy2' L - 4 a^2 Hw1' L Hy2' L -
4 a^2 Hw1'' L Hy2' L - 2 Hy1' L Hy2' L - Hw1' L Hy2'' L - 4 a^2 Hw1' L Hy2'' L - 8 a^2 Hw1'' L Hy2'' L -
5 a^2 Hw1' L Hy2'' L - a^2 Hw1' L Hy2'''' L - y3' + 2 HD0 y1 L HD0 y2 L + D0^2 w3 + 2 HD0 y1 L HD1 y1 L +
2 HD0 D1 w2 L + D1^2 w1 + 2 HD0 D2 w1 L + Hw1' L^2 w1 + Hw2' L w1 + 11 a^2 Hw2' L w1 + 4 a^2 Hw2'' L w1 -
Hw1' L Hy1' L w1 + Hy1' L^2 w1 - Hw1' L Hy1'' L w1 + 2 Hy2' L w1 - a^2 Hy2'' L w1 - HD0 y1 L^2 w1 + Hw1' L w1^2 +
Hw1' L w2 + 11 a^2 Hw1' L w2 + 4 a^2 Hw1'' L w2 + 2 Hy1' L w2 - a^2 Hy1'' L w2 + 6 a^2 w1 w2 + w3 + a^2 w3 == 0,
-  $\frac{1}{2}$  Hw1' L^3 - Hw1' L Hw2' L - a^2 Hw1' L Hw2' L - a^2 Hw1'' L Hw2' L - Hw1' L Hw2'' L - a^2 Hw1' L Hw2'' L +
a^2 Hw1'' L Hw2' L + a^2 Hw1'' L Hw2'' L - a^2 Hw1' L Hw2'''' L + w3' + 2 Hw1' L Hw1' L Hy1' L +
2 Hw2' L Hy1' L + Hw1' L^2 Hy1' L + 2 Hw1' L Hy2' L - y3'' - 2 HD0 w2 L HD0 y1 L - 2 HD0 w1 L HD0 y2 L + D0^2 y3 -
2 HD0 y1 L HD1 w1 L - 2 HD0 w1 L HD1 y1 L + 2 HD0 D1 y2 L + D1^2 y1 + 2 HD0 D2 y1 L - Hw1' L Hw1'' L w1 -
2 Hw2' L w1 + a^2 Hw2'' L w1 - 2 Hw1' L Hy1' L w1 + 2 Hy2' L w1 + 2 HD0 w1 L HD0 y1 L w1 - 2 HD0^2 y2 L w1 -
4 HD0 D1 y1 L w1 - Hy1' L w1^2 + HD0^2 y1 L w1^2 - 2 Hw1' L w2 + a^2 Hw1'' L w2 + 2 Hy1' L w2 - 2 HD0^2 y1 L w2 == 0==
```

§ 8.2.1 First-Order Solution

To determine the solution of the first-order problem, [eqEps\[\[1\]\]](#), we use the method of separation of variables and let

```
svRule = 9w1 -> I A E^I m #1 E^I w1 #2 &M, y1 -> I B E^I m #1 E^I w1 #2 &M=;
```

Substituting the [svRule](#) into [eqEps\[\[1\]\]](#), we obtain the characteristic equations

```
eqEps1lhs = ExpandAE-I m q - I wm T0 #@@1DD •. svRuleE & •ž eqEps@@1DD
```

$$8A - I B m + A a^2 - 2 A m^2 a^2 + A m^4 a^2 - A w_m^2, I A m + B m^2 - B w_m^2 <$$

Setting the determinant of the coefficient matrix in **eqEps1lhs** equal to zero yields the following equation governing the natural frequencies w_m of the shell:

```
HOuter@D, eqEps1lhs, 8A, B<D •• Det •• Collect@#, wmD &L == 0
```

$$m^2 a^2 - 2 m^4 a^2 + m^6 a^2 + H - 1 - m^2 - a^2 + 2 m^2 a^2 - m^4 a^2 L w_m^2 + w_m^4 == 0$$

It follows from **eqEps1lhs** that the ratio $G_m = I \frac{B}{A}$ of the amplitudes of y_1 and w_1 is given by

```
GRule = Gm -> FactorAI B/A •. Solve@eqEps1lhs@@1DD == 0, BD@@1DDE
```

$$G_m @ \frac{1 + a^2 - 2 m^2 a^2 + m^4 a^2 - w_m^2}{m}$$

Hence, the solution of **eqEps[[1]]** can be expressed in terms of the linear free-vibration modes

```
modeShapes = ComplexExpand@8Re@#D, Im@#D<D & •ž 8Exp@I n qD, - I Gn Exp@I n qD<
```

$$88 \cos@n qD, \sin@n qD<, 8 \sin@n qD G_n, - \cos@n qD G_n <<$$

as

```
sol1Form =
```

```
modeShapes.TableAAi,n@T1, T2D Exp@I wn T0D + •Ai,n@T1, T2D Exp@- I wn T0D, 8i, 2<E •• Expand
```

$$8E^{I T_0 w_n} \cos@n qD A_{1,n}@T_1, T_2D + E^{I T_0 w_n} \sin@n qD A_{2,n}@T_1, T_2D + \\ E^{-I T_0 w_n} \cos@n qD A_{1,n}@T_1, T_2D + E^{-I T_0 w_n} \sin@n qD A_{2,n}@T_1, T_2D, E^{I T_0 w_n} \sin@n qD G_n A_{1,n}@T_1, T_2D - \\ E^{I T_0 w_n} \cos@n qD G_n A_{2,n}@T_1, T_2D + E^{-I T_0 w_n} \sin@n qD G_n A_{1,n}@T_1, T_2D - E^{-I T_0 w_n} \cos@n qD G_n A_{2,n}@T_1, T_2D <$$

It follows from the **sol1Form** that all of the flexural modes are degenerate because two linearly independent eigenfunctions, namely **Cos@n qD** and **Sin@n qD**, correspond to the same frequency w_n for each n .

We consider the case of a primary-resonance excitation of the **Cos@n qD** mode corresponding to the frequency w_n ; it is called the **driven mode**. Because of the degeneracy, this mode is involved in a one-to-one internal resonance with the **Sin@n qD** mode, which is called the **companion mode**. We assume that neither of these two modes is involved in an internal resonance with any other mode. Therefore, in the presence of viscous damping, out of the infinite number of modes present in w_1 and y_1 , only the driven mode **Cos@n qD** and its companion mode **Sin@n qD** will contribute to the steady-state response (Nayfeh and Mook, 1979). Consequently, we take the solution of the first-order problem to consist only of these two modes and express it as

```
sol1 = 8w1 -> Function@8q, T0, T1, T2<, sol1Form@@1DD •• Evaluated, \\ y1 -> Function@8q, T0, T1, T2<, sol1Form@@2DD •• Evaluated<;
```

To describe the nearness of the primary resonance, we introduce the detuning parameter S defined by

$$\text{ResonanceCond} = 9W == w_n + e^2 s;$$

where the detuning is ordered at e^2 because secular terms appear first at order e^3 .

8.2.2 Second-Order Solution

Substituting the `sol1` into the second-order problem, `eqEps[[2]]`, we obtain

```
order2Eq =
  Table@HeqEps@@1, i, 1DD . 8w1 -> w2, y1 -> y2<L == HeqEps@@1, i, 1DD . 8w1 -> w2, y1 -> y2<L -
  HSubtract @@ eqEps@@2, iDD . sol1 . ExpandL, 8i, 2<D;
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_n T_0}$, we have

```
ST1 = CoefficientA#@2DD, E^{I w_n T_0} E & . order2Eq

9 - 2 I Cos[n qD w_n A_{1,n}^{H1,OL}@T1, T2D - 2 I Sin[n qD w_n A_{2,n}^{H1,OL}@T1, T2D,
- 2 I Sin[n qD G_n w_n A_{1,n}^{H1,OL}@T1, T2D + 2 I Cos[n qD G_n w_n A_{2,n}^{H1,OL}@T1, T2D =
```

Because the homogeneous parts of `order2Eq` have a nontrivial solution, the corresponding nonhomogeneous equations have a solution only if solvability conditions are satisfied. These conditions demand that `ST1` be orthogonal to every solution of the adjoint homogeneous equations. In this case, these equations are self-adjoint and hence solutions of the adjoint equations are given by the transpose of the `modeShapes`. Demanding that `ST1` be orthogonal to the transpose of the `modeShapes`, we obtain the solvability conditions

```
SCond1 =
  SolveAHint@#, 8q, 0, 2 p<D & . HTranspose@modeShapesD.ST1L . intRule1 . . intRule2 . .
  8int@Cos@s_D Sin@s_D, _D -> 0, int@f_@n qD^2, arg_D -> p<L == 0 . .
  Thread, 9A_{1,n}^{H1,OL}@T1, T2D, A_{2,n}^{H1,OL}@T1, T2D=E@@1DD

9A_{1,n}^{H1,OL}@T1, T2D @ 0, A_{2,n}^{H1,OL}@T1, T2D @ 0 =
```

Having determined the solvability conditions of the second-order equations, `order2Eq`, we use a combination of the superposition principle and the method of undetermined coefficients to determine a particular solution of them. To this end, we identify the forms of all of the possible terms on the right-hand sides of `order2Eq` as follows:

```
collectForm = OuterATimes, 8Cos[n qD, Sin[n qD<,
  TableA9A_{i,n}@T1, T2D E^{I w_n T_0}, A_{i,n}@T1, T2D E^{-I w_n T_0} =, 8i, 2<E . FlattenE . Flatten

8E^{I T_0 w_n} Cos[n qD A_{1,n}@T1, T2D, E^{-I T_0 w_n} Cos[n qD A_{1,n}@T1, T2D,
E^{I T_0 w_n} Cos[n qD A_{2,n}@T1, T2D, E^{-I T_0 w_n} Cos[n qD A_{2,n}@T1, T2D, E^{I T_0 w_n} Sin[n qD A_{1,n}@T1, T2D,
E^{-I T_0 w_n} Sin[n qD A_{1,n}@T1, T2D, E^{I T_0 w_n} Sin[n qD A_{2,n}@T1, T2D, E^{-I T_0 w_n} Sin[n qD A_{2,n}@T1, T2D<
```



```
sol2temp = 8w2 -> Function@8q, T0, T1, T2<, symbolList2@dD.possibleTerms •• EvaluatedD,
y2 -> Function@8q, T0, T1, T2<, symbolList2@eD.possibleTerms •• EvaluatedD<;
```

Substituting the **sol2temp** into the left-hand sides of the second-order equations, **order2Eq**, setting the coefficient of the j th term in **possibleTerms** in the i th equation equal to $c_{i,j}$, and solving the resulting algebraic equations, we obtain

```
deRule = Solve@
  Flatten@Table@Coefficient@order2Eq@@@i, 1DD •. sol2temp, possibleTermsD •. f_@2 n qD -> 0,
    8i, 2<DD == Flatten@symbolList1D •• Thread •• Union,
    8symbolList2@dD, symbolList2@eD< •• Flatten •• Union •• RestD@@1DD
```

Solve::svars: Equations may not give solutions for all "solve" variables.

```
:e1,1 ® 0, e1,7 ® 0, d1,10 ®  $\frac{C_{1,10}}{1+a^2}$ , d1,22 ®  $\frac{C_{1,22}}{1+a^2}$ , e1,11 ® 0, e1,15 ® 0,
e1,23 ® 0, e2,12 ®  $-\frac{2 n C_{1,11} - C_{2,12}}{4 n^2 H - 1 + 4 n^2 L^2 a^2} + \frac{C_{2,12}}{4 n^2}$ , e2,14 ®  $\frac{C_{2,14}}{4 n^2} + \frac{-2 n C_{1,15} + C_{2,14}}{4 n^2 H - 1 + 4 n^2 L^2 a^2}$ ,
e2,24 ®  $-\frac{2 n C_{1,23} - C_{2,24}}{4 n^2 H - 1 + 4 n^2 L^2 a^2} + \frac{C_{2,24}}{4 n^2}$ , d1,1 ®  $\frac{C_{1,1}}{1+a^2 - 4 w_n^2}$ , d1,7 ®  $\frac{C_{1,7}}{1+a^2 - 4 w_n^2}$ , e1,2 ® 0, e1,6 ® 0,
e1,8 ® 0, e2,3 ®  $\frac{C_{2,3}}{4 n^2 - 4 w_n^2} - \frac{2 n H - H^4 n^2 - 4 w_n^2 L C_{1,2} - 2 n C_{2,3} L}{H^4 n^2 - 4 w_n^2 L H - 4 n^2 + H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L L}$ ,
e2,5 ®  $\frac{C_{1,6}}{2 n} + \frac{H^4 n^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L H^4 n^2 - 4 w_n^2 L C_{1,6} - 2 n C_{2,5} L}{2 n H^4 n^2 - H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L L}$ ,
e2,9 ®  $-\frac{C_{1,8}}{2 n} - \frac{H^4 n^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L H^4 n^2 - 4 w_n^2 L C_{1,8} + 2 n C_{2,9} L}{2 n H^4 n^2 - H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L L}$ ,
d1,11 ®  $-\frac{2 n C_{1,11} - C_{2,12}}{2 n H - 1 + 4 n^2 L^2 a^2}$ , d1,15 ®  $-\frac{2 n C_{1,15} + C_{2,14}}{2 n H - 1 + 4 n^2 L^2 a^2}$ , d1,23 ®  $-\frac{2 n C_{1,23} - C_{2,24}}{2 n H - 1 + 4 n^2 L^2 a^2}$ ,
d2,12 ® 0, d2,14 ® 0, d2,24 ® 0, d1,2 ®  $-\frac{-H^4 n^2 - 4 w_n^2 L C_{1,2} - 2 n C_{2,3}}{-4 n^2 + H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L}$ ,
d1,6 ®  $-\frac{H^4 n^2 - 4 w_n^2 L C_{1,6} - 2 n C_{2,5}}{4 n^2 - H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L}$ ,
d1,8 ®  $-\frac{H^4 n^2 - 4 w_n^2 L C_{1,8} + 2 n C_{2,9}}{4 n^2 - H^4 n^2 - 4 w_n^2 L H + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 w_n^2 L}$ , d2,3 ® 0, d2,5 ® 0, d2,9 ® 0>
```

To simplify the expressions in the second-order solution, we introduce the notation

paperNotationRule = ThreadAHSymbolList2@CD •• Union •• RestL ->

$$\begin{aligned}
 & 9c_2, c_1, 2c_1, c_2, -c_1, -c_4, -\frac{1}{2}c_3, -c_3, -c_4, \frac{1}{2}c_3, d_1, -2d_1, -d_1, -\frac{1}{2}d_2, d_2, \frac{1}{2}d_2 = E \sim \text{Join} \sim \\
 & 9(1 + a^2 - 8n^2 a^2 + 16n^4 a^2 - 4w_n^2) \rightarrow \frac{D_n + 4n^2}{4n^2 - 4w_n^2}, \text{PowerA} - 1 + 4n^2, -2E \text{Power}@a, -2D \rightarrow \frac{4n^2}{Q_n} = \\
 & : C_{1,1} \otimes C_2, C_{1,2} \otimes C_1, C_{1,6} \otimes 2C_1, C_{1,7} \otimes C_2, C_{1,8} \otimes -C_1, C_{1,10} \otimes -C_4, C_{1,11} \otimes -\frac{C_3}{2}, C_{1,15} \otimes -C_3, \\
 & C_{1,22} \otimes -C_4, C_{1,23} \otimes \frac{C_3}{2}, C_{2,3} \otimes d_1, C_{2,5} \otimes -2d_1, C_{2,9} \otimes -d_1, C_{2,12} \otimes -\frac{d_2}{2}, C_{2,14} \otimes d_2, \\
 & C_{2,24} \otimes \frac{d_2}{2}, 1 + a^2 - 8n^2 a^2 + 16n^4 a^2 - 4w_n^2 \otimes \frac{4n^2 + D_n}{4n^2 - 4w_n^2}, \frac{1}{H - 1 + 4n^2 L^2 a^2} \otimes \frac{4n^2}{Q_n} >
 \end{aligned}$$

Then, we express the second-order solution as

w2Sol = symbolList2@dD.possibleTerms •. deRule ••. paperNotationRule;

w2Sol •. displayRule

$$\begin{aligned}
 & \frac{E^{2IT_0 w_n} C_2 A_2^2}{1 + a^2 - 4w_n^2} - \frac{E^{2IT_0 w_n} \cos@2n q D H - 2n d_1 - c_1 H 4n^2 - 4w_n^2 L L A_1^2}{D_n} + \\
 & \frac{E^{2IT_0 w_n} \sin@2n q D H 4n d_1 + 2c_1 H 4n^2 - 4w_n^2 L L A_{1,n} A_{2,n}}{D_n} + \frac{E^{2IT_0 w_n} C_2 A_2^2}{1 + a^2 - 4w_n^2} + \\
 & \frac{E^{2IT_0 w_n} \cos@2n q D H - 2n d_1 - c_1 H 4n^2 - 4w_n^2 L L A_2^2}{D_n} - \frac{C_4 A_{1,n} \dot{A}_{1,n}}{1 + a^2} - \frac{2n \cos@2n q D \ln c_3 + \frac{d_2}{Q_n} M A_{1,n} \dot{A}_{1,n}}{Q_n} - \\
 & \frac{2n \sin@2n q D H 2n c_3 + d_2 L A_{2,n} \dot{A}_{1,n}}{Q_n} - \frac{C_4 A_{2,n} \dot{A}_{2,n}}{1 + a^2} - \frac{2n \cos@2n q D \ln c_3 - \frac{d_2}{Q_n} M A_{2,n} \dot{A}_{2,n}}{Q_n}
 \end{aligned}$$

y2Sol = symbolList2@eD.possibleTerms •. deRule ••. paperNotationRule •. e_i_ -> 0;

y2Sol •. displayRule

$$\begin{aligned}
 & E^{2IT_0 w_n} \sin@2n q D \left\{ \frac{d_1}{4n^2 - 4w_n^2} - \frac{2n H - 2n d_1 - c_1 H 4n^2 - 4w_n^2 L L y}{D_n H 4n^2 - 4w_n^2 L} \right\} A_{1,n}^2 + \\
 & E^{2IT_0 w_n} \cos@2n q D \left\{ \frac{c_1}{n} - \frac{H 4n^2 + D_n L H 4n d_1 + 2c_1 H 4n^2 - 4w_n^2 L L y}{2n D_n H 4n^2 - 4w_n^2 L} \right\} A_{1,n} A_{2,n} + \\
 & E^{2IT_0 w_n} \sin@2n q D \left\{ \frac{c_1}{2n} + \frac{H 4n^2 + D_n L H - 2n d_1 - c_1 H 4n^2 - 4w_n^2 L L y}{2n D_n H 4n^2 - 4w_n^2 L} \right\} A_{2,n}^2 + \\
 & \sin@2n q D \left\{ -\frac{d_2}{8n^2} - \frac{n c_3 + \frac{d_2}{Q_n} y}{Q_n} \right\} A_{1,n} \dot{A}_{1,n} + \\
 & \cos@2n q D \left\{ \frac{d_2}{4n^2} + \frac{2n c_3 + d_2 y}{Q_n} \right\} A_{2,n} \dot{A}_{1,n} + \sin@2n q D \left\{ \frac{d_2}{8n^2} - \frac{-n c_3 - \frac{d_2}{Q_n} y}{Q_n} \right\} A_{2,n} \dot{A}_{2,n}
 \end{aligned}$$

We express this solution in pure function form as

sol2 = 8w2 -> Function@8q, T0, T1, T2<, w2Sol + Hw2Sol •. conjugateRuleL •• Evaluated,
y2 -> Function@8q, T0, T1, T2<, y2Sol + Hy2Sol •. conjugateRuleL •• Evaluated<;

8.2.3 Solvability Conditions

Substituting the **sol1** and **sol2** into **eqEps[[3]]** yields

```
order3Eq =
  Table@HeqEps@@1, i, 1DD . 8w1 -> w3, y1 -> y3<L == HeqEps@@1, i, 1DD . 8w1 -> w3, y1 -> y3<L -
    HSubtract žž eqEps@@3, iDD . sol1 . sol2 •• TrigToExp •• ExpandL, 8i, 2<D; •• Timing
890.5 Second, Null<
```

To convert the terms that lead to small-divisor terms in the third-order equations, **order3Eq**, into terms that lead to secular terms, we introduce the rule

```
expRule1 = Exp@arg_D :> ExpAExpand@arg . HResonanceCond . Equal -> RuleLD . e2T0 -> T2E;
```

Then, the sources of secular terms in **order3Eq**, the terms proportional to $\exp(\pm i n \varphi + i \omega_n T_0)$, are

```
ST2 =
  Expand@Coefficient@#@@2DD . Exp@c1_. + Complex@0, s_ .; Abs@sD != 1D n qD -> 0 . expRule1,
  Exp@I wn T0DD . Exp@c1_. + Complex@0, m_D n qD ->
  HCos@n qD + I m Sin@n qDL Exp@c1DD & •ž order3Eq; •• Timing
864.343 Second, Null<
```

These terms consist of linear and cubic terms in the complex-valued amplitudes A_j . The cubic terms are proportional to

```
cubicTerms =
  Flatten@If@Head@#D === Plus, List žž #, #D & •ž HNest@Outer@Times, collectForm, #D &,
    collectForm, 2D •• Flatten •• Union •• TrigReduce •• ExpandLD .
  8f_s n qD -> 0, Exp@c1_. + Complex@0, s_ .; s != 1D wn T0D -> 0< .
  Exp@_D -> 1 . c_? NumberQ form_ -> form •• Union •• Rest;
cubicTerms . displayRule
8Cos@n qD A1,n2  $\dot{A}_{1,n}$ , Sin@n qD A1,n2  $\dot{A}_{1,n}$ , Cos@n qD A1,n A2,n  $\dot{A}_{1,n}$ , Sin@n qD A1,n A2,n  $\dot{A}_{1,n}$ ,
  Cos@n qD A2,n2  $\dot{A}_{1,n}$ , Sin@n qD A2,n2  $\dot{A}_{1,n}$ , Cos@n qD A1,n2  $\dot{A}_{2,n}$ , Sin@n qD A1,n2  $\dot{A}_{2,n}$ ,
  Cos@n qD A1,n A2,n  $\dot{A}_{2,n}$ , Sin@n qD A1,n A2,n  $\dot{A}_{2,n}$ , Cos@n qD A2,n2  $\dot{A}_{2,n}$ , Sin@n qD A2,n2  $\dot{A}_{2,n}$ <
```

And, their coefficients in **ST2** are

```
coef2 = Outer@Coefficient, ST2, cubicTermsD;
```

which we denote by

```
symbolList3 = MapIndexed@If@#1 != 0, gSequence žž #2, 0D &, coef2, 82<D;
```

```

notationRule = HsymbolList3 •• Flatten •• Union •• RestL ->
  8E2, E1, E3, E3, E1, E2, -G2, G1, -G3, G3, -G1, G2< •• Thread

8g1,1 ® E2, g1,4 ® E1, g1,5 ® E3, g1,8 ® E3, g1,9 ® E1, g1,12 ® E2,
  g2,2 ® -G2, g2,3 ® G1, g2,6 ® -G3, g2,7 ® G3, g2,10 ® -G1, g2,11 ® G2<

```

In terms of this notation, **ST2** can be rewritten as

```

ST2New = H# •. Thread@cubicTerms -> 0D & •ž ST2L + symbolList3.cubicTerms •. notationRule •.
  D@SCond1, T1D;
ST2New •. displayRule


$$\begin{aligned}
& \frac{1}{2} E^{T_2} f \cos \theta \cos \phi - 2 \int \cos \theta \cos \phi \, HD_2 A_{1,n} L w_n - 2 \int HD_2 A_{2,n} L \sin \theta \cos \phi \, w_n + \\
& \cos \theta \cos \phi \, E_2 A_{1,n}^2 \dot{A}_{1,n} + \sin \theta \cos \phi \, E_1 A_{1,n} A_{2,n} \dot{A}_{1,n} + \cos \theta \cos \phi \, E_3 A_{2,n}^2 \dot{A}_{1,n} + \sin \theta \cos \phi \, E_3 A_{1,n}^2 \dot{A}_{2,n} + \\
& \cos \theta \cos \phi \, E_1 A_{1,n} A_{2,n} \dot{A}_{2,n} + \sin \theta \cos \phi \, E_2 A_{2,n}^2 \dot{A}_{2,n}, \quad 2 \int \cos \theta \cos \phi \, HD_2 A_{2,n} L G_n w_n - \\
& 2 \int HD_2 A_{1,n} L \sin \theta \cos \phi \, G_n w_n - \sin \theta \cos \phi \, G_2 A_{1,n}^2 \dot{A}_{1,n} + \cos \theta \cos \phi \, G_1 A_{1,n} A_{2,n} \dot{A}_{1,n} - \\
& \sin \theta \cos \phi \, G_3 A_{2,n}^2 \dot{A}_{1,n} + \cos \theta \cos \phi \, G_3 A_{1,n}^2 \dot{A}_{2,n} - \sin \theta \cos \phi \, G_1 A_{1,n} A_{2,n} \dot{A}_{2,n} + \cos \theta \cos \phi \, G_2 A_{2,n}^2 \dot{A}_{2,n} =
\end{aligned}$$


```

The solvability conditions of **order3Eq** demand that the vector **ST2New** be orthogonal to the transpose of the **modeShapes**; that is,

```

SCond2 = ExpandA[ $\frac{1}{2} E^{T_2} f \cos \theta \cos \phi - 2 \int \cos \theta \cos \phi \, HD_2 A_{1,n} L w_n - 2 \int HD_2 A_{2,n} L \sin \theta \cos \phi \, w_n +$ 
   $\cos \theta \cos \phi \, E_2 A_{1,n}^2 \dot{A}_{1,n} + \sin \theta \cos \phi \, E_1 A_{1,n} A_{2,n} \dot{A}_{1,n} + \cos \theta \cos \phi \, E_3 A_{2,n}^2 \dot{A}_{1,n} + \sin \theta \cos \phi \, E_3 A_{1,n}^2 \dot{A}_{2,n} +$ 
   $\cos \theta \cos \phi \, E_1 A_{1,n} A_{2,n} \dot{A}_{2,n} + \sin \theta \cos \phi \, E_2 A_{2,n}^2 \dot{A}_{2,n}, \quad 2 \int \cos \theta \cos \phi \, HD_2 A_{2,n} L G_n w_n -$ 
   $2 \int HD_2 A_{1,n} L \sin \theta \cos \phi \, G_n w_n - \sin \theta \cos \phi \, G_2 A_{1,n}^2 \dot{A}_{1,n} + \cos \theta \cos \phi \, G_1 A_{1,n} A_{2,n} \dot{A}_{1,n} -$ 
   $\sin \theta \cos \phi \, G_3 A_{2,n}^2 \dot{A}_{1,n} + \cos \theta \cos \phi \, G_3 A_{1,n}^2 \dot{A}_{2,n} - \sin \theta \cos \phi \, G_1 A_{1,n} A_{2,n} \dot{A}_{2,n} + \cos \theta \cos \phi \, G_2 A_{2,n}^2 \dot{A}_{2,n} =$ 
  int@#, 8q, 0, 2 p<D & •ž HTranspose@modeShapesD.ST2NewL •. intRule1 •.
  intRule2 •. 8int@Cos@s_D Sin@s_D, _D -> 0, int@f_@n qD^2, _D -> p<E == 0 •• Thread;
SCond2 •. displayRule


$$\begin{aligned}
& \frac{1}{2} E^{T_2} f \cos \theta \cos \phi - 2 \int HD_2 A_{1,n} L w_n - 2 \int HD_2 A_{1,n} L G_n^2 w_n + E_2 A_{1,n}^2 \dot{A}_{1,n} - \\
& G_2 G_n A_{1,n}^2 \dot{A}_{1,n} + E_3 A_{2,n}^2 \dot{A}_{1,n} - G_3 G_n A_{2,n}^2 \dot{A}_{1,n} + E_1 A_{1,n} A_{2,n} \dot{A}_{2,n} - G_1 G_n A_{1,n} A_{2,n} \dot{A}_{2,n} == 0, \\
& - 2 \int HD_2 A_{2,n} L w_n - 2 \int HD_2 A_{2,n} L G_n^2 w_n + E_1 A_{1,n} A_{2,n} \dot{A}_{1,n} - G_1 G_n A_{1,n} A_{2,n} \dot{A}_{1,n} + \\
& E_3 A_{1,n}^2 \dot{A}_{2,n} - G_3 G_n A_{1,n}^2 \dot{A}_{2,n} + E_2 A_{2,n}^2 \dot{A}_{2,n} - G_2 G_n A_{2,n}^2 \dot{A}_{2,n} == 0 =
\end{aligned}$$


```

à 8.3 Near-Square Plates

We consider the nonlinear transverse vibrations of a near-square simply supported plate that is initially flat. We introduce a Cartesian coordinate system $x y z$, with the x and y axes lying in the undisturbed plane of the middle surface of the plate, and let u , v , and w denote the displacements of the middle surface in the x , y , and z directions, respectively. Then, the equations of motion are (Chu and Herrmann, 1956)

```

Needs@"Calculus`VectorAnalysis`"D

BH@wD = Biharmonic@w@x, y, tD, Cartesian@x, y, zDD

w^H0,4,0L@x, y, tD + 2 w^H2,2,0L@x, y, tD + w^H4,0,0L@x, y, tD

```

EOM = 9

$$\begin{aligned}
 &u_{x,x} + w_x w_{x,x} + n H v_{x,y} + w_y w_{x,y} L + \frac{1}{2} H^2 - n L H u_{y,y} + v_{x,y} + w_x w_{y,y} + w_y w_{x,y} L = c_p^{-2} u_{t,t}, \\
 &v_{y,y} + w_y w_{y,y} + n H u_{x,y} + w_x w_{x,y} L + \frac{1}{2} H^2 - n L H v_{x,x} + u_{x,y} + w_y w_{x,x} + w_x w_{x,y} L = c_p^{-2} v_{t,t}, \\
 &\frac{1}{12} h^2 B H^2 D - N_1^i w_{x,x} - 2 N_{12}^i w_{x,y} - N_2^i w_{y,y} - u_x w_{x,x} - \\
 &\frac{1}{2} w_x^2 w_{x,x} - v_y w_{y,y} - \frac{1}{2} w_y^2 w_{y,y} - n \int_K v_y w_{x,x} + \frac{1}{2} w_y^2 w_{x,x} + u_x w_{y,y} + \frac{1}{2} w_x^2 w_{y,y} - \\
 &H^2 - n L H u_y + v_x + w_x w_y L w_{x,y} = -c_p^{-2} H w_x u_{t,t} + w_y v_{t,t} + w_{t,t} L + \frac{F L^2 - n^2 M}{E_s h} \\
 &= \bullet. Hs : u \in v \in wL_m \rightarrow \mathbb{I}_m s @ x, y, t D;
 \end{aligned}$$

and the boundary conditions are

$$\begin{aligned}
 &BC = 88u @ x, y, t D == 0, w @ x, y, t D == 0, \mathbb{I}_{x,x} w @ x, y, t D == 0 \bullet. 88x \rightarrow 0 <, 8x \rightarrow a < <, \\
 &8v @ x, y, t D == 0, w @ x, y, t D == 0, \mathbb{I}_{y,y} w @ x, y, t D == 0 \bullet. 88y \rightarrow 0 <, 8y \rightarrow b < < \bullet \bullet Flatten \\
 &8u @ 0, y, t D == 0, w @ 0, y, t D == 0, w^{H^2, 0, 0L} @ 0, y, t D == 0, u @ a, y, t D == 0, \\
 &w @ a, y, t D == 0, w^{H^2, 0, 0L} @ a, y, t D == 0, v @ x, 0, t D == 0, w @ x, 0, t D == 0, \\
 &w^{H^0, 2, 0L} @ x, 0, t D == 0, v @ x, b, t D == 0, w @ x, b, t D == 0, w^{H^0, 2, 0L} @ x, b, t D == 0 <
 \end{aligned}$$

where $c_p^2 = E_s \cdot H r H^2 - n^2 L$, E_s is Young's modulus, n is Poisson's ratio, r and h are the density and thickness of the plate, respectively, F is the transverse load, the N_k^i are the uniformly distributed in-plane applied edge loads, and a and b are the edge lengths, with $b \gg a$. **EOM** are usually called the [dynamic analog of the von Karman equations](#) because they reduce to von Karman's equations (1910) in the absence of the time derivatives.

For thin plates, $h \cdot a$ is very small, which we denote by ϵ . Next, we introduce the aspect ratio $c = a \cdot b$, define the rule

$$scaleRule1 = 9h^2 \rightarrow \epsilon a^2, a \rightarrow c b;$$

and nondimensionalize the variables as

$$\begin{aligned}
 &charT = \frac{1}{12} \epsilon a \cdot c_p; \quad H \cdot \text{characteristic time} \cdot L \\
 &nondimRule = 9Hdv : u \in v \in wL \rightarrow Ha dv @ \#1 \cdot a, \#2 \cdot b, \#3 \cdot charTD \&L, \\
 &F \rightarrow F E_s h \cdot L^2 a L^2 - n^2 MM, x \rightarrow a x, y \rightarrow b y, t \rightarrow charT t;
 \end{aligned}$$

Substituting **nondimRule** and **scaleRule1** into **EOM** and **BC** and considering the case of no in-plane edge loads, we obtain

$$\begin{aligned}
 &EOM1 = la HSubtract \check{z} \check{z} \#L \bullet. N_k^i \rightarrow 0 \bullet. nondimRule \bullet \bullet. scaleRule1 \bullet \bullet ExpandM == 0 \& \bullet \check{z} EOM; \\
 &BC1 = BC \bullet. nondimRule \bullet. c_h @ a, tD \rightarrow h @ a, tD \\
 &8u @ 0, y, t D == 0, w @ 0, y, t D == 0, w^{H^2, 0, 0L} @ 0, y, t D == 0, u @ 1, y, t D == 0, \\
 &w @ 1, y, t D == 0, w^{H^2, 0, 0L} @ 1, y, t D == 0, v @ x, 0, t D == 0, w @ x, 0, t D == 0, \\
 &w^{H^0, 2, 0L} @ x, 0, t D == 0, v @ x, 1, t D == 0, w @ x, 1, t D == 0, w^{H^0, 2, 0L} @ x, 1, t D == 0 <
 \end{aligned}$$

To write **EOM1** in a more concise way, we introduce the display rule

```
display1@expr_D := expr . 8Derivative@a_, b_, c_D@w_D@x, y, tD :>
  Subscript@w, Sequence @@ Join@Table@x, 8a<D, Table@y, 8b<D, Table@t, 8c<DDD<
```

and obtain

```
EOM1@@1DD •• display1
EOM1@@2DD •• display1
EOM1@@3DD •• display1
```

$$\begin{aligned}
 & -\frac{1}{12} e u_{t,t} + u_{x,x} + \frac{1}{2} c^2 u_{y,y} - \frac{1}{2} c^2 n u_{y,y} + \frac{1}{2} c v_{x,y} + \frac{1}{2} c n v_{x,y} + \\
 & w_x w_{x,x} + \frac{1}{2} c^2 w_y w_{x,y} + \frac{1}{2} c^2 n w_y w_{x,y} + \frac{1}{2} c^2 w_x w_{y,y} - \frac{1}{2} c^2 n w_x w_{y,y} == 0 \\
 & \frac{1}{2} c u_{x,y} + \frac{1}{2} c n u_{x,y} - \frac{1}{12} e v_{t,t} + \frac{v_{x,x}}{2} - \frac{1}{2} n v_{x,x} + c^2 v_{y,y} + \\
 & \frac{1}{2} c w_y w_{x,x} - \frac{1}{2} c n w_y w_{x,x} + \frac{1}{2} c w_x w_{x,y} + \frac{1}{2} c n w_x w_{x,y} + c^3 w_y w_{y,y} == 0 \\
 & -\frac{F}{12} + \frac{1}{12} e w_x u_{t,t} + \frac{1}{12} c e w_y v_{t,t} + \frac{1}{12} e w_{t,t} - u_x w_{x,x} - c n v_y w_{x,x} - \frac{1}{2} w_x^2 w_{x,x} - \frac{1}{2} c^2 n w_y^2 w_{x,x} - \\
 & c^2 u_y w_{x,y} + c^2 n u_y w_{x,y} - c v_x w_{x,y} + c n v_x w_{x,y} - c^2 w_x w_y w_{x,y} + c^2 n w_x w_y w_{x,y} - c^2 n u_x w_{y,y} - \\
 & c^3 v_y w_{y,y} - \frac{1}{2} c^2 n w_x^2 w_{y,y} - \frac{1}{2} c^4 w_y^2 w_{y,y} + \frac{1}{12} e w_{x,x,x,x} + \frac{1}{6} c^2 e w_{x,x,y,y} + \frac{1}{12} c^4 e w_{y,y,y,y} == 0
 \end{aligned}$$

We use the method of multiple scales to directly attack **EOM1** and **BC1**. To transform the time derivatives in **EOM1** in terms of the scales T_0 and T_1 , we let

```
multiScales = 8Hdv : u È v È wL@x_, y_, tD -> dv@x, y, T0, T1D,
  Derivative@m_, n_, o_D@u_D@x_, y_, tD -> dt@oD@D@u@x, y, T0, T1D, 8x, m<, 8y, n<DD<;
```

Then, we seek a second-order approximate solution in the form

```
solRule = 9Hdv : u È vL -> I EvaluateASum Ae^{j+1} dv_j@#1, #2, #3, #4D, 8j, 2<EE &M,
  w -> I EvaluateASum Ae^j w_j@#1, #2, #3, #4D, 8j, 3<EE &M=
8dv : u È v È He^2 dv_1@#1, #2, #3, #4D + e^3 dv_2@#1, #2, #3, #4D &L,
  w È He w_1@#1, #2, #3, #4D + e^2 w_2@#1, #2, #3, #4D + e^3 w_3@#1, #2, #3, #4D &L<
```

where #1, #2, #3, and #4 stand for x , y , T_0 , and T_1 , respectively.

We introduce the detuning s_1 to describe the nearness of the edge lengths a and b , consider the case of primary resonance, and hence let

```
scaleRule2 = 9c -> 1 + e s_1, F -> e^3 F@x, yD Cos@W T0D=;
```

Substituting **multiScales**, **solRule**, and **scaleRule2** into **EOM1** and **BC1**, expanding the result for small e , and discarding terms of order higher than e^3 , we obtain

```
eq83a =
  HEOM1 . multiScales . solRule . scaleRule2 . TrigToExp . ExpandAllL . en-·;n>3 -> 0;

bc83a = BC1 . multiScales . solRule;
```

Equating coefficients of like powers of e in **eq83a** and **bc83a** yields

```
eqEps =
  Thread@CoefficientList@Subtract žž #, eD == 0D & . ž eq83a . Transpose . Rest . Rest;

bcEpsuv =
  Thread@CoefficientList@Subtract žž #, eD == 0D & . ž Select@bc83a, FreeQ@#, wD &D .
  Transpose . Rest . Rest

88u1@0, y, T0, T1D == 0, u1@1, y, T0, T1D == 0, v1@x, 0, T0, T1D == 0, v1@x, 1, T0, T1D == 0<,
8u2@0, y, T0, T1D == 0, u2@1, y, T0, T1D == 0, v2@x, 0, T0, T1D == 0, v2@x, 1, T0, T1D == 0<<

bcEpsw = Thread@CoefficientList@Subtract žž #, eD == 0D & . ž
  Select@bc83a, ! FreeQ@#, wD &D . Transpose . Rest

99w1@0, y, T0, T1D == 0, w1H2,0,0,0L@0, y, T0, T1D == 0,
  w1@1, y, T0, T1D == 0, w1H2,0,0,0L@1, y, T0, T1D == 0, w1@x, 0, T0, T1D == 0,
  w1H0,2,0,0L@x, 0, T0, T1D == 0, w1@x, 1, T0, T1D == 0, w1H0,2,0,0L@x, 1, T0, T1D == 0=,
9w2@0, y, T0, T1D == 0, w2H2,0,0,0L@0, y, T0, T1D == 0, w2@1, y, T0, T1D == 0,
  w2H2,0,0,0L@1, y, T0, T1D == 0, w2@x, 0, T0, T1D == 0, w2H0,2,0,0L@x, 0, T0, T1D == 0,
  w2@x, 1, T0, T1D == 0, w2H0,2,0,0L@x, 1, T0, T1D == 0=,
9w3@0, y, T0, T1D == 0, w3H2,0,0,0L@0, y, T0, T1D == 0, w3@1, y, T0, T1D == 0,
  w3H2,0,0,0L@1, y, T0, T1D == 0, w3@x, 0, T0, T1D == 0, w3H0,2,0,0L@x, 0, T0, T1D == 0,
  w3@x, 1, T0, T1D == 0, w3H0,2,0,0L@x, 1, T0, T1D == 0==
```

Next, we introduce the display rule

```
displayRule1 = 9Derivative@a_, b_, c_ DAw_i_ E@x_, y_, T0, __D :=>
  SequenceFormAIfAarg1 = Times žž MapIndexedAD#1#2@1DD-1 &, 8c<E; arg1 != 1, arg1, ""E,
  Subscript@w, Sequence žž Join@8i<, Table@x, 8a<D, Table@y, 8b<DDDE,
  Derivative@a_ D@A_i_ D@__D :=> SequenceFormATimes žž MapIndexedAD#1#2@1DD &, 8a<E, A_i E,
  Derivative@a_ D@Ā_i_ D@__D :=> SequenceFormATimes žž MapIndexedAD#1#2@1DD &, 8a<E, Ā_i E,
  w_i_@__, T0, __D -> w_i, A_i_@__D -> A_i, Ā_i_@__D -> Ā_i=;
```

and write **eqEps** in a more concise way as

HeqEpsm = 8eqEps@@1, 3DD, eqEps@@1, 81, 2<DD, eqEps@@2, 3DD<L •. displayRule1

$$\begin{aligned}
 & 9 \frac{1}{12} Hw_{1,x,x,x,x}L + \frac{1}{6} Hw_{1,x,x,y,y}L + \frac{1}{12} Hw_{1,y,y,y,y}L + \frac{1}{12} HD_0^2 w_1 L == 0, \\
 & 9u_{1,x,x} + \frac{1}{2} Hu_{1,y,y}L - \frac{1}{2} n Hu_{1,y,y}L + \frac{1}{2} Hv_{1,x,y}L + \frac{1}{2} n Hv_{1,x,y}L + Hw_{1,x}L Hw_{1,x,x}L + \\
 & \quad \frac{1}{2} Hw_{1,y}L Hw_{1,x,y}L + \frac{1}{2} n Hw_{1,y}L Hw_{1,x,y}L + \frac{1}{2} Hw_{1,x}L Hw_{1,y,y}L - \frac{1}{2} n Hw_{1,x}L Hw_{1,y,y}L == 0, \\
 & \frac{1}{2} Hu_{1,x,y}L + \frac{1}{2} n Hu_{1,x,y}L + \frac{1}{2} Hv_{1,x,x}L - \frac{1}{2} n Hv_{1,x,x}L + v_{1,y,y} + \frac{1}{2} Hw_{1,y}L Hw_{1,x,x}L - \\
 & \quad \frac{1}{2} n Hw_{1,y}L Hw_{1,x,x}L + \frac{1}{2} Hw_{1,x}L Hw_{1,x,y}L + \frac{1}{2} n Hw_{1,x}L Hw_{1,x,y}L + Hw_{1,y}L Hw_{1,y,y}L == 0=, \\
 & - \frac{1}{24} E^{-I T_0 W} F@x, yD - \frac{1}{24} E^{I T_0 W} F@x, yD - Hu_{1,x}L Hw_{1,x,x}L - n Hv_{1,y}L Hw_{1,x,x}L - \frac{1}{2} Hw_{1,x}L^2 Hw_{1,x,x}L - \\
 & \quad \frac{1}{2} n Hw_{1,y}L^2 Hw_{1,x,x}L - Hu_{1,y}L Hw_{1,x,y}L + n Hu_{1,y}L Hw_{1,x,y}L - Hv_{1,x}L Hw_{1,x,y}L + n Hv_{1,x}L Hw_{1,x,y}L - \\
 & \quad Hw_{1,x}L Hw_{1,y}L Hw_{1,x,y}L + n Hw_{1,x}L Hw_{1,y}L Hw_{1,x,y}L - n Hu_{1,x}L Hw_{1,y,y}L - Hv_{1,y}L Hw_{1,y,y}L - \\
 & \quad \frac{1}{2} n Hw_{1,x}L^2 Hw_{1,y,y}L - \frac{1}{2} Hw_{1,y}L^2 Hw_{1,y,y}L + \frac{1}{12} Hw_{2,x,x,x,x}L + \frac{1}{6} Hw_{2,x,x,y,y}L + \\
 & \quad \frac{1}{12} Hw_{2,y,y,y,y}L + \frac{1}{12} HD_0^2 w_2 L + \frac{1}{6} HD_0 D_1 w_1 L + \frac{1}{3} Hw_{1,x,x,y,y}L s_1 + \frac{1}{3} Hw_{1,y,y,y,y}L s_1 == 0=
 \end{aligned}$$

§ 8.3.1 First-Order Solution

Linear vibrations are governed by **eqEpsm[[1]]** and **bcEpsw[[1]]**; that is,

Horder1Eq = eqEpsm@@1DDL •. displayRule1

$$\frac{1}{12} Hw_{1,x,x,x,x}L + \frac{1}{6} Hw_{1,x,x,y,y}L + \frac{1}{12} Hw_{1,y,y,y,y}L + \frac{1}{12} HD_0^2 w_1 L == 0$$

order1BC = bcEpsw@@1DD

$$\begin{aligned}
 & 9w_{1@0, y, T_0, T_1}D == 0, w_1^{H2, 0, 0, 0L}@0, y, T_0, T_1D == 0, \\
 & w_{1@1, y, T_0, T_1}D == 0, w_1^{H2, 0, 0, 0L}@1, y, T_0, T_1D == 0, w_{1@x, 0, T_0, T_1}D == 0, \\
 & w_1^{H0, 2, 0, 0L}@x, 0, T_0, T_1D == 0, w_{1@x, 1, T_0, T_1}D == 0, w_1^{H0, 2, 0, 0L}@x, 1, T_0, T_1D == 0=
 \end{aligned}$$

It follows from **order1Eq** and **order1BC** that the linear modeshapes and associated frequencies are given by

modeshapes = 8f_{m, n} :> HSin@m p #1D Sin@n p #2D &L<;

frequencies = 9w_{m, n} :> I m² + n² M p² =;

To investigate the case of one-to-one internal resonance between the m th and n th modes ($m \neq n$), we let

$$\begin{aligned}
 w1sol = 9w_1 \rightarrow & \text{FunctionA8x, y, } T_0, T_1<, A_1@T_1D E^{I w_{m,n} T_0} f_{m,n}@x, yD + \\
 & \dot{A}_1@T_1D E^{-I w_{m,n} T_0} f_{m,n}@x, yD + A_2@T_1D E^{I w_{m,n} T_0} f_{n,m}@x, yD + \dot{A}_2@T_1D E^{-I w_{m,n} T_0} f_{n,m}@x, yD =;
 \end{aligned}$$

§ 8.3.2 Second-Order Solution

Substituting the `w1sol` into `eqEpsm[[2]]`, we obtain

```
order2Equ = HeqEpsm@@2, 1, 1DD •. w1 -> H0 &LL ==
  HHeqEpsm@@2, 1, 1DD •. w1 -> H0 &LL - eqEpsm@@2, 1, 1DD •. w1sol •. modeshapes ••
  TrigReduce •• ExpandL; •• Timing
```

84.39 Second, Null<

```
order2Eqv = HeqEpsm@@2, 2, 1DD •. w1 -> H0 &LL ==
  HHeqEpsm@@2, 2, 1DD •. w1 -> H0 &LL - eqEpsm@@2, 2, 1DD •. w1sol •. modeshapes ••
  TrigReduce •• ExpandL; •• Timing
```

84.28 Second, Null<

The associated boundary conditions are

```
order2BC = bcEpsuv@@1DD
```

```
8u1@0, y, T0, T1D == 0, u1@1, y, T0, T1D == 0, v1@x, 0, T0, T1D == 0, v1@x, 1, T0, T1D == 0<
```

To obtain particular solutions for u_1 and v_1 , we use the method of undetermined coefficients. To accomplish this, we first look for all possible terms that appear on the right-hand sides of `order2Equ` and `order2Eqv` as follows:

```
possibleTerms1 =
  CasesA8order2Equ@@2DD, order2Eqv@@2DD<, _ Sin@a_D E^b_ -> Sin@aD E^b, ¥E •• Union;

possibleTerms2 = possibleTerms1 •. Exp@_D -> 1 •• Union;
```



```
possibleTerms = Join@possibleTerms1, possibleTerms2D
```

```
8E-2IT0Wm,n Sin@2mp xD, E2IT0Wm,n Sin@2mp xD, E-2IT0Wm,n Sin@2np xD, E2IT0Wm,n Sin@2np xD,
E-2IT0Wm,n Sin@2mp yD, E2IT0Wm,n Sin@2mp yD, E-2IT0Wm,n Sin@2np yD, E2IT0Wm,n Sin@2np yD,
E-2IT0Wm,n Sin@2np x - 2mp yD, E2IT0Wm,n Sin@2np x - 2mp yD, E-2IT0Wm,n Sin@2np x + 2mp yD,
E2IT0Wm,n Sin@2np x + 2mp yD, E-2IT0Wm,n Sin@2mp x - 2np yD, E2IT0Wm,n Sin@2mp x - 2np yD,
E-2IT0Wm,n Sin@mp x - np x - mp y - np yD, E2IT0Wm,n Sin@mp x - np x - mp y - np yD,
E-2IT0Wm,n Sin@mp x + np x - mp y - np yD, E2IT0Wm,n Sin@mp x + np x - mp y - np yD,
E-2IT0Wm,n Sin@mp x - np x + mp y - np yD, E2IT0Wm,n Sin@mp x - np x + mp y - np yD,
E-2IT0Wm,n Sin@mp x + np x + mp y - np yD, E2IT0Wm,n Sin@mp x + np x + mp y - np yD,
E-2IT0Wm,n Sin@mp x - np x - mp y + np yD, E2IT0Wm,n Sin@mp x - np x - mp y + np yD,
E-2IT0Wm,n Sin@mp x + np x - mp y + np yD, E2IT0Wm,n Sin@mp x + np x - mp y + np yD,
E-2IT0Wm,n Sin@mp x - np x + mp y + np yD, E2IT0Wm,n Sin@mp x - np x + mp y + np yD,
E-2IT0Wm,n Sin@mp x + np x + mp y + np yD, E2IT0Wm,n Sin@mp x + np x + mp y + np yD,
E-2IT0Wm,n Sin@2mp x + 2np yD, E2IT0Wm,n Sin@2mp x + 2np yD, Sin@2mp xD, Sin@2np xD,
Sin@2mp yD, Sin@2np yD, Sin@2np x - 2mp yD, Sin@2np x + 2mp yD, Sin@2mp x - 2np yD,
Sin@mp x - np x - mp y - np yD, Sin@mp x + np x - mp y - np yD, Sin@mp x - np x + mp y - np yD,
Sin@mp x + np x + mp y - np yD, Sin@mp x - np x - mp y + np yD, Sin@mp x + np x - mp y + np yD,
Sin@mp x - np x + mp y + np yD, Sin@mp x + np x + mp y + np yD, Sin@2mp x + 2np yD<
```

which satisfy **order2BC**. We then define the following symbollists to represent the undetermined coefficients.

```
usymbolList = Table@ci, 8i, Length@possibleTermsD<D
```

```
8C1, C2, C3, C4, C5, C6, C7, C8, C9, C10, C11, C12, C13, C14, C15, C16,
C17, C18, C19, C20, C21, C22, C23, C24, C25, C26, C27, C28, C29, C30, C31, C32,
C33, C34, C35, C36, C37, C38, C39, C40, C41, C42, C43, C44, C45, C46, C47, C48<
```

```
vsymbolList = usymbolList •. c -> d
```

```
8d1, d2, d3, d4, d5, d6, d7, d8, d9, d10, d11, d12, d13, d14, d15, d16,
d17, d18, d19, d20, d21, d22, d23, d24, d25, d26, d27, d28, d29, d30, d31, d32,
d33, d34, d35, d36, d37, d38, d39, d40, d41, d42, d43, d44, d45, d46, d47, d48<
```

Then, we express the solution as

```
u1sol = u1 -> Function@8x, y, T0, T1<, usymbolList.possibleTerms •• EvaluatedD;
v1sol = v1 -> Function@8x, y, T0, T1<, vsymbolList.possibleTerms •• EvaluatedD;
```

Substituting **u1sol** and **v1sol** into **order2Equ** and **order2Eqv**, collecting coefficients of **possibleTerms**, and solving for the **usymbolList** and **vsymbolList**, we obtain

```
algEq1 = Coefficient@Subtract @@ order2Equ •. u1sol •. v1sol, possibleTermsD == 0 •.
Exp@_ T0D -> 0 •• Thread; •• Timing
```

```
867.45 Second, Null<
```

```
algEq2 = Coefficient@Subtract @@ order2Eqv •. u1sol •. v1sol, possibleTermsD == 0 •.
```

```
Exp@_T0D -> 0 •• Thread; •• Timing
```

```
865.91 Second, Null<
```

```
symbolRule1 = Solve@Join@algEq1, algEq2D, Join@symbolList, vsymbolListDD@1DD •• Simplify
```

```
: c1 ® -  $\frac{p H m^2 - n^2 n L A_1 @ T_1 D^2}{16 m}$ , c2 ® -  $\frac{p H m^2 - n^2 n L A_1 @ T_1 D^2}{16 m}$ , c3 ® -  $\frac{p H n^2 - m^2 n L A_2 @ T_1 D^2}{16 n}$ ,
c4 ® -  $\frac{p H n^2 - m^2 n L A_2 @ T_1 D^2}{16 n}$ , c33 ® -  $\frac{p H m^2 - n^2 n L A_1 @ T_1 D A_1 @ T_1 D}{8 m}$ , c34 ® -  $\frac{p H n^2 - m^2 n L A_2 @ T_1 D A_2 @ T_1 D}{8 n}$ ,
d5 ® -  $\frac{p H m^2 - n^2 n L A_1 @ T_1 D^2}{16 m}$ , d6 ® -  $\frac{p H m^2 - n^2 n L A_2 @ T_1 D^2}{16 m}$ , d7 ® -  $\frac{p H n^2 - m^2 n L A_1 @ T_1 D^2}{16 n}$ ,
d8 ® -  $\frac{p H n^2 - m^2 n L A_2 @ T_1 D^2}{16 n}$ , d35 ® -  $\frac{p H m^2 - n^2 n L A_2 @ T_1 D A_2 @ T_1 D}{8 m}$ , d36 ® -  $\frac{p H n^2 - m^2 n L A_1 @ T_1 D A_1 @ T_1 D}{8 n}$ ,
c5 ® 0, c6 ® 0, c7 ® 0, c8 ® 0, c35 ® 0, c36 ® 0, d1 ® 0, d2 ® 0, d3 ® 0, d4 ® 0,
d33 ® 0, d34 ® 0, c9 ®  $\frac{1}{32} n p A_2 @ T_1 D^2$ , c10 ®  $\frac{1}{32} n p A_2 @ T_1 D^2$ , c11 ®  $\frac{1}{32} n p A_2 @ T_1 D^2$ ,
c12 ®  $\frac{1}{32} n p A_2 @ T_1 D^2$ , c13 ®  $\frac{1}{32} m p A_1 @ T_1 D^2$ , c14 ®  $\frac{1}{32} m p A_1 @ T_1 D^2$ , c31 ®  $\frac{1}{32} m p A_1 @ T_1 D^2$ ,
c32 ®  $\frac{1}{32} m p A_1 @ T_1 D^2$ , c37 ®  $\frac{1}{16} n p A_2 @ T_1 D A_2 @ T_1 D$ , c38 ®  $\frac{1}{16} n p A_2 @ T_1 D A_2 @ T_1 D$ ,
c39 ®  $\frac{1}{16} m p A_1 @ T_1 D A_1 @ T_1 D$ , c48 ®  $\frac{1}{16} m p A_1 @ T_1 D A_1 @ T_1 D$ , d9 ® -  $\frac{1}{32} m p A_2 @ T_1 D^2$ ,
d10 ® -  $\frac{1}{32} m p A_2 @ T_1 D^2$ , d11 ®  $\frac{1}{32} m p A_2 @ T_1 D^2$ , d12 ®  $\frac{1}{32} m p A_2 @ T_1 D^2$ , d13 ® -  $\frac{1}{32} n p A_1 @ T_1 D^2$ ,
d14 ® -  $\frac{1}{32} n p A_1 @ T_1 D^2$ , d31 ®  $\frac{1}{32} n p A_1 @ T_1 D^2$ , d32 ®  $\frac{1}{32} n p A_1 @ T_1 D^2$ , d37 ® -  $\frac{1}{16} m p A_2 @ T_1 D A_2 @ T_1 D$ ,
d38 ®  $\frac{1}{16} m p A_2 @ T_1 D A_2 @ T_1 D$ , d39 ® -  $\frac{1}{16} n p A_1 @ T_1 D A_1 @ T_1 D$ , d48 ®  $\frac{1}{16} n p A_1 @ T_1 D A_1 @ T_1 D$ ,
c15 ®  $\frac{1}{32} H m - n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ , c16 ®  $\frac{1}{32} H m - n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ ,
c21 ®  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ , c22 ®  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ ,
c25 ®  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ , c26 ®  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ ,
c27 ®  $\frac{1}{32} H m - n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ , c28 ®  $\frac{1}{32} H m - n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ ,
c40 ®  $\frac{1}{32} H m - n L p H - 1 + n L H A_2 @ T_1 D A_1 @ T_1 D + A_1 @ T_1 D A_2 @ T_1 D L$ ,
c43 ®  $\frac{1}{32} H m + n L p H - 1 + n L H A_2 @ T_1 D A_1 @ T_1 D + A_1 @ T_1 D A_2 @ T_1 D L$ ,
c45 ®  $\frac{1}{32} H m + n L p H - 1 + n L H A_2 @ T_1 D A_1 @ T_1 D + A_1 @ T_1 D A_2 @ T_1 D L$ ,
c46 ®  $\frac{1}{32} H m - n L p H - 1 + n L H A_2 @ T_1 D A_1 @ T_1 D + A_1 @ T_1 D A_2 @ T_1 D L$ ,
d15 ® -  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ , d16 ® -  $\frac{1}{32} H m + n L p H - 1 + n L A_1 @ T_1 D A_2 @ T_1 D$ ,
```

$$\begin{aligned}
d_{21} &\otimes \frac{1}{32} \text{Hm} - nL p H - 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D, & d_{22} &\otimes \frac{1}{32} \text{Hm} - nL p H - 1 + nL A_1 @ T_1 D A_2 @ T_1 D, \\
d_{25} &\otimes - \frac{1}{32} \text{Hm} - nL p H - 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D, & d_{26} &\otimes - \frac{1}{32} \text{Hm} - nL p H - 1 + nL A_1 @ T_1 D A_2 @ T_1 D, \\
d_{27} &\otimes \frac{1}{32} \text{Hm} + nL p H - 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D, & d_{28} &\otimes \frac{1}{32} \text{Hm} + nL p H - 1 + nL A_1 @ T_1 D A_2 @ T_1 D, \\
d_{40} &\otimes - \frac{1}{32} \text{Hm} + nL p H - 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL, \\
d_{43} &\otimes \frac{1}{32} \text{Hm} - nL p H - 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL, \\
d_{45} &\otimes - \frac{1}{32} \text{Hm} - nL p H - 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL, \\
d_{46} &\otimes \frac{1}{32} \text{Hm} + nL p H - 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL, \\
C_{17} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} + nL}, \\
C_{18} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL A_1 @ T_1 D A_2 @ T_1 D}{32 \text{Hm} + nL}, \\
C_{19} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} - nL}, \\
C_{20} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL A_1 @ T_1 D A_2 @ T_1 D}{32 \text{Hm} - nL}, \\
C_{23} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} - nL}, \\
C_{24} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL A_1 @ T_1 D A_2 @ T_1 D}{32 \text{Hm} - nL}, \\
C_{29} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} + nL}, \\
C_{30} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL A_1 @ T_1 D A_2 @ T_1 D}{32 \text{Hm} + nL}, \\
C_{41} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL}{32 \text{Hm} + nL}, \\
C_{42} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL}{32 \text{Hm} - nL}, \\
C_{44} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL}{32 \text{Hm} - nL}, \\
C_{47} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL HA_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 DL}{32 \text{Hm} + nL}, \\
d_{17} &\otimes \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} + nL}, \\
d_{18} &\otimes \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL - 2 m n H 1 + nL A_1 @ T_1 D A_2 @ T_1 D}{32 \text{Hm} + nL}, \\
d_{19} &\otimes - \frac{p \text{Hm}^2 H - 1 + nL + n^2 H - 1 + nL + 2 m n H 1 + nL \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 \text{Hm} - nL},
\end{aligned}$$

$$\begin{aligned}
d_{20} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L A_1 @ T_1 D A_2 @ T_1 D}{32 H m - n L}, \\
d_{23} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 H m - n L}, \\
d_{24} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L A_1 @ T_1 D A_2 @ T_1 D}{32 H m - n L}, \\
d_{29} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L \dot{A}_1 @ T_1 D \dot{A}_2 @ T_1 D}{32 H m + n L}, \\
d_{30} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L A_1 @ T_1 D A_2 @ T_1 D}{32 H m + n L}, \\
d_{41} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L H A_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 D L}{32 H m + n L}, \\
d_{42} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L H A_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 D L}{32 H m - n L}, \\
d_{44} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L H A_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 D L}{32 H m - n L}, \\
d_{47} &= \frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L H A_2 @ T_1 D \dot{A}_1 @ T_1 D + A_1 @ T_1 D \dot{A}_2 @ T_1 D L}{32 H m + n L} >
\end{aligned}$$

8.3.3 Solvability Conditions

Substituting the **w1sol**, **u1sol**, and **v1sol** into **eqEpsm[[3]]**, we obtain

```

order3Eq = Horder1Eq@@1DD •. w1 -> w2L ==
  HHorder1Eq@@1DD •. w1 -> w2L - eqEpsm@@3, 1DD •. w1sol •. u1sol •. v1sol •. modeshapes ••
  ExpandL; •• Timing

84.18 Second, Null<

```

The associated boundary conditions are

```

order3BC = bcEpsw@@2DD

9w2@0, y, T0, T1D == 0, w2^H2,0,0,0L@0, y, T0, T1D == 0,
w2@1, y, T0, T1D == 0, w2^H2,0,0,0L@1, y, T0, T1D == 0, w2@x, 0, T0, T1D == 0,
w2^H0,2,0,0L@x, 0, T0, T1D == 0, w2@x, 1, T0, T1D == 0, w2^H0,2,0,0L@x, 1, T0, T1D == 0=

```

To describe the nearness of the primary resonance, we introduce the detuning parameter s_2 defined by

```

ResonanceCond = 8W -> w_m, n + e s2<;

```

Because the homogeneous parts of **order3Eq** and **order3BC** have a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. These conditions demand that the right-hand side of **order3Eq** be orthogonal to every solution of the adjoint homogeneous problem. In this case, the problem is self-adjoint.

To determine the solvability conditions, we collect the terms that may lead to secular terms, the terms proportional to $E^{I_{w_m, n} T_0}$. To this end, we use the rule

```
expRule = Exp@a_D :=> Exp@Expand@a •. ResonanceCondD •. e T0 -> T1D;
```

and obtain

```
ST = CoefficientAorder3Eq@@2DD •. expRule, EIwm, n T0E •. TrigReduce; •. Timing
8132.53 Second, Null<
```

Among all the spatial-dependent non-forcing terms in **ST**, only the following terms are not orthogonal to the **modeshapes**:

```
terms1 = Cases@ST, Cos@m p x + a_. n p y •; Abs@aD == 1D, ¥D •. Union
8Cos@m p x - n p yD, Cos@m p x + n p yD<
terms2 = Cases@ST, Cos@n p x + a_. m p y •; Abs@aD == 1D, ¥D •. Union
8Cos@n p x - m p yD, Cos@n p x + m p yD<
```

The coefficients of these terms in **ST** are

```
terms1Coef = Coefficient@ST, terms1D;
terms2Coef = Coefficient@ST, terms2D;
```

The projection of **terms1** onto $\sin@m p x \sin@n p y$ can be obtained as

```
trigRule = 8Cos@2_. pD -> 1, Sin@_. pD -> 0<;
proj1Coef =  $\hat{a}_0^1 \hat{a}_0^1 \sin@m p x \sin@n p y \# \hat{a} x \hat{a} y \& \cdot \sum \text{terms1} \cdot \text{trigRule}$ 
 $9 \frac{1}{4}, -\frac{1}{4} =$ 
```

and the projection of **terms2** onto $\sin@n p x \sin@m p y$ can be obtained as

```
proj2Coef =  $\hat{a}_0^1 \hat{a}_0^1 \sin@n p x \sin@m p y \# \hat{a} x \hat{a} y \& \cdot \sum \text{terms2} \cdot \text{trigRule}$ 
 $9 \frac{1}{4}, -\frac{1}{4} =$ 
```

The forcing term in **ST** is

```
f0 = ST •. Cos@_D -> 0
 $\frac{1}{24} E^{I_{T_1} s_2} F@x, yD$ 
```

Then, the solvability conditions demand that **ST** be orthogonal to every linear eigenfunction, **modeshapes**; that is,

```

intRule3 = 8int@a_fun_, a1_, b1_D := a int@fun, a1, b1D •; FreeQ@a, First@a1D & First@b1DD<;

SCond1 =
  Expand@24 Hint@Sin@m p xD Sin@n p yD f0, 8x, 0, 1<, 8y, 0, 1<D + proj1Coef.terms1Coef •.
  symbolRule1 ••. intRule3 •. int -> IntegrateLD == 0;

SCond2 =
  Expand@24 Hint@Sin@n p xD Sin@m p yD f0, 8x, 0, 1<, 8y, 0, 1<D + proj2Coef.terms2Coef •.
  symbolRule1 ••. intRule3 •. int -> IntegrateLD == 0;

```

To express these solvability conditions in a more readable form, we define

```

basicTerms = 9A1@T1D EI Wm, n T0, A1@T1D E-I Wm, n T0, A2@T1D EI Wm, n T0, A2@T1D E-I Wm, n T0;

cubicTerms =
  I E-I Wm, n T0 Nest@Outer@Times, basicTerms, #D &, basicTerms, 2D •• FlattenM •. Exp@_ T0D -> 0 ••
  Union •• Rest

8A1@T1D2 A1@T1D, A1@T1D A2@T1D A1@T1D, A2@T1D2 A1@T1D,
A1@T1D2 A2@T1D, A1@T1D A2@T1D A2@T1D, A2@T1D2 A2@T1D<

```

Then, we collect the coefficients of **cubicTerms** from **SCond1** and **SCond2** and obtain

```

coef1 = Coefficient@SCond1@@1DD, cubicTermsD •• Simplify

9/8 p4 H-4 m2 n2 n + m4 H-3 + n2L + n4 H-3 + n2LL, 0,
3/8 p4 H4 m2 n2 H-2 + n2L + m4 H-1 - 2 n + n2L + n4 H-1 - 2 n + n2LL, 0,
3/4 p4 H4 m2 n2 H-2 + n2L + m4 H-1 - 2 n + n2L + n4 H-1 - 2 n + n2LL, 0=

coef2 = Coefficient@SCond2@@1DD, cubicTermsD •• Simplify

90, 3/4 p4 H4 m2 n2 H-2 + n2L + m4 H-1 - 2 n + n2L + n4 H-1 - 2 n + n2LL,
0, 3/8 p4 H4 m2 n2 H-2 + n2L + m4 H-1 - 2 n + n2L + n4 H-1 - 2 n + n2LL,
0, 9/8 p4 H-4 m2 n2 n + m4 H-3 + n2L + n4 H-3 + n2LL=

```

Hence, **SCond1** and **SCond2** can be rewritten and combined as

```

SCond = 8Collect@SCond1@@1DD •. Thread@cubicTerms -> 0D, s1 A1@T1DD + coef1.cubicTerms == 0,
Collect@SCond2@@1DD •. Thread@cubicTerms -> 0D, s1 A2@T1DD + coef2.cubicTerms == 0<;
SCond •. displayRule1

```

$$\begin{aligned}
& :E^{T_1 s_2} \hat{a}_0^1 \hat{a}_0^1 F(x, y) \sin(m p x) \sin(n p y) \hat{a} y \hat{a} x + \\
& H - 2 m^2 n^2 p^4 - 2 n^4 p^4 L A_1 s_1 + \frac{9}{8} p^4 H - 4 m^2 n^2 n + m^4 H - 3 + n^2 L + n^4 H - 3 + n^2 L L A_1^2 \dot{A}_1 + \\
& \frac{3}{8} p^4 H 4 m^2 n^2 H - 2 + n^2 L + m^4 H - 1 - 2 n + n^2 L + n^4 H - 1 - 2 n + n^2 L L A_2^2 \dot{A}_1 + \\
& \frac{3}{4} p^4 H 4 m^2 n^2 H - 2 + n^2 L + m^4 H - 1 - 2 n + n^2 L + n^4 H - 1 - 2 n + n^2 L L A_1 A_2 \dot{A}_2 - I H D_1 A_1 L w_{m,n} == 0, \\
& E^{T_1 s_2} \hat{a}_0^1 \hat{a}_0^1 F(x, y) \sin(n p x) \sin(m p y) \hat{a} y \hat{a} x + H - 2 m^4 p^4 - 2 m^2 n^2 p^4 L A_2 s_1 + \\
& \frac{3}{4} p^4 H 4 m^2 n^2 H - 2 + n^2 L + m^4 H - 1 - 2 n + n^2 L + n^4 H - 1 - 2 n + n^2 L L A_1 A_2 \dot{A}_1 + \\
& \frac{3}{8} p^4 H 4 m^2 n^2 H - 2 + n^2 L + m^4 H - 1 - 2 n + n^2 L + n^4 H - 1 - 2 n + n^2 L L A_1^2 \dot{A}_2 + \\
& \frac{9}{8} p^4 H - 4 m^2 n^2 n + m^4 H - 3 + n^2 L + n^4 H - 3 + n^2 L L A_2^2 \dot{A}_2 - I H D_1 A_2 L w_{m,n} == 0>
\end{aligned}$$

§ 8.3.4 Mixed Approach

Alternatively, we can use a mixed approach, a combination of discretization and direct approaches, to attack the problem, **EOM1** and **BC1**. We first define

```

scaleRule3 = 9Hdv : u ∈ vL -> I e^2 dv@#1, #2, #3D &M, w -> He w@#1, #2, #3D &L=;

```

Substituting **scaleRule2** and **scaleRule3** into **EOM1**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```

eq83b = HEOM1 •. scaleRule2 •. scaleRule3 •• TrigToExp •• ExpandAllL •. e^N; #3 -> 0;

```

To write **eq83b** in a more concise way, we introduce the display rule

```

display2@expr_D := expr •. 8Derivative@a_, b_, c_D@w_D@x, y, tD :=>
Subscript@w, Sequence @@ Join@Table@x, 8a<D, Table@y, 8b<D, Table@t, 8c<DDD<

```

Hence,

```

IuEOM = CoefficientAeq83b@@1, 1DD, e^2E == 0M •• display2

```

$$u_{x,x} + \frac{u_{y,y}}{2} - \frac{1}{2} n u_{y,y} + \frac{v_{x,y}}{2} + \frac{1}{2} n v_{x,y} + w_x w_{x,x} + \frac{1}{2} w_y w_{x,y} + \frac{1}{2} n w_y w_{x,y} + \frac{1}{2} w_x w_{y,y} - \frac{1}{2} n w_x w_{y,y} == 0$$

```

IvEOM = CoefficientAeq83b@@2, 1DD, e^2E == 0M •• display2

```

$$\frac{u_{x,y}}{2} + \frac{1}{2} n u_{x,y} + \frac{v_{x,x}}{2} - \frac{1}{2} n v_{x,x} + v_{y,y} + \frac{1}{2} w_y w_{x,x} - \frac{1}{2} n w_y w_{x,x} + \frac{1}{2} w_x w_{x,y} + \frac{1}{2} n w_x w_{x,y} + w_y w_{y,y} == 0$$

lweom = eq83b@@3DD •• ThreadAe⁻²#, EqualE & •• ExpandAllM •• display2

$$\begin{aligned}
 & -\frac{1}{24} E^{-I T_0 W} e F @ x, y D - \frac{1}{24} E^{I T_0 W} e F @ x, y D + \frac{W_{t,t}}{12} - e u_x w_{x,x} - \\
 & e n v_y w_{x,x} - \frac{1}{2} e w_x^2 w_{x,x} - \frac{1}{2} e n w_y^2 w_{x,x} - e u_y w_{x,y} + e n u_y w_{x,y} - e v_x w_{x,y} + e n v_x w_{x,y} - \\
 & e w_x w_y w_{x,y} + e n w_x w_y w_{x,y} - e n u_x w_{y,y} - e v_y w_{y,y} - \frac{1}{2} e n w_x^2 w_{y,y} - \frac{1}{2} e w_y^2 w_{y,y} + \\
 & \frac{1}{12} w_{x,x,x,x} + \frac{1}{6} w_{x,x,y,y} + \frac{1}{3} e s_1 w_{x,x,y,y} + \frac{1}{12} w_{y,y,y,y} + \frac{1}{3} e s_1 w_{y,y,y,y} == 0
 \end{aligned}$$

Linear vibrations are governed by **wEOM** with $e = 0$; that is,

Horder1Eqw = wEOM •. e -> 0L •• display2

$$\frac{W_{t,t}}{12} + \frac{1}{12} w_{x,x,x,x} + \frac{1}{6} w_{x,x,y,y} + \frac{1}{12} w_{y,y,y,y} == 0$$

subject to the boundary conditions **BC1**. It follows from **order1Eqw** and **BC1** that the linear modeshapes and associated frequencies are given by

f_{m,n}@x, yD •. modeshapes

$$\sin @ m p x D \sin @ n p y D$$

w_{m,n} •. frequencies

$$H m^2 + n^2 L p^2$$

To investigate the case of one-to-one internal resonance between the m th and n th modes ($m \approx n$), we let

$$w_{sol} = 8w \rightarrow \text{Function} @ 8x, y, t \langle, h @ t D \sin @ m p x D \sin @ n p y D + z @ t D \sin @ n p x D \sin @ m p y D \rangle;$$

Substituting the **w1sol** into **uEOM** and **vEOM**, we obtain

order2Equ =

HuEOM@@1DD •. w -> H0 &LL == HHuEOM@@1DD •. w -> H0 &LL - uEOM@@1DD •. w1sol •• TrigReduceL;
order2Equ •• display2

$$\begin{aligned}
 & u_{x,x} + \frac{u_{y,y}}{2} - \frac{1}{2} n u_{y,y} + \frac{v_{x,y}}{2} + \frac{1}{2} n v_{x,y} == \\
 & \frac{1}{16} H^4 n^3 p^3 \sin @ 2 n p x D z @ t D^2 - 4 m^2 n p^3 n \sin @ 2 n p x D z @ t D^2 - \\
 & 2 m^2 n p^3 \sin @ 2 n p x - 2 m p y D z @ t D^2 - 2 n^3 p^3 \sin @ 2 n p x - 2 m p y D z @ t D^2 - \\
 & 2 m^2 n p^3 \sin @ 2 n p x + 2 m p y D z @ t D^2 - 2 n^3 p^3 \sin @ 2 n p x + 2 m p y D z @ t D^2 + \\
 & m^3 p^3 \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D - m^2 n p^3 \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D + \\
 & m n^2 p^3 \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D - n^3 p^3 \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D - \\
 & m^3 p^3 n \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D + m^2 n p^3 n \sin @ m p x - n p x - m p y - n p y D \\
 & z @ t D h @ t D - m n^2 p^3 n \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D + \\
 & n^3 p^3 n \sin @ m p x - n p x - m p y - n p y D z @ t D h @ t D - m^3 p^3 \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D - \\
 & 3 m^2 n p^3 \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D -
 \end{aligned}$$

$$\begin{aligned}
& 3 m n^2 p^3 \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D - \\
& n^3 p^3 \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D + m^3 p^3 n \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D - \\
& m^2 n p^3 n \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D - \\
& m n^2 p^3 n \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D + \\
& n^3 p^3 n \sin @ m p x + n p x - m p y - n p y D z @ t D h @ t D - m^3 p^3 \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D + \\
& 3 m^2 n p^3 \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D - \\
& 3 m n^2 p^3 \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D + \\
& n^3 p^3 \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D + m^3 p^3 n \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D + \\
& m^2 n p^3 n \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D - m n^2 p^3 n \sin @ m p x - n p x + m p y - n p y D \\
& z @ t D h @ t D - n^3 p^3 n \sin @ m p x - n p x + m p y - n p y D z @ t D h @ t D + \\
& m^3 p^3 \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D + m^2 n p^3 \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D + \\
& m n^2 p^3 \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D + n^3 p^3 \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D - \\
& m^3 p^3 n \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D - m^2 n p^3 n \sin @ m p x + n p x + m p y - n p y D \\
& z @ t D h @ t D - m n^2 p^3 n \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D - \\
& n^3 p^3 n \sin @ m p x + n p x + m p y - n p y D z @ t D h @ t D - m^3 p^3 \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D + \\
& 3 m^2 n p^3 \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D - \\
& 3 m n^2 p^3 \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D + \\
& n^3 p^3 \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D + m^3 p^3 n \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D + \\
& m^2 n p^3 n \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D - \\
& m n^2 p^3 n \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D - \\
& n^3 p^3 n \sin @ m p x - n p x - m p y + n p y D z @ t D h @ t D + m^3 p^3 \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D + \\
& m^2 n p^3 \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D + \\
& m n^2 p^3 \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D + n^3 p^3 \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D - \\
& m^3 p^3 n \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D - m^2 n p^3 n \sin @ m p x + n p x - m p y + n p y D \\
& z @ t D h @ t D - m n^2 p^3 n \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D - \\
& n^3 p^3 n \sin @ m p x + n p x - m p y + n p y D z @ t D h @ t D + m^3 p^3 \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D - \\
& m^2 n p^3 \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D + \\
& m n^2 p^3 \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D - n^3 p^3 \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D - \\
& m^3 p^3 n \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D + m^2 n p^3 n \sin @ m p x - n p x + m p y + n p y D \\
& z @ t D h @ t D - m n^2 p^3 n \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D + \\
& n^3 p^3 n \sin @ m p x - n p x + m p y + n p y D z @ t D h @ t D - m^3 p^3 \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& 3 m^2 n p^3 \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& 3 m n^2 p^3 \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& n^3 p^3 \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D + m^3 p^3 n \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& m^2 n p^3 n \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& m n^2 p^3 n \sin @ m p x + n p x + m p y + n p y D z @ t D h @ t D + n^3 p^3 n \sin @ m p x + n p x + m p y + n p y D \\
& z @ t D h @ t D + 4 m^3 p^3 \sin @ 2 m p x D h @ t D^2 - 4 m n^2 p^3 n \sin @ 2 m p x D h @ t D^2 - \\
& 2 m^3 p^3 \sin @ 2 m p x - 2 n p y D h @ t D^2 - 2 m n^2 p^3 \sin @ 2 m p x - 2 n p y D h @ t D^2 - \\
& 2 m^3 p^3 \sin @ 2 m p x + 2 n p y D h @ t D^2 - 2 m n^2 p^3 \sin @ 2 m p x + 2 n p y D h @ t D^2 L
\end{aligned}$$

order2Eqv =

HvEOM@@1DD •. w -> H0 &LL == HHvEOM@@1DD •. w -> H0 &LL - vEOM@@1DD •. w1sol •• TrigReduceL;

order2Eqv •• display2

$$\frac{u_{x,y}}{2} + \frac{1}{2} n u_{x,y} + \frac{v_{x,x}}{2} - \frac{1}{2} n v_{x,x} + v_{y,y} ==$$

$$\begin{aligned} & \frac{1}{16} H^4 m^3 p^3 \sin(2 m p y D z @ t D^2 - 4 m n^2 p^3 n \sin(2 m p y D z @ t D^2 + \\ & 2 m^3 p^3 \sin(2 n p x - 2 m p y D z @ t D^2 + 2 m n^2 p^3 \sin(2 n p x - 2 m p y D z @ t D^2 - \\ & 2 m^3 p^3 \sin(2 n p x + 2 m p y D z @ t D^2 - 2 m n^2 p^3 \sin(2 n p x + 2 m p y D z @ t D^2 - \\ & m^3 p^3 \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D - m^2 n p^3 \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D - \\ & m n^2 p^3 \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D - n^3 p^3 \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D + \\ & m^3 p^3 n \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D + m^2 n p^3 n \sin(m p x - n p x - m p y - n p y D \\ & z @ t D h @ t D + m n^2 p^3 n \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D + \\ & n^3 p^3 n \sin(m p x - n p x - m p y - n p y D z @ t D h @ t D + m^3 p^3 \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D + \\ & 3 m^2 n p^3 \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D + \\ & 3 m n^2 p^3 \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D + \\ & n^3 p^3 \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D - m^3 p^3 n \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D + \\ & m^2 n p^3 n \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D + \\ & m n^2 p^3 n \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D - \\ & n^3 p^3 n \sin(m p x + n p x - m p y - n p y D z @ t D h @ t D - m^3 p^3 \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D + \\ & 3 m^2 n p^3 \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D - \\ & 3 m n^2 p^3 \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D + \\ & n^3 p^3 \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D + m^3 p^3 n \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D + \\ & m^2 n p^3 n \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D - m n^2 p^3 n \sin(m p x - n p x + m p y - n p y D \\ & z @ t D h @ t D - n^3 p^3 n \sin(m p x - n p x + m p y - n p y D z @ t D h @ t D + \\ & m^3 p^3 \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D - m^2 n p^3 \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D + \\ & m n^2 p^3 \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D - n^3 p^3 \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D - \\ & m^3 p^3 n \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D + m^2 n p^3 n \sin(m p x + n p x + m p y - n p y D \\ & z @ t D h @ t D - m n^2 p^3 n \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D + \\ & n^3 p^3 n \sin(m p x + n p x + m p y - n p y D z @ t D h @ t D + m^3 p^3 \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D - \\ & 3 m^2 n p^3 \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D + \\ & 3 m n^2 p^3 \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D - \\ & n^3 p^3 \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D - m^3 p^3 n \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D - \\ & m^2 n p^3 n \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D + \\ & m n^2 p^3 n \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D + \\ & n^3 p^3 n \sin(m p x - n p x - m p y + n p y D z @ t D h @ t D - m^3 p^3 \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D + \\ & m^2 n p^3 \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D - \\ & m n^2 p^3 \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D + n^3 p^3 \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D + \\ & m^3 p^3 n \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D - m^2 n p^3 n \sin(m p x + n p x - m p y + n p y D \\ & z @ t D h @ t D + m n^2 p^3 n \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D - \\ & n^3 p^3 n \sin(m p x + n p x - m p y + n p y D z @ t D h @ t D + m^3 p^3 \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D + \\ & m^2 n p^3 \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D + \\ & m n^2 p^3 \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D + n^3 p^3 \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D - \\ & m^3 p^3 n \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D - m^2 n p^3 n \sin(m p x - n p x + m p y + n p y D \\ & z @ t D h @ t D - m n^2 p^3 n \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D - \\ & n^3 p^3 n \sin(m p x - n p x + m p y + n p y D z @ t D h @ t D - m^3 p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D - \\ & 3 m^2 n p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D - \\ & 3 m n^2 p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D - \\ & n^3 p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D + m^3 p^3 n \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D - \end{aligned}$$

$$\begin{aligned}
& m^2 n p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D - \\
& m n^2 p^3 \sin(m p x + n p x + m p y + n p y D z @ t D h @ t D + n^3 p^3 \sin(m p x + n p x + m p y + n p y D \\
& \quad z @ t D h @ t D + 4 n^3 p^3 \sin(2 n p y D h @ t D^2 - 4 m^2 n p^3 \sin(2 n p y D h @ t D^2 + \\
& 2 m^2 n p^3 \sin(2 m p x - 2 n p y D h @ t D^2 + 2 n^3 p^3 \sin(2 m p x - 2 n p y D h @ t D^2 - \\
& 2 m^2 n p^3 \sin(2 m p x + 2 n p y D h @ t D^2 - 2 n^3 p^3 \sin(2 m p x + 2 n p y D h @ t D^2 L
\end{aligned}$$

To obtain the particular solutions for u and v , we use the method of undetermined coefficients. To accomplish this, we first look for all possible terms that appear on the right-hand sides of **order2Equ** and **order2Eqv** as follows:

```
possibleTerms =
Cases@8order2Equ@@2DD, order2Eqv@@2DD<, #, ¥D & .ž 9_ Sin@a_D h1_@tD^2 -> Sin@aD h1@tD^2,
_ Sin@a_D h1_@tD h2_@tD -> Sin@aD h1@tD h2@tD= •• Flatten •• Union

8Sin@2 n p xD z@tD^2, Sin@2 m p yD z@tD^2, Sin@2 n p x - 2 m p yD z@tD^2, Sin@2 n p x + 2 m p yD z@tD^2,
Sin@m p x - n p x - m p y - n p yD z@tD h@tD, Sin@m p x + n p x - m p y - n p yD z@tD h@tD,
Sin@m p x - n p x + m p y - n p yD z@tD h@tD, Sin@m p x + n p x + m p y - n p yD z@tD h@tD,
Sin@m p x - n p x - m p y + n p yD z@tD h@tD, Sin@m p x + n p x - m p y + n p yD z@tD h@tD,
Sin@m p x - n p x + m p y + n p yD z@tD h@tD, Sin@m p x + n p x + m p y + n p yD z@tD h@tD,
Sin@2 m p xD h@tD^2, Sin@2 n p yD h@tD^2, Sin@2 m p x - 2 n p yD h@tD^2, Sin@2 m p x + 2 n p yD h@tD^2<
```

which satisfy **BC1**. We then define the following symbol lists to represent the undetermined coefficients.

```
usymbolList = Table@c_i, 8i, Length@possibleTermsD<D

8C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_10, C_11, C_12, C_13, C_14, C_15, C_16<

vsymbolList = usymbolList •. c -> d

8d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_10, d_11, d_12, d_13, d_14, d_15, d_16<
```

Then, we express the solution as

```
usol = u -> Function@8x, y, t<, usymbolList.possibleTerms •• Evaluated;
vsol = v -> Function@8x, y, t<, vsymbolList.possibleTerms •• Evaluated;
```

Substituting **usol** and **vsol** into **order2Equ** and **order2Eqv**, collecting coefficients of **possibleTerms**, and solving for **usymbolList** and **vsymbolList**, we obtain

```
algEq1 = Coefficient@Subtract žž order2Equ •. usol •. vsol, possibleTermsD == 0 •• Thread;
algEq2 = Coefficient@Subtract žž order2Eqv •. usol •. vsol, possibleTermsD == 0 •• Thread;
```

HsymbolRule2 =

```
Solve@Join@algEq1, algEq2D, Join@usymbolList, vsymbolListDD@@1DD •• SimplifyL •• Timing
:2.48 Second, :C1 ® -  $\frac{p H n^2 - m^2 n L}{16 n}$ , C13 ® -  $\frac{p H m^2 - n^2 n L}{16 m}$ , d2 ® -  $\frac{p H m^2 - n^2 n L}{16 m}$ , d14 ® -  $\frac{p H n^2 - m^2 n L}{16 n}$ ,
C2 ® 0, C14 ® 0, d1 ® 0, d13 ® 0, C3 ®  $\frac{n p}{32}$ , C4 ®  $\frac{n p}{32}$ , C15 ®  $\frac{m p}{32}$ , C16 ®  $\frac{m p}{32}$ , d3 ® -  $\frac{m p}{32}$ ,
d4 ®  $\frac{m p}{32}$ , d15 ® -  $\frac{n p}{32}$ , d16 ®  $\frac{n p}{32}$ , C7 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L}{32 H m - n L}$ ,
C12 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L}{32 H m + n L}$ ,
d7 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L}{32 H m - n L}$ ,
d12 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L}{32 H m + n L}$ ,
C5 ®  $\frac{1}{32} H m - n L p H - 1 + n L$ , C6 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L}{32 H m + n L}$ ,
C8 ®  $\frac{1}{32} H m + n L p H - 1 + n L$ , C9 ® -  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L}{32 H m - n L}$ ,
C10 ®  $\frac{1}{32} H m + n L p H - 1 + n L$ , C11 ®  $\frac{1}{32} H m - n L p H - 1 + n L$ ,
d5 ® -  $\frac{1}{32} H m + n L p H - 1 + n L$ , d6 ®  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L - 2 m n H 1 + n L L}{32 H m + n L}$ ,
d8 ®  $\frac{1}{32} H m - n L p H - 1 + n L$ , d9 ®  $\frac{p H m^2 H - 1 + n L + n^2 H - 1 + n L + 2 m n H 1 + n L L}{32 H m - n L}$ ,
d10 ® -  $\frac{1}{32} H m - n L p H - 1 + n L$ , d11 ®  $\frac{1}{32} H m + n L p H - 1 + n L$ >>
```

Y Solvability Conditions

We introduce two time scales T_0 and T_1 and seek a first-order uniform asymptotic expansion of the solution of **wEOM** in the form

```
solRule = 9w -> FunctionA8x, y, T0, T1<, SumAej-1 wj@x, y, T0, T1D, 8j, 2<EE=;
```

Transforming the time derivatives in **wEOM** in terms of the scales T_0 and T_1 , substituting **solRule** into **wEOM**, expanding the result for small ϵ , and discarding terms of order higher than ϵ , we obtain

eq83c =

```
HwEOM •. 8Hs : u È v È wL@x, y, tD -> s@x, y, T0, T1D, Derivative@a_, b_, c_D@h_D@x, y, tD ->
dt@cD@D@h@x, y, T0, T1D, 8x, a<, 8y, b<DD< •. solRule •• ExpandAllL •. en_•; n>1 -> 0;
```

Equating coefficients of like powers of ϵ , we have

```
eqEps = CoefficientList@eq83c@@1DD, eD == 0 •• Thread;
```

Accounting for the two time scales T_0 and T_1 , we create a rule for $h(t)$ and $z(t)$

$$\text{ampRule} = 9h(t) \rightarrow A_1(t) D E^{I w_m, n T_0} + \dot{A}_1(t) D E^{-I w_m, n T_0}, \quad z(t) \rightarrow A_2(t) D E^{I w_m, n T_0} + \dot{A}_2(t) D E^{-I w_m, n T_0};$$

and rewrite **w1sol**, **usol**, and **vsol** as

$$\begin{aligned} \text{sol1} &= 8w_1 \rightarrow \text{Function}[8x, y, T_0, T_1, w[x, y, t] \cdot \text{w1sol} \cdot \text{ampRule} \cdot \text{Evaluated}, \\ u &\rightarrow \text{Function}[8x, y, T_0, T_1, u[x, y, t] \cdot \text{usol} \cdot \text{ampRule} \cdot \text{Evaluated}, \\ v &\rightarrow \text{Function}[8x, y, T_0, T_1, v[x, y, t] \cdot \text{vsol} \cdot \text{ampRule} \cdot \text{Evaluated}]; \end{aligned}$$

Substituting the **sol1** into the second-order problem, **eqEps[[2]]**, we obtain

$$\begin{aligned} \text{order2Eqw} &= \text{HeqEps}[[1, 1DD] \cdot w_1 \rightarrow w_2L == \text{HeqEps}[[1, 1DD] \cdot w_1 \rightarrow w_2L - \text{eqEps}[[2, 1DD] \cdot \text{sol1} \cdot \\ &\quad \text{ExpandAll}; \cdot \cdot \text{Timing} \\ &811.09 \text{ Second}, \text{Null}] \end{aligned}$$

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_m, n T_0}$, we have

$$\begin{aligned} \text{ST} &= \text{CoefficientAorder2Eqw}[[2DD] \cdot \text{expRule}, E^{I w_m, n T_0} E \cdot \cdot \text{TrigReduce}; \cdot \cdot \text{Timing} \\ &8156.76 \text{ Second}, \text{Null}] \end{aligned}$$

Among all the spatial-dependent non-forcing terms in **ST**, only the following terms are not orthogonal to the **modes** **hapes**:

$$\begin{aligned} \text{terms1} &= \text{Cases}[\text{ST}, \text{Cos}[m p x + a \cdot n p y]; \text{Abs}[aD] == 1D, \text{YD} \cdot \cdot \text{Union} \\ &8 \text{Cos}[m p x - n p yD], \text{Cos}[m p x + n p yD] \\ \text{terms2} &= \text{Cases}[\text{ST}, \text{Cos}[n p x + a \cdot m p y]; \text{Abs}[aD] == 1D, \text{YD} \cdot \cdot \text{Union} \\ &8 \text{Cos}[n p x - m p yD], \text{Cos}[n p x + m p yD] \end{aligned}$$

The coefficients of these terms in **ST** are

$$\begin{aligned} \text{terms1Coef} &= \text{Coefficient}[\text{ST}, \text{terms1D}; \\ \text{terms2Coef} &= \text{Coefficient}[\text{ST}, \text{terms2D}; \end{aligned}$$

The projection of **terms1** onto $\sin(m p x) \sin(n p y)$ can be obtained as

$$\begin{aligned} \text{p1Coef} &= \frac{1}{9} \frac{1}{4} \sin(m p x) \sin(n p y) \# \hat{a} x \hat{a} y \& \cdot \text{terms1} \cdot \text{trigRule} \\ &\frac{1}{9} \frac{1}{4}, -\frac{1}{4} = \end{aligned}$$

and the projection of **terms2** onto $\sin(n p x) \sin(m p y)$ can be obtained as

$$p2Coef = \frac{1}{a_0} \frac{1}{a_0} \sin(n p x) \sin(m p y) \# \hat{a} x \hat{a} y \& \cdot \check{z} \text{ terms2} \cdot \text{trigRule}$$

$$9 \frac{1}{4}, -\frac{1}{4} =$$

The forcing term in **ST** is

$$f_0 = ST \cdot \cos(_D) \rightarrow 0$$

$$\frac{1}{24} E^{I T_1 s_2} F(x, y)$$

Then, the solvability conditions demand that **ST** be orthogonal to every linear eigenfunction, **modeshapes**; that is,

$$SC1 = \text{Expand} @ 24 \text{ Hint} @ \sin(m p x) \sin(n p y) f_0, 8x, 0, 1<, 8y, 0, 1<D + p1Coef.terms1Coef \cdot \text{symbolRule2} \cdot \cdot \text{intRule3} \cdot \text{int} \rightarrow \text{IntegrateLD} == 0;$$

$$SC2 = \text{Expand} @ 24 \text{ Hint} @ \sin(m p x) \sin(n p y) f_0, 8x, 0, 1<, 8y, 0, 1<D + p2Coef.terms2Coef \cdot \text{symbolRule2} \cdot \cdot \text{intRule3} \cdot \text{int} \rightarrow \text{IntegrateLD} == 0;$$

They agree with the results obtained in the previous section:

$$8SC1, SC2 < === 8SCond1, SCond2 <$$

True

Chapter 9

Higher Approximations of Continuous Systems Having Two-to-One Internal Resonances

In this chapter, we use the method of multiple scales to determine second-order uniform asymptotic expansions of the solutions of continuous systems with quadratic and cubic nonlinearities possessing [two-to-one internal resonances](#). Higher-order treatments of such systems lead to inconsistent results if the time derivatives in their governing equations are expressed in second-order rather than first-order form. Therefore, we express the time derivatives in the governing equations in first-order form before treating them with the method of multiple scales. We describe the methodology using two examples: two-mode interactions in hinged-hinged buckled beams and four-mode interactions in suspended cables.

à Preliminaries

Off@General::spell1, Integrate::generD

To determine second-order uniform asymptotic expansions of the solutions of buckled beams and suspended cables by using the method of multiple scales, taking into account two-to-one internal resonances, we introduce the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, where ϵ is a small nondimensional parameter and the order of the amplitude of oscillations. Moreover, we symbolize these scales according to

Needs@"Utilities`Notation`"

Symbolize@T₀D; Symbolize@T₁D; Symbolize@T₂D;

timeScales = {T₀, T₁, T₂};

In terms of the time scales T_0 , T_1 , and T_2 , the time derivatives can be expressed as

dt@0D@expr_D := expr; dt@1D@expr_D := SumAeⁱ D@expr, timeScales@@i + 1DDD, {i, 0, 2}<E;

To speed up manipulations of the integrals involved in the governing equations with *Mathematica*, we introduce the rules

**intRule1 = {int@fun_, arg2__D := int@Expand@funD, arg2D<;
intRule2 = {int@a_ + b_, arg2__D := int@a, arg2D + int@b, arg2D,
int@eⁿ· fun_, arg2__D := eⁿ int@fun, arg2D,
int@a fun_, a1_, b1__D := a int@fun, a1, b1D •; FreeQ@a, First@a1DD,
int@int@a1_, a2_D fun_, a3__D := int@a1, a2D int@fun, a3D •; FreeQ@Rest@a2D, First@a2DD<;**

In the ensuing analysis, we express the solutions in terms of complex-variable functions and hence we introduce the complex-conjugate rule

```
conjugateRule = 9A_s_ :> A_s_, A_s_ :> A_s_, Complex@m_, n_D :> Complex@m, -nD;
```

To display the outputs in easily read expressions, we introduce the rules

```
displayRule = 9Derivative@a_, b_DAw_i_ E@_, T_0, __D :>
SequenceFormAIfAarg1 = Times @@ MapIndexedAD^#1_#2@@1DD-1 &, 8b<E; arg1 != 1, arg1, ""E,
w_i_ SequenceForm @@ Table["", 8a<DE,
Derivative@a_D@A_i_D@__D :> SequenceFormATimes @@ MapIndexedAD^#1_#2@@1DD &, 8a<E, A_iE,
Derivative@a_D@A_i_D@__D :> SequenceFormATimes @@ MapIndexedAD^#1_#2@@1DD &, 8a<E, A_iE,
w_i_@_, T_0, __D -> w_i, A_i_@__D -> A_i, A_i_@__D -> A_i,
Exp@a_. + b_. Complex@0, m_D T_0 + c_. Complex@0, n_D T_0D -> Exp@a + Hm * b + n * cL I T_0D=;
```

à 9.1 Two-Mode Interactions in Buckled Beams

In this section, we consider the nonlinear response of a hinged-hinged buckled beam possessing a two-to-one internal resonance to a principal parametric resonance of the higher mode. The analysis assumes a unimodal static buckled deflection. The nondimensional equation of motion is (Section 8.1.1)

$$\begin{aligned} \text{EOM} = & u_{t,t} + u_{x,x,x,x} + P_k u_{x,x} - 2b^2 a \int_0^1 u_x f_k^e(x) dx \hat{a} x \frac{1}{f_k^2(x)} f_k^2(x) dx = \\ & -c u_t + b a \int_0^1 u_x^2 \hat{a} x \frac{1}{f_k^2(x)} f_k^2(x) dx + 2b a \int_0^1 u_x f_k^e(x) dx \hat{a} x \frac{1}{f_k^2(x)} u_{x,x} + \\ & a \int_0^1 u_x^2 \hat{a} x \frac{1}{f_k^2(x)} u_{x,x} + \cos t \omega_D f u_{x,x} + b \cos t \omega_D f f_k^2(x) dx \cdot u_m :> \mathbb{1}_m u(x, tD); \end{aligned}$$

and the associated boundary conditions are

$$\text{BC} = 9u@0, tD == 0, u^{H2,0L}@0, tD == 0, u@1, tD == 0, u^{H2,0L}@1, tD == 0=;$$

where P_k is the k th critical Euler buckling load, f_k is the k th buckling mode shape, and b is the nondimensional buckling level.

Because the two-to-one internal resonance is activated by the quadratic nonlinearities, as indicated in Chapter 5, second- and higher-order treatments of such systems lead to inconsistent results if the time derivatives in their governing equations are expressed in second-order rather than first-order form. Therefore, we express the time derivatives in first-order form by letting $v(x, t) = \mathbb{1}_t u(x, t)$, adding it to **EOM**, and obtaining

$$\text{EOM1} = 8\mathbb{1}_t u(x, t) - v(x, t) == 0, \text{EOM} \cdot \mathbb{1}_{t,t} u(x, t) -> \mathbb{1}_t v(x, tD);$$

We use the method of multiple scales to directly attack **EOM1** and **BC**. To transform the time derivatives in **EOM1** in terms of the scales T_0 , T_1 , and T_2 , we define


```

multiScales = 8u@x_, tD -> u@x, T0, T1, T2D, v@x, tD -> v@x, T0, T1, T2D,
Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;

```

Then, we seek a second-order approximate solution of **EOM1** and **BC** in the form

```

solRule = h : u È v -> I EvaluateASumAej hj@#1, #2, #3, #4D, 8j, 3<EE &M;

```

where #1 stands for x and #2, #3, and #4 stand for T_0 , T_1 , and T_2 , respectively.

We consider the case of a two-to-one internal resonance between the lowest two modes and a principal parametric resonance of the second mode; that is,

```

omgList = 8w1, w2<;
ResonanceConds = 8w2 == 2 w1 + e s1, W == 2 w2 + e s2<;

```

where the S_i are detuning parameters that describe the nearness of the resonances. Because the influence of the two-to-one internal resonance appears at $O(\epsilon)$, we scale the damping and forcing terms as

```

scaling = 8c -> 2 e m, f -> e f<;

```

Substituting **multiScales**, **solRule**, and **scaling** into **EOM1**, expanding the result for small ϵ , and discarding terms of order higher than ϵ^3 , we obtain

```

eq91a =
HEOM1 . . Integrate -> int . . multiScales . . solRule . . scaling . . intRule1 . . . intRule2 . .
TrigToExp . . ExpandAllL . . en_*; n>3 -> 0;

```

Equating coefficients of like powers of ϵ , we obtain

```

eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & . ž eq91a . . Transpose . . Rest;
eqEps . . displayRule

```

$$\begin{aligned}
99D_0u_1 - v_1 &= 0, \quad u_1'''' + D_0v_1 + Hu_1' L P_k - \frac{1}{2} b E^{-IT_0 W} f f_k^2 @xD - \\
&\quad \frac{1}{2} b E^{IT_0 W} f f_k^2 @xD - 2 b^2 a \text{int}@Hu_1' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0=, \\
9D_0u_2 + D_1u_1 - v_2 &= 0, \quad -\frac{1}{2} E^{-IT_0 W} f Hu_1' L - \frac{1}{2} E^{IT_0 W} f Hu_1' L - \\
&\quad 2 b a \text{int}@Hu_1' L f_k^e @xD, 8x, 0, 1<D Hu_1' L + u_2'''' + 2 m HD_0u_1 L + D_0v_2 + D_1v_1 + Hu_2' L P_k - \\
&\quad b a \text{int}BHu_1' L^2, 8x, 0, 1<F f_k^2 @xD - 2 b^2 a \text{int}@Hu_2' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0=, \\
9D_0u_3 + D_1u_2 + D_2u_1 - v_3 &= 0, \quad -a \text{int}BHu_1' L^2, 8x, 0, 1<F Hu_1' L - \\
&\quad 2 b a \text{int}@Hu_2' L f_k^e @xD, 8x, 0, 1<D Hu_1' L - \frac{1}{2} E^{-IT_0 W} f Hu_2' L - \frac{1}{2} E^{IT_0 W} f Hu_2' L - \\
&\quad 2 b a \text{int}@Hu_1' L f_k^e @xD, 8x, 0, 1<D Hu_2' L + u_3'''' + 2 m HD_0u_2 L + D_0v_3 + 2 m HD_1u_1 L + D_1v_2 + D_2v_1 + \\
&\quad Hu_3' L P_k - 2 b a \text{int}@Hu_1' L Hu_2' L, 8x, 0, 1<D f_k^2 @xD - 2 b^2 a \text{int}@Hu_3' L f_k^e @xD, 8x, 0, 1<D f_k^2 @xD == 0==
\end{aligned}$$

9.1.1 First-Order Solution

The homogeneous part of the first-order equations, `eqEps[[1]]`, can be written as

```
linearSys = #@@1DD & . ž eqEps@@1DD . f -> 0;
linearSys . displayRule

8D0u1 - v1, u1'''' + D0v1 + Hu1'' L Pk - 2 b2 a int@Hu1' fkε@xD, 8x, 0, 1<D fk2@xD<
```

Because in the presence of damping all of the modes that are not directly or indirectly excited decay with time (Nayfeh and Mook, 1979), the solution of `eqEps[[1]]` is taken to consist of the lowest two modes; that is,

```
solForm = Y@xD Exp@I W T0D + Y@xD Exp@-I W T0D +
SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Āi@T1, T2D Fi@xD Exp@-I wi T0D, 8i, 2<E;

sol1 = 8u1 -> Function@8x, T0, T1, T2<, solForm . . EvaluatedD,
v1 -> Function@8x, T0, T1, T2<, D@solForm, T0D . . EvaluatedD<

8u1 @ Function@8x, T0, T1, T2<, E-I T0 W Y@xD + EI T0 W Y@xD + EI T0 w1 A1@T1, T2D F1@xD +
EI T0 w2 A2@T1, T2D F2@xD + E-I T0 w1 F1@xD Ā1@T1, T2D + E-I T0 w2 F2@xD Ā2@T1, T2DD,
v1 @ Function@8x, T0, T1, T2<, -I E-I T0 W W Y@xD + I EI T0 W W Y@xD + I EI T0 w1 w1 A1@T1, T2D F1@xD +
I EI T0 w2 w2 A2@T1, T2D F2@xD - I E-I T0 w1 w1 F1@xD Ā1@T1, T2D - I E-I T0 w2 w2 F2@xD Ā2@T1, T2DD<
```

where the F_i are the eigenmodes, which satisfy the orthonormality condition $\int_0^1 F_n(x) F_m(x) dx = d_{n,m}$, and Y is uniquely determined by the following boundary-value problem:

```
eq91b = CoefficientA
eqEps@@1, 2, 1DD . Isol1 . . 9Ai -> H0 &L, Āi -> H0 &L=M . . intRule1 . . . intRule2,
EI W T0 E == 0

-W2 Y@xD + Pk Y2@xD -  $\frac{1}{2}$  b fk2@xD - 2 b2 a int@Yε@xD fkε@xD, 8x, 0, 1<D fk2@xD + YH4L@xD == 0

bc91b = BC . . u -> Y
8Y@0, tD == 0, YH2, 0L@0, tD == 0, Y@1, tD == 0, YH2, 0L@1, tD == 0<
```

In what follows, we consider nonlinear vibrations of a first-mode buckled beam; hence, the critical buckling load is

```
bucklingLoad = Pk -> p2;
```

We note that, as the buckling level increases from zero, the symmetric mode is the first mode. However, as the buckling level exceeds the first cross-over value, the symmetric mode becomes the second mode. Hence, the second natural frequency w_2 depends on the buckling level. For buckling levels greater than the cross-over value, the first static buckled deflection and the first two normalized mode shapes and associated natural frequencies are

```

modeshapes = 9 fk -> H Sin@p #D &L, F1 -> I 1/2 Sin@2 p #D &M, F2 -> I 1/2 Sin@p #D &M;
frequencies = 9 w2 -> b p2 1/a, w1 -> 2 1/3 p2 =;

```

The buckling level at which $w_2 = 2 w_1$ is

```

values1 = Solve@w2 == 2 w1 . frequencies, bD@@1DD
: b 4/24 1/a >

```

Substituting the **bucklingLoad** and **modeshapes** into **eq91b** yields

```

eq91c = eq91b . bucklingLoad . modeshapes
1/2 b f p2 Sin@p xD +
2 b2 p2 a int@p Cos@p xD Y4@xD, 8x, 0, 1<D Sin@p xD - W2 Y@xD + p2 Y2@xD + YH4L@xD == 0

```

Inspection of **eq91c** and **bc91b** suggests that $Y(x)$ has the form

```

Ysol = Y -> HG Sin@p #D &L;

```

Substituting **Ysol** into **eq91c**, collecting the coefficient of Sin@p xD , solving the resulting equation for G , and simplifying it using **frequencies** and **values1**, we obtain

```

GRule = Solve@Coefficient@eq91c@@1DD . Ysol . int -> Integrate, Sin@p xDD == 0, GD@@1DD .
W -> 2 w2 . frequencies . values1
: G 1/24 1/3 p2 1/a >

```

Because the buckled-beam problem is self-adjoint, we have

```

adjoint = 88 - I w1 F1@xD, F1@xD<, 8 - I w2 F2@xD, F2@xD<<;

```

whose complex conjugate is

```

adjointC = adjoint . conjugateRule
88 I w1 F1@xD, F1@xD<, 8 I w2 F2@xD, F2@xD<<

```

9.1.2 Second-Order Solution

Substituting **sol1**, **Ysol**, and some of the parameter values into **eqEps[[2]]** yields

```

order2Eq =
  HlinearSys . u_1 -> u_2L == HHlinearSys . u_1 -> u_2L - H#@@1DD & . ž eqEps@@2DDL . sol1 .
    Ysol . GRule . intRule1 . . intRule2 . . modeshapes .
      int -> Integrate . . ExpandL . . bucklingLoad . . values1 . . Thread;
order2Eq . . displayRule

: D_0 u_2 - v_2 == - . 1/2 E^{I T_0 w_2} HD_1 A_2 L Sin@p xD - . 1/2 E^{-I T_0 w_2} HD_1 A_2 L Sin@p xD -
  . 1/2 E^{I T_0 w_1} HD_1 A_1 L Sin@2 p xD - . 1/2 E^{-I T_0 w_1} HD_1 A_1 L Sin@2 p xD,
p^2 Hu_2'' L + u_2'''' + D_0 v_2 - 96 int@Hu_2' f_k^c @xD, 8x, 0, 1<D f_k^z @xD ==
- f^2 Sin@p xD / (16 . 3 . a^3) - E^{-2 I T_0 W} f^2 Sin@p xD / (32 . 3 . a^3) - E^{2 I T_0 W} f^2 Sin@p xD / (32 . 3 . a^3) + I E^{-I T_0 W} f_m W Sin@p xD / (12 . 3 . p^2 . a^3) -
I E^{I T_0 W} f_m W Sin@p xD / (12 . 3 . p^2 . a^3) - 8/3 . 1/2 E^{I T_0 H-W+w_1 L} f p^2 Sin@2 p xD A_1 - 8/3 . 1/2 E^{I T_0 H+W+w_1 L} f p^2 Sin@2 p xD A_1 -
16 . 1/3 E^{2 I T_0 w_1} p^4 . 1/3 Sin@p xD A_1^2 - . 1/2 E^{I T_0 H-W+w_2 L} f p^2 Sin@p xD A_2 - . 1/2 E^{I T_0 H+W+w_2 L} f p^2 Sin@p xD A_2 -
32 . 1/3 E^{I T_0 Hw_1+w_2 L} p^4 . 1/3 Sin@2 p xD A_1 A_2 - 12 . 1/3 E^{2 I T_0 w_2} p^4 . 1/3 Sin@p xD A_2^2 -
I . 1/2 E^{I T_0 w_1} HD_1 A_1 L Sin@2 p xD w_1 + I . 1/2 E^{-I T_0 w_1} HD_1 A_1 L Sin@2 p xD w_1 -
2 I . 1/2 E^{I T_0 w_1} m Sin@2 p xD A_1 w_1 - I . 1/2 E^{I T_0 w_2} HD_1 A_2 L Sin@p xD w_2 + I . 1/2 E^{-I T_0 w_2} HD_1 A_2 L Sin@p xD w_2 -
2 I . 1/2 E^{I T_0 w_2} m Sin@p xD A_2 w_2 - 8/3 . 1/2 E^{I T_0 H-W-w_1 L} f p^2 Sin@2 p xD A_1 -
8/3 . 1/2 E^{I T_0 H+W-w_1 L} f p^2 Sin@2 p xD A_1 - 32 . 1/3 p^4 . 1/3 Sin@p xD A_1 A_1 -
32 . 1/3 E^{I T_0 H-W+w_2 L} p^4 . 1/3 Sin@2 p xD A_2 A_1 + 2 I . 1/2 E^{-I T_0 w_1} m Sin@2 p xD w_1 A_1 -
16 . 1/3 E^{-2 I T_0 w_1} p^4 . 1/3 Sin@p xD A_1^2 - . 1/2 E^{I T_0 H-W-w_2 L} f p^2 Sin@p xD A_2 -
. 1/2 E^{I T_0 H+W-w_2 L} f p^2 Sin@p xD A_2 - 32 . 1/3 E^{I T_0 Hw_1-w_2 L} p^4 . 1/3 Sin@2 p xD A_1 A_2 -
24 . 1/3 p^4 . 1/3 Sin@p xD A_2 A_2 + 2 I . 1/2 E^{-I T_0 w_2} m Sin@p xD w_2 A_2 -
32 . 1/3 E^{I T_0 H-W-w_2 L} p^4 . 1/3 Sin@2 p xD A_1 A_2 - 12 . 1/3 E^{-2 I T_0 w_2} p^4 . 1/3 Sin@p xD A_2^2>

```

In order to collect the terms that may lead to secular terms from the right-hand sides of **order2Eq**, we define the rules

```
OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D~Join~8W<D@@1DD & . ž omgList
```

```
98w_2 @ e s_1 + 2 w_1, W @ 2 e s_1 + e s_2 + 4 w_1<, 9w_1 @ 1/2 H-e s_1 + w_2 L, W @ e s_2 + 2 w_2==
```

```
expRule1@i_D := Exp@arg_D :> Exp@Expand@arg . . OmgRule@@iDDD . . e T_0 -> T_1 D
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_1 T_0}$, we have

```
ST11 = CoefficientA#@@2DD & . ž order2Eq . . expRule1@1D, E^{I w_1 T_0} E;
```

```
ST11 . . displayRule
```

```

9- . 1/2 HD_1 A_1 L Sin@2 p xD,
- I . 1/2 HD_1 A_1 L Sin@2 p xD w_1 - 2 I . 1/2 m Sin@2 p xD A_1 w_1 - 32 . 1/3 E^{I T_1 s_1} p^4 . 1/3 Sin@2 p xD A_2 A_1=

```

```

ST12 = CoefficientA#@2DD & . 2 order2Eq . expRule1@2D, EI w2 T0 E;
ST12 . displayRule

9- 2 HD1A2L Sin@p xD, -16 3 E-I T1 s1 p4 a Sin@p xD A12 -
I 2 HD1A2L Sin@p xD w2 - 2 I 2 m Sin@p xD A2 w2 - 2 EI T1 s2 f p2 Sin@p xD A22 =

```

Demanding that **ST11** be orthogonal to the **adjointC[[1]]**, we obtain the solvability condition

```

SCond11 =
SolveAint@adjointC@@1DD.ST11, 8x, 0, 1<D == 0 . intRule1 . intRule2 . modeshapes .
int -> Integrate, A1H1,0L@T1, T2DE@@1DD . ExpandAll

:A1H1,0L@T1, T2D ® -m A1@T1, T2D + 8 I 6 EI T1 s1 p4 a A2@T1, T2D A1@T1, T2D >
w1

```

Demanding that **ST12** be orthogonal to the **adjointC[[2]]**, we obtain the solvability condition

```

SCond12 =
SolveAint@adjointC@@2DD.ST12, 8x, 0, 1<D == 0 . intRule1 . intRule2 . modeshapes .
int -> Integrate, A2H1,0L@T1, T2DE@@1DD . ExpandAll

:A2H1,0L@T1, T2D ® 4 I 6 E-I T1 s1 p4 a A1@T1, T2D2 - m A2@T1, T2D + I EI T1 s2 f p2 A2@T1, T2D >
w2 2 w2

```

Hence, we have the solvability conditions

```
SCond1 = Join@SCond11, SCond12D;
```

whose complex conjugates are

```
ccSCond1 = SCond1 . conjugateRule;
```

To remove $D_1 A_1$ and $D_1 A_2$ from the right-hand sides of **order2Eq**, we first define the rules

```

sigRule = Solve@ResonanceConds, 8s1, s2<D@@1DD

9s1 ® - 2 W1 - W2
e
s2 ® - W + 2 W2
e

expRule2 = Exp@a_D := Exp@a . sigRule . T1 -> e T0 . ExpandD;

```

Substituting **SCond1**, **ccSCond1**, and **expRule2** into **order2Eq**, we obtain

```
order2Eqm = order2Eq . SCond1 . ccSCond1 . expRule2 . ExpandAll;
order2Eqm . displayRule
```

$$\begin{aligned}
9D_0 u_2 - v_2 = & \frac{1}{2} E^{I T_0 w_1} m \sin @ 2 p x D A_1 + \frac{1}{2} E^{I T_0 w_2} m \sin @ p x D A_2 - \\
& \frac{8 I}{w_2} \frac{1}{3} E^{2 I T_0 w_1} p^4 \frac{1}{a} \sin @ p x D A_1^2 + \frac{I E^{I T_0 H - W + w_2 L} f p^2 \sin @ p x D A_2}{\frac{1}{2} w_2} + \\
& \frac{1}{2} E^{-I T_0 w_1} m \sin @ 2 p x D A_1 - \frac{16 I}{w_1} \frac{1}{3} E^{I T_0 H - w_1 + w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_2 \dot{A}_1 + \\
& \frac{8 I}{w_2} \frac{1}{3} E^{-2 I T_0 w_1} p^4 \frac{1}{a} \sin @ p x D A_1^2 + \frac{1}{2} E^{-I T_0 w_2} m \sin @ p x D A_2 + \\
& \frac{16 I}{w_1} \frac{1}{3} E^{I T_0 H w_1 - w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_1 \dot{A}_2 - \frac{I E^{I T_0 H w_1 - w_2 L} f p^2 \sin @ p x D A_2}{\frac{1}{2} w_2}, \\
p^2 H u_2'' L + u_2'''' + D_0 v_2 - 96 \text{int}@ H u_2' L f_k @ x D, 8 x, 0, 1 < D f_k @ x D = & \\
- \frac{f^2 \sin @ p x D}{16 \frac{1}{3} \frac{1}{a}} - \frac{E^{-2 I T_0 w_1} f^2 \sin @ p x D}{32 \frac{1}{3} \frac{1}{a}} - \frac{E^{-2 I T_0 w_2} f^2 \sin @ p x D}{32 \frac{1}{3} \frac{1}{a}} + \frac{I E^{-I T_0 w_1} f m w \sin @ p x D}{12 \frac{1}{3} p^2 \frac{1}{a}} - \\
\frac{I E^{I T_0 w_1} f m w \sin @ p x D}{12 \frac{1}{3} p^2 \frac{1}{a}} - \frac{8}{3} \frac{1}{2} E^{I T_0 H - W + w_1 L} f p^2 \sin @ 2 p x D A_1 - \frac{8}{3} \frac{1}{2} E^{I T_0 H w_1 + w_2 L} f p^2 \sin @ 2 p x D A_1 - \\
8 \frac{1}{3} E^{2 I T_0 w_1} p^4 \frac{1}{a} \sin @ p x D A_1^2 + \frac{E^{I T_0 H - W + w_2 L} f p^2 \sin @ p x D A_2}{\frac{1}{2}} - \frac{1}{2} E^{I T_0 H - W + w_2 L} f p^2 \sin @ p x D A_2 - \\
\frac{1}{2} E^{I T_0 H w_1 + w_2 L} f p^2 \sin @ p x D A_2 - 32 \frac{1}{3} E^{I T_0 H w_1 + w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_1 A_2 - \\
12 \frac{1}{3} E^{2 I T_0 w_2} p^4 \frac{1}{a} \sin @ p x D A_2^2 - I \frac{1}{2} E^{I T_0 w_1} m \sin @ 2 p x D A_1 w_1 - \\
I \frac{1}{2} E^{I T_0 w_2} m \sin @ p x D A_2 w_2 - \frac{8}{3} \frac{1}{2} E^{I T_0 H - W - w_1 L} f p^2 \sin @ 2 p x D A_1 - \\
\frac{8}{3} \frac{1}{2} E^{I T_0 H w_1 - w_2 L} f p^2 \sin @ 2 p x D A_1 - 32 \frac{1}{3} p^4 \frac{1}{a} \sin @ p x D A_1 \dot{A}_1 - \\
16 \frac{1}{3} E^{I T_0 H - w_1 + w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_2 \dot{A}_1 + I \frac{1}{2} E^{-I T_0 w_1} m \sin @ 2 p x D w_1 \dot{A}_1 - \\
8 \frac{1}{3} E^{-2 I T_0 w_1} p^4 \frac{1}{a} \sin @ p x D A_1^2 - \frac{1}{2} E^{I T_0 H - W - w_2 L} f p^2 \sin @ p x D A_2 + \frac{E^{I T_0 H w_1 - w_2 L} f p^2 \sin @ p x D A_2}{\frac{1}{2}} - \\
\frac{1}{2} E^{I T_0 H w_1 - w_2 L} f p^2 \sin @ p x D A_2 - 16 \frac{1}{3} E^{I T_0 H w_1 - w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_1 \dot{A}_2 - \\
24 \frac{1}{3} p^4 \frac{1}{a} \sin @ p x D A_2 \dot{A}_2 + I \frac{1}{2} E^{-I T_0 w_2} m \sin @ p x D w_2 \dot{A}_2 - \\
32 \frac{1}{3} E^{I T_0 H - w_1 - w_2 L} p^4 \frac{1}{a} \sin @ 2 p x D A_1 \dot{A}_2 - 12 \frac{1}{3} E^{-2 I T_0 w_2} p^4 \frac{1}{a} \sin @ p x D A_2^2 =
\end{aligned}$$

We use the method of undetermined coefficients to determine the solution of **order2Eqm** and their associated boundary conditions. To implement this, we first determine all possible forms of the terms on the right-hand sides of **order2Eqm** as follows:

```

sol2Form = TableACases@order2Eqm@@k, 2DD, #D & .ž 9_ Ea-T0+b· f@c_ xD -> EaT0+b f@c xD,
a_ f@c_ xD :> f@c xD •; FreeQ@a, T0D=, 8k, 2<E • Flatten • Union

8Sin@p xD, E-I T0W Sin@p xD, EI T0W Sin@p xD, E-2 I T0W Sin@p xD, E2 I T0W Sin@p xD,
E-2 I T0W1 Sin@p xD, E2 I T0W1 Sin@p xD, E-I T0W2 Sin@p xD, EI T0W2 Sin@p xD, E-2 I T0W2 Sin@p xD,
E2 I T0W2 Sin@p xD, E-I T0W-I T0W2 Sin@p xD, EI T0W-I T0W2 Sin@p xD, E-I T0W+I T0W2 Sin@p xD,
EI T0W+I T0W2 Sin@p xD, E-I T0W1 Sin@2 p xD, EI T0W1 Sin@2 p xD, E-I T0W-I T0W1 Sin@2 p xD,
EI T0W-I T0W1 Sin@2 p xD, E-I T0W+I T0W1 Sin@2 p xD, EI T0W+I T0W1 Sin@2 p xD, E-I T0W1-I T0W2 Sin@2 p xD,
EI T0W1-I T0W2 Sin@2 p xD, E-I T0W1+I T0W2 Sin@2 p xD, EI T0W1+I T0W2 Sin@2 p xD<

```

where all the spatial dependent functions satisfy the hinged-hinged boundary conditions. We note that **sol2Form** consists of (a) the terms that are proportional to $E^{-I w_1 T_0}$, $E^{I w_1 T_0}$, $E^{-I w_2 T_0}$, $E^{I w_2 T_0}$ and (b) the remainder of the terms. The positions of the former terms, which are orthogonal to the adjoint, in **sol2Form** are

```

pos@1D =
HPosition@# sol2Form •. expRule1@1D •. Exp@_ + _ T0D -> 0, a_ •; a != 0, 1D • Flatten •
RestL & .ž 9E-I w1 T0, EI w1 T0=

8817, 24<, 816, 23<<

pos@2D =
HPosition@# sol2Form •. expRule1@2D •. Exp@_ + _ T0D -> 0, a_ •; a != 0, 1D • Flatten •
RestL & .ž 9E-I w2 T0, EI w2 T0=

887, 9, 13<, 86, 8, 14<<

```

Next, we seek the solution of u_2 and v_2 in two parts. First, we consider the part of **sol2Form** that is not related to the secular terms

```

sol2Forma = Delete@sol2Form, 8#< & .ž Flatten@Array@pos, 2DDD

8Sin@p xD, E-I T0W Sin@p xD, EI T0W Sin@p xD, E-2 I T0W Sin@p xD, E2 I T0W Sin@p xD,
E-2 I T0W2 Sin@p xD, E2 I T0W2 Sin@p xD, E-I T0W-I T0W2 Sin@p xD, EI T0W+I T0W2 Sin@p xD,
E-I T0W-I T0W1 Sin@2 p xD, EI T0W-I T0W1 Sin@2 p xD, E-I T0W+I T0W1 Sin@2 p xD,
EI T0W+I T0W1 Sin@2 p xD, E-I T0W1-I T0W2 Sin@2 p xD, EI T0W1+I T0W2 Sin@2 p xD<

```

and define the undetermined coefficients as

```

uSymbola = Table@yi, 8i, Length@sol2FormaD<D;
vSymbola = uSymbola •. y -> h;

```

The general solution in terms of the **sol2Forma** can be written as

```

sol2a = 8u2 -> Function@8x, T0, T1, T2<, uSymbola.sol2Forma •• Evaluated,
v2 -> Function@8x, T0, T1, T2<, vSymbola.sol2Forma •• Evaluated<;

```

Second, we consider the part of **sol2Form** that is related to the secular terms

```
sol2Formb1 = sol2Form@@pos@1D@@1DDDD
```

```
8EI T0 w1 Sin@2 p xD, E-I T0 w1+I T0 w2 Sin@2 p xD<
```

```
sol2Formb2 = sol2Form@@pos@2D@@1DDDD
```

```
8E2 I T0 w1 Sin@p xD, EI T0 w2 Sin@p xD, EI T0 W-I T0 w2 Sin@p xD<
```

and define the undetermined coefficients as

```
uSymbolb1 = Table@ys1,i, 8i, Length@sol2Formb1D<D;
```

```
uSymbolb2 = Table@ys2,i, 8i, Length@sol2Formb2D<D;
```

where the undetermined coefficients for v_2 are related to **uSymbolb1** and **uSymbolb2** due to the orthogonality condition. Therefore, the general solution in terms of the **sol2Formb1** and **sol2Formb2** can be written as

```
sol2b =
```

```
8u2 -> Function@8x, T0, T1, T2<, uSymbolb1.sol2Formb1 + uSymbolb2.sol2Formb2 •• Evaluated,
```

```
v2 -> Function@8x, T0, T1, T2<,
```

```
- I w1 uSymbolb1.sol2Formb1 - I w2 uSymbolb2.sol2Formb2 •• EvaluatedD<;
```

The total solution is the sum of **sol2a**, **sol2b**, and the complexconjugate of **sol2b**.

Next, we solve for these undetermined coefficients. Substituting **sol2a** into **order2Eqm** and equating the coefficients of **sol2Forma**, we have

```
algEqa =
```

```
Flatten@Coefficient@Subtract žž # •. sol2a •. intRule1 •. intRule2 •. modeshapes •. int ->
```

```
Integrate, sol2FormaD & •ž order2EqmD == 0 •. Exp@_ T0 + _.D -> 0 •• Thread;
```

Substituting some of the parameter values into **algEqa** and solving for **uSymbola** and **vSymbola**, we obtain

symbolaRule =

Solve@algEq, uSymbola~Join~vSymbolaD@@1DD •. W -> 2 w₂ •. frequencies •. values1

$$\begin{aligned}
 &: h_1 \otimes 0, y_1 \otimes - \frac{\frac{1}{3} f^2 + 1536 \frac{1}{3} p^4 a A_1 \otimes T_1, T_2 D \dot{A}_1 \otimes T_1, T_2 D + 1152 \frac{1}{3} p^4 a A_2 \otimes T_1, T_2 D \dot{A}_2 \otimes T_1, T_2 D}{2304 p^4 \frac{1}{3} a}, \\
 &h_2 \otimes - \frac{f m}{9 \frac{1}{3} p^2 \frac{1}{3} a}, h_3 \otimes - \frac{f m}{9 \frac{1}{3} p^2 \frac{1}{3} a}, h_4 \otimes - \frac{I f^2}{1440 p^2 \frac{1}{3} a}, h_5 \otimes \frac{I f^2}{1440 p^2 \frac{1}{3} a}, \\
 &h_6 \otimes -2 I p^2 \frac{1}{3} a A_2 \otimes T_1, T_2 D^2, h_7 \otimes 2 I p^2 \frac{1}{3} a A_2 \otimes T_1, T_2 D^2, y_2 \otimes - \frac{I f m}{216 p^4 \frac{1}{3} a}, \\
 &y_3 \otimes \frac{I f m}{216 p^4 \frac{1}{3} a}, y_4 \otimes \frac{f^2}{23040 \frac{1}{3} p^4 \frac{1}{3} a}, y_5 \otimes \frac{f^2}{23040 \frac{1}{3} p^4 \frac{1}{3} a}, y_6 \otimes \frac{\frac{1}{3} a A_2 \otimes T_1, T_2 D^2}{4 \frac{1}{3}}, \\
 &y_7 \otimes \frac{\frac{1}{3} a A_2 \otimes T_1, T_2 D^2}{4 \frac{1}{3}}, h_8 \otimes - \frac{1}{16} I \$ \frac{3}{2} f A_2 \otimes T_1, T_2 D, h_9 \otimes \frac{1}{16} I \$ \frac{3}{2} f A_2 \otimes T_1, T_2 D, \\
 &h_{10} \otimes - \frac{5 I f A_1 \otimes T_1, T_2 D}{9 \frac{1}{3} 6}, h_{11} \otimes \frac{I f A_1 \otimes T_1, T_2 D}{9 \frac{1}{3} 6}, h_{12} \otimes - \frac{I f A_1 \otimes T_1, T_2 D}{9 \frac{1}{3} 6}, h_{13} \otimes \frac{5 I f A_1 \otimes T_1, T_2 D}{9 \frac{1}{3} 6}, \\
 &h_{14} \otimes -6 I p^2 \frac{1}{3} a A_1 \otimes T_1, T_2 D \dot{A}_2 \otimes T_1, T_2 D, h_{15} \otimes 6 I p^2 \frac{1}{3} a A_1 \otimes T_1, T_2 D \dot{A}_2 \otimes T_1, T_2 D, y_8 \otimes \frac{f A_2 \otimes T_1, T_2 D}{192 \frac{1}{3} 2 p^2}, \\
 &y_9 \otimes \frac{f A_2 \otimes T_1, T_2 D}{192 \frac{1}{3} 2 p^2}, y_{10} \otimes \frac{f A_1 \otimes T_1, T_2 D}{54 \frac{1}{3} 2 p^2}, y_{11} \otimes \frac{f A_1 \otimes T_1, T_2 D}{18 \frac{1}{3} 2 p^2}, y_{12} \otimes \frac{f A_1 \otimes T_1, T_2 D}{18 \frac{1}{3} 2 p^2}, \\
 &y_{13} \otimes \frac{f A_1 \otimes T_1, T_2 D}{54 \frac{1}{3} 2 p^2}, y_{14} \otimes \frac{\frac{1}{3} a A_2 \otimes T_1, T_2 D \dot{A}_2 \otimes T_1, T_2 D}{\frac{1}{3}}, y_{15} \otimes \frac{\frac{1}{3} a A_2 \otimes T_1, T_2 D \dot{A}_2 \otimes T_1, T_2 D}{\frac{1}{3}} >
 \end{aligned}$$

Substituting **sol2b** into either of the two equations in **order2Eqm**, equating the coefficients of **sol2Formb1** and **sol2Formb2**, and solving for **uSymbolb1** and **uSymbolb2**, we have

symbolbRule =

Solve@Coefficient@Subtract žž order2Eqm@@1DD •. sol2b, sol2Formb1~Join~sol2Formb2D ==

0 •. Exp@_ T₀ + _ .D -> 0 •• Thread,

uSymbolb1~Join~uSymbolb2D@@1DD •. W -> 2 w₂ •. frequencies •. values1

$$\begin{aligned}
 &: y_{S1,1} \otimes - \frac{I m A_1 \otimes T_1, T_2 D}{2 \frac{1}{3} 6 p^2}, y_{S1,2} \otimes - \frac{2 \frac{1}{3} a A_2 \otimes T_1, T_2 D \dot{A}_1 \otimes T_1, T_2 D}{3}, \\
 &y_{S2,2} \otimes - \frac{I m A_2 \otimes T_1, T_2 D}{4 \frac{1}{3} 6 p^2}, y_{S2,3} \otimes - \frac{f A_2 \otimes T_1, T_2 D}{96 \frac{1}{3} 2 p^2}, y_{S2,1} \otimes - \frac{\frac{1}{3} a A_1 \otimes T_1, T_2 D^2}{4 \frac{1}{3}} >
 \end{aligned}$$

Substituting these symbol values into **sol2a** and **sol2b** yields

sol2aForm = 8u₂@x, T₀, T₁, T₂D, v₂@x, T₀, T₁, T₂D< •. sol2a •. symbolaRule;

sol2bForm = 8u₂@x, T₀, T₁, T₂D, v₂@x, T₀, T₁, T₂D< •. sol2b •. symbolbRule;

sol2 = 8u₂ @ Function@8x, T₀, T₁, T₂<,

sol2aForm@@1DD + sol2bForm@@1DD + Hsol2bForm@@1DD . conjugateRuleL . Expand .

Evaluated, v₂ @ Function@8x, T₀, T₁, T₂<, sol2aForm@@2DD + sol2bForm@@2DD +

Hsol2bForm@@2DD . conjugateRuleL . Expand . Evaluated<

:u₂ @ FunctionB8x, T₀, T₁, T₂<,

$$\begin{aligned}
 & - \frac{f^2 \sin(p x D)}{768 \cdot 3 p^4 \cdot a} + \frac{E^{-2 I T_0 W} f^2 \sin(p x D)}{23040 \cdot 3 p^4 \cdot a} + \frac{E^{2 I T_0 W} f^2 \sin(p x D)}{23040 \cdot 3 p^4 \cdot a} - \frac{I E^{-I T_0 W} f m \sin(p x D)}{216 p^4 \cdot a} + \\
 & \frac{I E^{I T_0 W} f m \sin(p x D)}{216 p^4 \cdot a} + \frac{E^{-I T_0 W + I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{18 \cdot 2 p^2} + \frac{E^{I T_0 W + I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{54 \cdot 2 p^2} - \\
 & \frac{I E^{I T_0 W_1} m \sin(2 p x D A_1 @ T_1, T_2 D)}{2 \cdot 6 p^2} - \frac{E^{2 I T_0 W_1} \cdot a \sin(p x D A_1 @ T_1, T_2 D^2)}{4 \cdot 3} - \\
 & \frac{E^{-I T_0 W + I T_0 W_2} f \sin(p x D A_2 @ T_1, T_2 D)}{96 \cdot 2 p^2} + \frac{E^{I T_0 W + I T_0 W_2} f \sin(p x D A_2 @ T_1, T_2 D)}{192 \cdot 2 p^2} - \\
 & \frac{I E^{I T_0 W_2} m \sin(p x D A_2 @ T_1, T_2 D)}{4 \cdot 6 p^2} + \frac{E^{I T_0 W_1 + I T_0 W_2} \cdot a \sin(2 p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D)}{3} + \\
 & \frac{E^{2 I T_0 W_2} \cdot a \sin(p x D A_2 @ T_1, T_2 D^2)}{4 \cdot 3} + \frac{E^{-I T_0 W - I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{54 \cdot 2 p^2} + \\
 & \frac{E^{I T_0 W - I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{18 \cdot 2 p^2} + \frac{I E^{-I T_0 W_1} m \sin(2 p x D A_1 @ T_1, T_2 D)}{2 \cdot 6 p^2} - \\
 & \frac{2 \cdot a \sin(p x D A_1 @ T_1, T_2 D A_1 @ T_1, T_2 D)}{3} - \frac{2 E^{-I T_0 W_1 + I T_0 W_2} \cdot a \sin(2 p x D A_2 @ T_1, T_2 D A_1 @ T_1, T_2 D)}{3} - \\
 & \frac{E^{-2 I T_0 W_1} \cdot a \sin(p x D A_1 @ T_1, T_2 D^2)}{4 \cdot 3} + \frac{E^{-I T_0 W - I T_0 W_2} f \sin(p x D A_2 @ T_1, T_2 D)}{192 \cdot 2 p^2} - \\
 & \frac{E^{I T_0 W - I T_0 W_2} f \sin(p x D A_2 @ T_1, T_2 D)}{96 \cdot 2 p^2} + \frac{I E^{-I T_0 W_2} m \sin(p x D A_2 @ T_1, T_2 D)}{4 \cdot 6 p^2} - \\
 & \frac{2 E^{I T_0 W_1 - I T_0 W_2} \cdot a \sin(2 p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D)}{3} - \frac{1}{2} \cdot \frac{a \sin(p x D A_2 @ T_1, T_2 D A_2 @ T_1, T_2 D)}{3} + \\
 & \frac{E^{-I T_0 W_1 - I T_0 W_2} \cdot a \sin(2 p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D)}{3} + \frac{E^{-2 I T_0 W_2} \cdot a \sin(p x D A_2 @ T_1, T_2 D^2)}{4 \cdot 3} F,
 \end{aligned}$$

$$\begin{aligned}
 v_2 @ \text{FunctionB8x, } T_0, T_1, T_2 <, & - \frac{I E^{-2 I T_0 W} f^2 \sin(p x D)}{1440 p^2 \cdot a} + \frac{I E^{2 I T_0 W} f^2 \sin(p x D)}{1440 p^2 \cdot a} - \\
 & \frac{E^{-I T_0 W} f m \sin(p x D)}{9 \cdot 3 p^2 \cdot a} - \frac{E^{I T_0 W} f m \sin(p x D)}{9 \cdot 3 p^2 \cdot a} - \frac{I E^{-I T_0 W + I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{6} + \\
 & \frac{5 I E^{I T_0 W + I T_0 W_1} f \sin(2 p x D A_1 @ T_1, T_2 D)}{9 \cdot 6} - \frac{E^{I T_0 W_1} m \sin(2 p x D A_1 @ T_1, T_2 D)}{2 \cdot 6 p^2} + \\
 & \frac{I E^{2 I T_0 W_1} \cdot a \sin(p x D A_2 @ T_1, T_2 D^2)}{4 \cdot 3} + \frac{1}{16} I \$ \frac{3}{2} E^{I T_0 W + I T_0 W_2} f \sin(p x D A_2 @ T_1, T_2 D) -
 \end{aligned}$$

$$\begin{aligned}
& \frac{I E^{-I T_0 W_1 + I T_0 W_2} f \sin @ p \times D w_2 A_2 @ T_1, T_2 D}{96 \cdot \frac{1}{2} p^2} - \frac{E^{I T_0 W_2} m \sin @ p \times D w_2 A_2 @ T_1, T_2 D}{4 \cdot \frac{1}{6} p^2} + \\
& 6 I E^{I T_0 W_1 + I T_0 W_2} p^2 \cdot \frac{!!!}{a} \sin @ 2 p \times D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D + 2 I E^{2 I T_0 W_2} p^2 \cdot \frac{!!!}{a} \sin @ p \times D A_2 @ T_1, T_2 D^2 - \\
& \frac{5 I E^{-I T_0 W_1 - I T_0 W_2} f \sin @ 2 p \times D \dot{A}_1 @ T_1, T_2 D}{9 \cdot \frac{1}{6}} + \frac{I E^{I T_0 W_1 - I T_0 W_2} f \sin @ 2 p \times D \dot{A}_1 @ T_1, T_2 D}{6} - \\
& \frac{E^{-I T_0 W_1} m \sin @ 2 p \times D w_1 \dot{A}_1 @ T_1, T_2 D}{2 \cdot \frac{1}{6} p^2} + \frac{2 I E^{-I T_0 W_1 + I T_0 W_2} \cdot \frac{!!!}{a} \sin @ 2 p \times D w_1 A_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3} - \\
& \frac{I E^{-2 I T_0 W_1} \cdot \frac{!!!}{a} \sin @ p \times D w_2 \dot{A}_1 @ T_1, T_2 D^2}{4 \cdot \frac{1}{3}} - \frac{1}{16} I \$ \frac{3}{2} E^{-I T_0 W_1 - I T_0 W_2} f \sin @ p \times D \dot{A}_2 @ T_1, T_2 D + \\
& \frac{I E^{I T_0 W_1 - I T_0 W_2} f \sin @ p \times D w_2 \dot{A}_2 @ T_1, T_2 D}{96 \cdot \frac{1}{2} p^2} - \frac{E^{-I T_0 W_2} m \sin @ p \times D w_2 \dot{A}_2 @ T_1, T_2 D}{4 \cdot \frac{1}{6} p^2} - \\
& \frac{2 I E^{I T_0 W_1 - I T_0 W_2} \cdot \frac{!!!}{a} \sin @ 2 p \times D w_1 A_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3} - 6 I E^{-I T_0 W_1 - I T_0 W_2} p^2 \cdot \frac{!!!}{a} \\
& \sin @ 2 p \times D \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D - 2 I E^{-2 I T_0 W_2} p^2 \cdot \frac{!!!}{a} \sin @ p \times D \dot{A}_2 @ T_1, T_2 D^2 F >
\end{aligned}$$

§ 9.1.3 Solvability Conditions

Substituting **sol1**, **Ysol**, **sol2**, and some of the parameter values into **eqEps[[3]]** yields

```

order3Eq = HlinearSys . u-1 -> u3L ==
  HHlinearSys . u-1 -> u3L - H#@@1DD & . ž eqEps@@3DDL . sol1 . Ysol . GRule . sol2 .
  intrRule1 . . intrRule2 . . ExpandL . . values1 . . Thread; . . Timing

821.141 Second, Null<

```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_1 T_0}$, we have

```

ST21 =
  CoefficientA#@@2DD & . ž order3Eq . expRule1@1D, EI w1 T0 E . . modeshapes . . int -> Integrate;

ST22 =
  CoefficientA#@@2DD & . ž order3Eq . expRule1@2D, EI w2 T0 E . . modeshapes . . int -> Integrate;

```

Demanding that **ST21** be orthogonal to the **adjointC[[1]]**, we obtain the solvability condition

```

SCond21 = SolveAHint@adjointC@@1DD.ST21, 8x, 0, 1<D . intRule1 . . intRule2L == 0,
  A1HO,1L@T1, T2DE@@1DD . modeshapes .
  int -> Integrate . SCond1 . frequencies . values1 . . ExpandAll;
SCond21 . displayRule

```

$$:D_2 A_1 \otimes \left(\frac{13}{648} \frac{I f^2 A_1}{3 p^2} - \frac{I m^2 A_1}{4 \cdot 3 p^2} + I \cdot \frac{I}{3} p^2 a A_1^2 \dot{A}_1 - \frac{3}{2} E^{I T_1 s_1} \cdot \frac{I}{a} m A_2 \dot{A}_1 - \frac{8}{3} \frac{I p^2 a A_1 A_2 \dot{A}_2}{3} + \frac{31}{72} \frac{I E^{I T_1 s_1 + I T_1 s_2} f}{2} \cdot \frac{I}{a} A_1 \dot{A}_2 \right)$$

Demanding that **ST22** be orthogonal to the **adjointC[[2]]**, we obtain the solvability condition

```

SCond22 = SolveAHint@adjointC@@2DD.ST22, 8x, 0, 1<D . intRule1 . . intRule2L == 0,
  A2HO,1L@T1, T2DE@@1DD . modeshapes . int -> Integrate .
  SCond1 . W -> 2 w2 . frequencies . values1 . . ExpandAll;
SCond22 . displayRule

```

$$:D_2 A_2 \otimes \left(\frac{1}{4} \frac{3}{2} E^{-I T_1 s_1} \cdot \frac{I}{a} m A_1^2 - \frac{19}{9216} \frac{I f^2 A_2}{3 p^2} - \frac{I m^2 A_2}{8 \cdot 3 p^2} - \frac{4}{3} \frac{I p^2 a A_1 A_2 \dot{A}_1}{3} + \frac{31}{288} \frac{I E^{I T_1 s_1 + I T_1 s_2} f}{2} \cdot \frac{I}{a} A_1^2 \dot{A}_2 - \frac{5}{288} \frac{E^{I T_1 s_2} f m \dot{A}_2}{p^2} - \frac{1}{2} I \cdot \frac{I}{3} p^2 a A_2^2 \dot{A}_2 \right)$$

Because in the absence of damping the system is conservative, the modulation equations, solvability conditions, must satisfy symmetry conditions. To check for these symmetries, we let

```

formList1a = 9A1@T1, T2D A2@T1, T2D, A1@T1, T2D A2@T1, T2D A2@T1, T2D=;
a1coefs = 8s11, s12<->Coefficient@H2 w1 IL SCond21@@1, 2DD, formList1aD . . Thread

```

$$:S_{11} \otimes - \frac{31}{36} \frac{E^{I T_1 s_1 + I T_1 s_2} f}{2} \cdot \frac{I}{a} w_1, S_{12} \otimes \frac{16}{3} \frac{p^2 a w_1}{3}$$

```

formList2a = 9A1@T1, T2D A2@T1, T2D A1@T1, T2D, A1@T1, T2D^2=;
a2coefs = 8s21, s22<->Coefficient@H2 w2 IL SCond22@@1, 2DD, formList2aD . . Thread

```

$$:S_{21} \otimes \frac{8}{3} \frac{p^2 a w_2}{3}, S_{22} \otimes - \frac{31}{144} \frac{E^{I T_1 s_1 + I T_1 s_2} f}{2} \cdot \frac{I}{a} w_2$$

The symmetry condition demands that $S_{12} = S_{21}$, which is true because

```

s12 - s21 . a1coefs . a2coefs . frequencies . values1
0

```

à 9.2 Four-Mode Interactions in Suspended Cables

We determine a second-order uniform asymptotic expansion of the three-dimensional response of a suspended cable to a transverse harmonic distributed excitation of one of the first two vertical or horizontal modes when their frequencies are in the ratio of either two-to-one or one-to-one and no other internal resonance is activated. Benedettini, Rega, and Alaggio (1995) derived nondimensional equations governing three-dimensional finite-amplitude vibrations of a suspended homogeneous elastic cable excited by harmonic distributed vertical and horizontal loads. The time derivatives are expressed in second-order form. As discussed in Chapter 5, second- and higher-order treatments of such equations lead to inconsistent results unless the time derivatives are expressed in first-order rather than second-order form. Therefore, we rewrite these equations as

```

di_Integer,j_Integer := If@i == j, 1, 0D

eq1 = Table@uj,t - vj == 0, 8j, 2<D;
eq2 = TableAvj,t + 2 mj vj - uj,x,x - a Hb y2@xD d1,j + uj,x,x La  $\sum_{k=0}^1 b y^k @xD u_{1,x} + \frac{1}{2} |u_{1,x}^2 + u_{2,x}^2| M \hat{a} x ==$ 
Pj@xD Cos@W t + tjD, 8j, 2<E;

EOM = Transpose@8eq1, eq2<D . {u-s,m :> {us@x, tD, vs :> vs@x, tD} // Flatten

:- v1@x, tD + u1H0,1L@x, tD == 0, 2 m1 v1@x, tD + v1H0,1L@x, tD - u1H2,0L@x, tD -
a  $\sum_{k=0}^1 \int b y^k @xD u_1^{H1,0L} @x, tD + \frac{1}{2} J u_1^{H1,0L} @x, tD^2 + u_2^{H1,0L} @x, tD^2 N \hat{a} x \frac{y}{2} | b y^2 @xD + u_1^{H2,0L} @x, tD ==$ 
Cos@t W + t1D P1@xD, -v2@x, tD + u2H0,1L@x, tD == 0, 2 m2 v2@x, tD + v2H0,1L@x, tD - u2H2,0L@x, tD -
a  $\sum_{k=0}^1 \int b y^k @xD u_1^{H1,0L} @x, tD + \frac{1}{2} J u_1^{H1,0L} @x, tD^2 + u_2^{H1,0L} @x, tD^2 N \hat{a} x \frac{y}{2} u_2^{H2,0L} @x, tD ==$ 
Cos@t W + t2D P2@xD>

yRule = 8y -> H4 # H1 - #L &L<;

```

Moreover, we write the associated boundary conditions as

```
BC = 8uj@0, tD == 0, uj@1, tD == 0<;
```

Here, the index $j = 1$ refers to the vertical component of the displacement and $j = 2$ refers to the horizontal component of the displacement; the m_j are the viscous damping coefficients; the $P_j @xD$ are the distributed amplitudes of the excitation with the frequency W ; $d_{i,j}$ denotes the Kronecker delta; and $b y @xD$, where $y @xD = 4xH1 - xL$, defined in the **yRule**, is the initial static configuration. The spatial and temporal variables were nondimensionalized using the span of the cable $\{$ and the characteristic time $\{ \cdot \cdot \cdot E$, where $\cdot$ and E are the cable density and modulus of elasticity, respectively.

We seek an asymptotic expansion of the response of the cable when its span is such that the first vertical and horizontal natural frequencies are in the ratio of two-to-one; that is, $b^2 a \gg p^2 \cdot 16$. When $b^2 a = p^2 \cdot 16$, the natural frequencies are

```
params = 9wj ; j! = 3 :> 2 p, w3 -> p, b2 a -> p2 • 16=;
```

and the corresponding normalized modeshapes are

```

modeshapes = 9F1 -> { $ {H1 - Cos@2 p #DL &Z,
                        {
F2 -> I * 2 Sin@2 p #D &M, F3 -> I * 2 Sin@p #D &M, F4 -> I * 2 Sin@2 p #D &M=;

```

We use the method of multiple scales and seek a second-order uniform asymptotic expansion of the response of the cable to a primary-resonance excitation of the first vertical mode in the form

```

multiScales = 9u_{j_}@x, tD -> u_j@x, T_0, T_1, T_2D,
Derivative@m_, n_D@u_D@x, tD -> dt@nD@D@u@x, T_0, T_1, T_2D, 8x, m<DD, t -> T_0=;

solRule = 9u_{i_} -> IEvaluateASumAe^j u_{i,j}@#1, #2, #3, #4D, 8j, 3<EE &M,
v_{i_} -> IEvaluateASumAe^j v_{i,j}@#1, #2, #3, #4D, 8j, 3<EE &M=;

```

Taking into account the resonance conditions, we scale the damping and forcing terms as

```

scaling = 9m_{j_} -> e^2 m_j, P_{j_}@xD -> e^3 P_j@xD=;

```

where the forcing has been scaled at order e^3 so that its influence first appears at the same order as the nonlinear shift in the frequencies in the absence of the internal resonance.

We consider the case in which the orders of magnitude of the forcing levels for both the vertical and horizontal motions are the same. Otherwise, a primary resonance of the vertical modes (current study) could also initiate a subharmonic resonance of order one-half of the first horizontal mode and a primary resonance of the second horizontal mode at different levels of approximation.

Substituting the **multiScales**, **solRule**, and **scaling** into **EOM**, expanding the result for small e , and discarding terms of order higher than e^3 , we obtain

```

eq92a =
HHEOM . Integrate -> int . multiScales . solRule . scaling . ExpandAllL . . intRule2 . .
ExpandAllL . . e^{n_};n>3 -> 0;

```

Equating coefficients of like powers of e in **eq92a**, we obtain

```

eqEps =
Rest@Thread@CoefficientList@Subtract @@ #, eD == 0DD & . eq92a . . Transpose . . TrigToExp;

```

§ 9.2.1 First-Order Solution

The homogeneous parts of the first-order equations are

```

HlinearSys = #@@1DD & . eqEps@@1DDL . displayRule

8D_0 u_{1,1} - v_{1,1}, -Hu'_{1,1}L + D_0 v_{1,1} - b^2 a int@Hu'_{1,1}L y^e@xD, 8x, 0, 1<D y^2@xD,
D_0 u_{2,1} - v_{2,1}, -Hu'_{2,1}L + D_0 v_{2,1}<

```

The first-order problems are identical to the linear eigenvalue problems. To account for the interactions arising from the multiple internal resonances and the primary resonance, we include the lowest two modes in each plane; the other modes will decay due to damping. Hence, the solution of `eqEps[[1]]` can be expressed as

```
sol1u = 9u1,1 -> FunctionA8x, T0, T1, T2<,
    SumAAi@T1, T2D Fi@xD Exp@I wi T0D +  $\dot{A}_i$ @T1, T2D Fi@xD Exp@- I wi T0D, 8i, 1, 2<E • EvaluateE,
u2,1 -> FunctionA8x, T0, T1, T2<,
    SumAAi@T1, T2D Fi@xD Exp@I wi T0D +  $\dot{A}_i$ @T1, T2D Fi@xD Exp@- I wi T0D, 8i, 3, 4<E • EvaluateE;

sol1v = Table@
    vk,1 ® Function@8x, T0, T1, T2<, D@uk,1@x, T0, T1, T2D, T0D •. sol1u • EvaluateD, 8k, 2<D;

sol1 = Join@sol1u, sol1vD

8u1,1 ® Function@8x, T0, T1, T2<, EI T0 w1 A1@T1, T2D F1@xD +
    EI T0 w2 A2@T1, T2D F2@xD + E-I T0 w1 F1@xD  $\dot{A}_1$ @T1, T2D + E-I T0 w2 F2@xD  $\dot{A}_2$ @T1, T2DD,
u2,1 ® Function@8x, T0, T1, T2<, EI T0 w3 A3@T1, T2D F3@xD + EI T0 w4 A4@T1, T2D F4@xD +
    E-I T0 w3 F3@xD  $\dot{A}_3$ @T1, T2D + E-I T0 w4 F4@xD  $\dot{A}_4$ @T1, T2DD,
v1,1 ® Function@8x, T0, T1, T2<, I EI T0 w1 w1 A1@T1, T2D F1@xD + I EI T0 w2 w2 A2@T1, T2D F2@xD -
    I E-I T0 w1 w1 F1@xD  $\dot{A}_1$ @T1, T2D - I E-I T0 w2 w2 F2@xD  $\dot{A}_2$ @T1, T2DD,
v2,1 ® Function@8x, T0, T1, T2<, I EI T0 w3 w3 A3@T1, T2D F3@xD + I EI T0 w4 w4 A4@T1, T2D F4@xD -
    I E-I T0 w3 w3 F3@xD  $\dot{A}_3$ @T1, T2D - I E-I T0 w4 w4 F4@xD  $\dot{A}_4$ @T1, T2DD<
```

where $F_1(x)$ and $F_2(x)$ are the lowest symmetric and antisymmetric vertical eigenmodes, and $F_3(x)$ and $F_4(x)$ are the lowest symmetric and antisymmetric horizontal eigenmodes. The $F_n(x)$ are orthogonal; they are normalized so that they satisfy the orthonormality condition $\int_0^1 F_n(x) F_m(x) dx = \delta_{n,m}$.

Because the problem is self-adjoint, we have

```
adjoint = Table@8- I wk Fk@xD, Fk@xD<, 8k, 4<D;
```

whose complex conjugate is

```
adjointc = adjoint •. conjugateRule
```

```
88I w1 F1@xD, F1@xD<, 8I w2 F2@xD, F2@xD<, 8I w3 F3@xD, F3@xD<, 8I w4 F4@xD, F4@xD<<
```

9.2.2 Second-Order Solution

Substituting `sol1` into `eqEps[[2]]` yields

```
order2Eq = llinearSys •. u-k,1 -> uk,2M ==
```

```
llinearSys •. u-k,1 -> uk,2M - H#@@1DD & •ž eqEps@@2DDL •. sol1 •. intRule1 ••. intRule2 •.
```

```
modeshapes •. yRule •. int -> Integrate •• ExpandM •• Thread;
```

```
order2Eq •. displayRule
```

$$\begin{aligned}
& :D_0 u_{1,2} - v_{1,2} == - \frac{2}{3} E^{I T_0 w_1} H D_1 A_1 L + \frac{2}{3} E^{I T_0 w_1} \cos 2 p x D H D_1 A_1 L - \frac{2}{3} E^{-I T_0 w_1} H D_1 \dot{A}_1 L + \\
& \frac{2}{3} E^{-I T_0 w_1} \cos 2 p x D H D_1 \dot{A}_1 L - \frac{1}{2} E^{I T_0 w_2} H D_1 A_2 L \sin 2 p x D - \frac{1}{2} E^{-I T_0 w_2} H D_1 \dot{A}_2 L \sin 2 p x D, \\
& - H u_{1,2}' L + D_0 v_{1,2} - b^2 a \text{int}@H u_{1,2}' L y^t @ x D, 8 x, 0, 1 < D y^2 @ x D == \\
& - \frac{16}{3} b E^{2 I T_0 w_1} p^2 a A_1^2 + \frac{64}{3} b E^{2 I T_0 w_1} p^2 a \cos 2 p x D A_1^2 - \frac{64 b E^{I T_0 H w_1 + w_2 L} p^2 a \sin 2 p x D A_1 A_2}{3} - \\
& 16 b E^{2 I T_0 w_2} p^2 a A_2^2 - 4 b E^{2 I T_0 w_3} p^2 a A_3^2 - 16 b E^{2 I T_0 w_4} p^2 a A_4^2 - I \frac{2}{3} E^{I T_0 w_1} H D_1 A_1 L w_1 + \\
& I \frac{2}{3} E^{I T_0 w_1} \cos 2 p x D H D_1 A_1 L w_1 + I \frac{2}{3} E^{-I T_0 w_1} H D_1 \dot{A}_1 L w_1 - I \frac{2}{3} E^{-I T_0 w_1} \cos 2 p x D H D_1 \dot{A}_1 L w_1 - \\
& I \frac{1}{2} E^{I T_0 w_2} H D_1 A_2 L \sin 2 p x D w_2 + I \frac{1}{2} E^{-I T_0 w_2} H D_1 \dot{A}_2 L \sin 2 p x D w_2 - \frac{32}{3} b p^2 a A_1 \dot{A}_1 + \\
& \frac{128}{3} b p^2 a \cos 2 p x D A_1 \dot{A}_1 - \frac{64 b E^{I T_0 H - w_1 + w_2 L} p^2 a \sin 2 p x D A_2 \dot{A}_1}{3} - \frac{16}{3} b E^{-2 I T_0 w_1} p^2 a \dot{A}_1^2 + \\
& \frac{64}{3} b E^{-2 I T_0 w_1} p^2 a \cos 2 p x D \dot{A}_1^2 - \frac{64 b E^{I T_0 H w_1 - w_2 L} p^2 a \sin 2 p x D A_1 \dot{A}_2}{3} - \\
& 32 b p^2 a A_2 \dot{A}_2 - \frac{64 b E^{I T_0 H - w_1 - w_2 L} p^2 a \sin 2 p x D \dot{A}_1 \dot{A}_2}{3} - 16 b E^{-2 I T_0 w_2} p^2 a \dot{A}_2^2 - \\
& 8 b p^2 a A_3 \dot{A}_3 - 4 b E^{-2 I T_0 w_3} p^2 a \dot{A}_3^2 - 32 b p^2 a A_4 \dot{A}_4 - 16 b E^{-2 I T_0 w_4} p^2 a \dot{A}_4^2, \\
& D_0 u_{2,2} - v_{2,2} == - \frac{1}{2} E^{I T_0 w_3} H D_1 A_3 L \sin p x D - \frac{1}{2} E^{-I T_0 w_3} H D_1 \dot{A}_3 L \sin p x D - \\
& \frac{1}{2} E^{I T_0 w_4} H D_1 A_4 L \sin 2 p x D - \frac{1}{2} E^{-I T_0 w_4} H D_1 \dot{A}_4 L \sin 2 p x D, \\
& - H u_{2,2}' L + D_0 v_{2,2} == - \frac{16 b E^{I T_0 H w_1 + w_3 L} p^2 a \sin p x D A_1 A_3}{3} - \frac{64 b E^{I T_0 H w_1 + w_4 L} p^2 a \sin 2 p x D A_1 A_4}{3} - \\
& I \frac{1}{2} E^{I T_0 w_3} H D_1 A_3 L \sin p x D w_3 + I \frac{1}{2} E^{-I T_0 w_3} H D_1 \dot{A}_3 L \sin p x D w_3 - \\
& I \frac{1}{2} E^{I T_0 w_4} H D_1 A_4 L \sin 2 p x D w_4 + I \frac{1}{2} E^{-I T_0 w_4} H D_1 \dot{A}_4 L \sin 2 p x D w_4 - \\
& \frac{16 b E^{I T_0 H - w_1 + w_3 L} p^2 a \sin p x D A_3 \dot{A}_1}{3} - \frac{64 b E^{I T_0 H - w_1 + w_4 L} p^2 a \sin 2 p x D A_4 \dot{A}_1}{3} - \\
& \frac{16 b E^{I T_0 H w_1 - w_3 L} p^2 a \sin p x D A_1 \dot{A}_3}{3} - \frac{16 b E^{I T_0 H - w_1 - w_3 L} p^2 a \sin p x D \dot{A}_1 \dot{A}_3}{3} - \\
& \frac{64 b E^{I T_0 H w_1 - w_4 L} p^2 a \sin 2 p x D A_1 \dot{A}_4}{3} - \frac{64 b E^{I T_0 H - w_1 - w_4 L} p^2 a \sin 2 p x D \dot{A}_1 \dot{A}_4}{3} >
\end{aligned}$$

We consider the case of primary resonance of the first vertical mode, a two-to-one internal resonance between the first vertical and horizontal modes, a one-to-one internal resonance between the first and second vertical modes, and a one-to-one

internal resonance between the first vertical and second horizontal modes. In order to collect the terms that may lead to secular terms from the right-hand sides of **order2Eq**, we define the rules

```
omgList = Table@wi, 8i, 4<D;
ResonanceConds = 8W == w1 + e s1, w2 == w1 + e s2, 2 w3 == w1 + e s3, w4 == w1 + e s4<;

OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D~Join~8W<D@@1DD & . Ź omgList

99w2 ® e s2 + w1, w3 ®  $\frac{1}{2}$  He s3 + w1L, w4 ® e s4 + w1, W ® e s1 + w1=,

9w1 ® -e s2 + w2, w3 ®  $\frac{1}{2}$  H - e s2 + e s3 + w2L, w4 ® -e s2 + e s4 + w2, W ® e s1 - e s2 + w2=,

8w1 ® -e s3 + 2 w3, w2 ® e s2 - e s3 + 2 w3, w4 ® -e s3 + e s4 + 2 w3, W ® e s1 - e s3 + 2 w3<,

9w1 ® -e s4 + w4, w2 ® e s2 - e s4 + w4, w3 ®  $\frac{1}{2}$  He s3 - e s4 + w4L, W ® e s1 - e s4 + w4==

expRule1@i_D := Exp@arg_D :> Exp@Expand@arg . OmgRule@@iDDD . e T0 -> T1D
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I w_1 T_0}$, from the equations governing the vertical modes, we have

```
ST1v = CoefficientA#@@2DD & . Ź order2Eq@@81, 2<DD . expRule1@#D, EI w1 T0E & . Ź 81, 2<;
ST1v . displayRule

:: - $  $\frac{2}{3}$  HD1A1L + $  $\frac{2}{3}$  Cos@2 p xD HD1A1L - •  $\frac{1}{2}$  EI T1 s2 HD1A2L Sin@2 p xD, - 4 b EI T1 s3 p2 a A32 -

I $  $\frac{2}{3}$  HD1A1L w1 + I $  $\frac{2}{3}$  Cos@2 p xD HD1A1L w1 - I •  $\frac{1}{2}$  EI T1 s2 HD1A2L Sin@2 p xD w2>,

:- $  $\frac{2}{3}$  E-I T1 s2 HD1A1L + $  $\frac{2}{3}$  E-I T1 s2 Cos@2 p xD HD1A1L - •  $\frac{1}{2}$  HD1A2L Sin@2 p xD,

- 4 b E-I T1 s2 + I T1 s3 p2 a A32 - I $  $\frac{2}{3}$  E-I T1 s2 HD1A1L w1 +

I $  $\frac{2}{3}$  E-I T1 s2 Cos@2 p xD HD1A1L w1 - I •  $\frac{1}{2}$  HD1A2L Sin@2 p xD w2>>
```

Collecting the terms that may lead to secular terms from the equations governing the horizontal modes, we have

```
ST1h = CoefficientA#@@2DD & . Ź order2Eq@@83, 4<DD . expRule1@#D, EI w1 T0E & . Ź 83, 4<;
ST1h . displayRule

:: - •  $\frac{1}{2}$  HD1A3L Sin@p xD, - I •  $\frac{1}{2}$  HD1A3L Sin@p xD w3 -  $\frac{16 b E^{-I T_1 s_3} p^2 a \sin@p xD A_1 A_3}{3}$ ,

9 - •  $\frac{1}{2}$  HD1A4L Sin@2 p xD, - I •  $\frac{1}{2}$  HD1A4L Sin@2 p xD w4=>
```

Demanding that **ST1v[[i]]** be orthogonal to the **adjointC[[i]]**, we obtain the solvability conditions

```

SCond1v =
  SolveAHint@adjointC@@#DD.ST1v@@#DD, 8x, 0, 1<D •. intRule1 •. intRule2 •. modeshapes •.
    params •. int -> IntegrateL == 0, AHL,OL#@T1, T2DE@@1DD & •ž 81, 2< •. Flatten;
SCond1v •. displayRule

: D1A1 ® I $  $\frac{2}{3}$  b EI T1 s3 p a A32, D1A2 ® 0>

```

Demanding that **ST1h[[i]]** be orthogonal to the **adjointC[[i+2]]**, we obtain the solvability conditions

```

SCond1h =
  SolveAHint@adjointC@@# + 2DD.ST1h@@#DD, 8x, 0, 1<D •. intRule1 •. intRule2 •. modeshapes •.
    params •. int -> IntegrateL == 0, AHL,OL#+2@T1, T2DE@@1DD & •ž 81, 2< •. Flatten;
SCond1h •. displayRule

: D1A3 ® 4 I $  $\frac{2}{3}$  b E-I T1 s3 p a A1 A3•, D1A4 ® 0>

SCond1 = Join@SCond1v, SCond1hD;

```

whose complex conjugates are

```
ccSCond1 = SCond1 •. conjugateRule;
```

Next, we use the solvability conditions to eliminate the $D_1 A_i$ from the right-hand sides of **order2Eq**. To this end, we define the rules

```

sigRule = Solve@ResonanceConds, Table@si, 8i, 4<DD@@1DD

9s1 ® -  $\frac{-W + w_1}{e}$ , s2 ® -  $\frac{W_1 - W_2}{e}$ , s3 ® -  $\frac{W_1 - 2W_3}{e}$ , s4 ® -  $\frac{W_1 - W_4}{e} =$ 

expRule2 = Exp@a_D :-> Exp@a •. sigRule •. T1 -> e T0 •. ExpandD;

```

Substituting **SCond1**, **ccSCond1**, and **expRule2** into **order2Eq**, we obtain

```
order2Eqm = order2Eq •. SCond1 •. ccSCond1 •. expRule2;
```

To determine the solution of **order2Eqm** and the associated boundary conditions, we use the method of undetermined coefficients. To accomplish this, we first determine the forms of the terms on the right-hand sides of **order2Eqm** by picking up the x dependent or x and T_0 dependent terms by defining the rule

```
baits = 9_ Ea_ T0 + b_ f_@c_ xD :-> Ea T0 + b f@c xD, a_ f_@b_ xD :-> f@b xD •; FreeQ@a, T0D=;
```

Then, we collect the solution forms for the horizontal modes according to

```
sol2Formh = Table@Cases@order2Eqm@@k, 2DD, #D & . ž baits, 8k, 3, 4<D • Flatten • Union
```

```
8E-I T0 w1 - I T0 w3 Sin@p xD, EI T0 w1 - I T0 w3 Sin@p xD, E-I T0 w1 + I T0 w3 Sin@p xD, EI T0 w1 + I T0 w3 Sin@p xD,  
E-I T0 w1 - I T0 w4 Sin@2 p xD, EI T0 w1 - I T0 w4 Sin@2 p xD, E-I T0 w1 + I T0 w4 Sin@2 p xD, EI T0 w1 + I T0 w4 Sin@2 p xD<
```

Each of these terms satisfies the boundary conditions. However, not all of the solution forms

```
sol2Formv = Table@Cases@order2Eqm@@k, 2DD, #D & . ž baits, 8k, 2<D • Flatten • Union
```

```
8Cos@2 p xD, E-2 I T0 w1 Cos@2 p xD, E2 I T0 w1 Cos@2 p xD, E-2 I T0 w3 Cos@2 p xD, E2 I T0 w3 Cos@2 p xD,  
E-I T0 w1 - I T0 w2 Sin@2 p xD, EI T0 w1 - I T0 w2 Sin@2 p xD, E-I T0 w1 + I T0 w2 Sin@2 p xD, EI T0 w1 + I T0 w2 Sin@2 p xD<
```

for the vertical modes satisfy the boundary conditions. Besides, there exist terms that are not functions of x on the right-hand sides of `order2Eqm[{{1,2}}]`, namely

```
#@@2DD & . ž order2Eqm@@81, 2<DD • Thread@sol2Formv -> 0D • . displayRule
```

```
9 -  $\frac{2}{3} I b E^{2 I T_0 w_3} p a A_3^2 + \frac{2}{3} I b E^{-2 I T_0 w_3} p a \dot{A}_3^2,$   
-  $\frac{16}{3} b E^{2 I T_0 w_1} p^2 a A_1^2 - 16 b E^{2 I T_0 w_2} p^2 a A_2^2 - 4 b E^{2 I T_0 w_3} p^2 a A_3^2 - 16 b E^{2 I T_0 w_4} p^2 a A_4^2 +$   
 $\frac{2}{3} b E^{2 I T_0 w_3} p a A_3^2 w_1 - \frac{32}{3} b p^2 a A_1 \dot{A}_1 - \frac{16}{3} b E^{-2 I T_0 w_1} p^2 a \dot{A}_1^2 - 32 b p^2 a A_2 \dot{A}_2 - 16 b E^{-2 I T_0 w_2} p^2 a \dot{A}_2^2 -$   
 $8 b p^2 a A_3 \dot{A}_3 - 4 b E^{-2 I T_0 w_3} p^2 a \dot{A}_3^2 + \frac{2}{3} b E^{-2 I T_0 w_3} p a w_1 \dot{A}_3^2 - 32 b p^2 a A_4 \dot{A}_4 - 16 b E^{-2 I T_0 w_4} p^2 a \dot{A}_4^2 =$ 
```

Hence, we need to either consider some algebraic forms or combined algebraic and sinusoidal forms for x to manually adjust the forms in `sol2Formv` or use a more general approach to let *Mathematica* pick up the right forms. The latter approach is used here.

Ÿ Vertical Modes

Instead of looking for the x dependent forms, we seek all of the T_0 dependent forms as

```
sol2Formv1 =
```

```
Join@81<, Table@Cases@order2Eqm@@k, 2DD, Exp@_D, žD, 8k, 2<D • Flatten • UnionD
```

```
81, E-2 I T0 w1, E2 I T0 w1, E-2 I T0 w2, E2 I T0 w2, E-I T0 w1 - I T0 w2,  
EI T0 w1 - I T0 w2, E-I T0 w1 + I T0 w2, EI T0 w1 + I T0 w2, E-2 I T0 w3, E2 I T0 w3, E-2 I T0 w4, E2 I T0 w4<
```

where the first argument in `sol2Formv1` is included to account for the T_0 independent terms.

We define the undetermined coefficients for $u_{1,2}$ and $v_{1,2}$ as

```
uSymbolv = Table@y1,i@xD, 8i, Length@sol2Formv1D<D;
```

```
vSymbolv = uSymbolv • y -> h;
```

and write the general solution in the form

```
sol2v = 8u1,2 -> Function@8x, T0, T1, T2<, uSymbolv.sol2Formv1 •• Evaluated,
v1,2 -> Function@8x, T0, T1, T2<, vSymbolv.sol2Formv1 •• Evaluated<;
```

Substituting **sol2v** into **order2Eqm[[1]]** and equating the coefficients of the **sol2Formv1** on both sides yields

```
expr1 = Subtract @@ order2Eqm@1DD •. sol2v;

eqv1 = Join@8expr1 == 0 •. Exp@_D -> 0<, Coefficient@expr1, Rest@sol2Formv1DD == 0 •• ThreadD

9 - h1,1@xD == 0, - h1,2@xD - 2 I w1 y1,2@xD == 0, - h1,3@xD + 2 I w1 y1,3@xD == 0,
- h1,4@xD - 2 I w2 y1,4@xD == 0, - h1,5@xD + 2 I w2 y1,5@xD == 0,
- h1,6@xD - I w1 y1,6@xD - I w2 y1,6@xD == 0, - h1,7@xD + I w1 y1,7@xD - I w2 y1,7@xD == 0,
- h1,8@xD - I w1 y1,8@xD + I w2 y1,8@xD == 0, - h1,9@xD + I w1 y1,9@xD + I w2 y1,9@xD == 0,
-  $\frac{2}{3}$  I b p a  $\dot{A}_3@T_1, T_2D^2 + \frac{2}{3}$  I b p a Cos@2 p xD  $\dot{A}_3@T_1, T_2D^2 - h_{1,10}@xD - 2 I w_3 y_{1,10}@xD == 0,$ 
 $\frac{2}{3}$  I b p a  $A_3@T_1, T_2D^2 - \frac{2}{3}$  I b p a Cos@2 p xD  $A_3@T_1, T_2D^2 - h_{1,11}@xD + 2 I w_3 y_{1,11}@xD == 0,$ 
- h1,12@xD - 2 I w4 y1,12@xD == 0, - h1,13@xD + 2 I w4 y1,13@xD == 0=
```

Solving for **vSymbolv** from **eqv1**, we obtain

```
eta1Sol = Solve@eqv1, vSymbolvD@1DD

9 h1,1@xD @ 0, h1,2@xD @ - 2 I w1 y1,2@xD, h1,3@xD @ 2 I w1 y1,3@xD, h1,4@xD @ - 2 I w2 y1,4@xD,
h1,5@xD @ 2 I w2 y1,5@xD, h1,6@xD @ - I H w1 y1,6@xD + w2 y1,6@xDL, h1,7@xD @ I H w1 y1,7@xD - w2 y1,7@xDL,
h1,8@xD @ - I H w1 y1,8@xD - w2 y1,8@xDL, h1,9@xD @ I H w1 y1,9@xD + w2 y1,9@xDL,
h1,10@xD @  $\frac{2}{3}$  I J - b p a  $\dot{A}_3@T_1, T_2D^2 + b p a \text{Cos}@2 p xD \dot{A}_3@T_1, T_2D^2 - 3 w_3 y_{1,10}@xDN,$ 
h1,11@xD @  $\frac{2}{3}$  I H - b p a  $A_3@T_1, T_2D^2 + b p a \text{Cos}@2 p xD A_3@T_1, T_2D^2 - 3 w_3 y_{1,11}@xDL,$ 
h1,12@xD @ - 2 I w4 y1,12@xD, h1,13@xD @ 2 I w4 y1,13@xD=
```

Substituting **sol2v** and **eta1Sol** into **order2Eqm[[2]]** and equating the coefficients of the **sol2Formv1** on both sides yields

```
expr2 = Subtract @@ order2Eqm@2DD •. sol2v •. eta1Sol •. intRule1 ••. intRule2 •.
int@yε@xD y1,kε@xD, _D -> bk •. yRule;
```

```

eqv2 = Join@8expr2 == 0 •. Exp@_D -> 0<,
  Coefficient@expr2, Rest@sol2Formv1DD == 0 •• ThreadD •. params
:

$$\frac{p^2 b_1}{2} + \frac{32}{3} b p^2 a A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D -$$


$$\frac{128}{3} b p^2 a \cos @ 2 p x D A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D + 32 b p^2 a A_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D +$$


$$8 b p^2 a A_3 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D + 32 b p^2 a A_4 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D - y_{1,1}^2 @ x D == 0,$$


$$\frac{p^2 b_2}{2} + \frac{16}{3} b p^2 a \dot{A}_1 @ T_1, T_2 D^2 - \frac{64}{3} b p^2 a \cos @ 2 p x D \dot{A}_1 @ T_1, T_2 D^2 - 16 p^2 y_{1,2} @ x D - y_{1,2}^2 @ x D == 0,$$


$$\frac{p^2 b_3}{2} + \frac{16}{3} b p^2 a A_1 @ T_1, T_2 D^2 - \frac{64}{3} b p^2 a \cos @ 2 p x D A_1 @ T_1, T_2 D^2 - 16 p^2 y_{1,3} @ x D - y_{1,3}^2 @ x D == 0,$$


$$\frac{p^2 b_4}{2} + 16 b p^2 a \dot{A}_2 @ T_1, T_2 D^2 - 16 p^2 y_{1,4} @ x D - y_{1,4}^2 @ x D == 0,$$


$$\frac{p^2 b_5}{2} + 16 b p^2 a A_2 @ T_1, T_2 D^2 - 16 p^2 y_{1,5} @ x D - y_{1,5}^2 @ x D == 0,$$


$$\frac{p^2 b_6}{2} + \frac{64 b p^2 a \sin @ 2 p x D \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3} - 16 p^2 y_{1,6} @ x D - y_{1,6}^2 @ x D == 0,$$


$$\frac{p^2 b_7}{2} + \frac{64 b p^2 a \sin @ 2 p x D A_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3} - y_{1,7}^2 @ x D == 0,$$


$$\frac{p^2 b_8}{2} + \frac{64 b p^2 a \sin @ 2 p x D A_2 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3} - y_{1,8}^2 @ x D == 0,$$


$$\frac{p^2 b_9}{2} + \frac{64 b p^2 a \sin @ 2 p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D}{3} - 16 p^2 y_{1,9} @ x D - y_{1,9}^2 @ x D == 0,$$


$$\frac{p^2 b_{10}}{2} + \frac{4}{3} b p^2 a \dot{A}_3 @ T_1, T_2 D^2 + \frac{8}{3} b p^2 a \cos @ 2 p x D \dot{A}_3 @ T_1, T_2 D^2 - 4 p^2 y_{1,10} @ x D - y_{1,10}^2 @ x D == 0,$$


$$\frac{p^2 b_{11}}{2} + \frac{4}{3} b p^2 a A_3 @ T_1, T_2 D^2 + \frac{8}{3} b p^2 a \cos @ 2 p x D A_3 @ T_1, T_2 D^2 - 4 p^2 y_{1,11} @ x D - y_{1,11}^2 @ x D == 0,$$


$$\frac{p^2 b_{12}}{2} + 16 b p^2 a \dot{A}_4 @ T_1, T_2 D^2 - 16 p^2 y_{1,12} @ x D - y_{1,12}^2 @ x D == 0,$$


$$\frac{p^2 b_{13}}{2} + 16 b p^2 a A_4 @ T_1, T_2 D^2 - 16 p^2 y_{1,13} @ x D - y_{1,13}^2 @ x D == 0 >$$


```

Next, we determine the solutions of the boundary-value problems: **eqv2** and the associated boundary conditions. First, we obtain the information about the solution forms **uSymbolv** according to

```
yForm = DSolve@8#@1DD, #@@2DD == 0 . x -> 0, #@@2DD == 0 . x -> 1<, #@@2DD, xD@@1DD .
```

```
C@i_D -> 0 & . ž Transpose@8eqv2, uSymbolv<D . Flatten . ExpandAll;
```

```
yForm . displayRule
```

```
: y1,1@xD ® - 1/4 p^2 x b1 + 1/4 p^2 x^2 b1 - 32/3 b a A1 A1 - 16/3 b p^2 x a A1 A1 +
16/3 b p^2 x^2 a A1 A1 + 32/3 b a Cos@2 p xD A1 A1 - 16 b p^2 x a A2 A2 + 16 b p^2 x^2 a A2 A2 -
4 b p^2 x a A3 A3 + 4 b p^2 x^2 a A3 A3 - 16 b p^2 x a A4 A4 + 16 b p^2 x^2 a A4 A4,
y1,2@xD ® b2/32 - 1/32 Cos@4 p xD b2 + 1/3 b a A1^2 - 16/9 b a Cos@2 p xD A1^2 + 13/9 b a Cos@4 p xD A1^2,
y1,3@xD ® 1/3 b a A1^2 - 16/9 b a Cos@2 p xD A1^2 + 13/9 b a Cos@4 p xD A1^2 + b3/32 - 1/32 Cos@4 p xD b3,
y1,4@xD ® b4/32 - 1/32 Cos@4 p xD b4 + b a A2^2 - b a Cos@4 p xD A2^2,
y1,5@xD ® b a A2^2 - b a Cos@4 p xD A2^2 + b5/32 - 1/32 Cos@4 p xD b5,
y1,6@xD ® b6/32 - 1/32 Cos@4 p xD b6 + 16 b a Sin@2 p xD A1 A2 / 3,
y1,7@xD ® - 1/4 p^2 x b7 + 1/4 p^2 x^2 b7 - 16 b a Sin@2 p xD A1 A2 / 3,
y1,8@xD ® - 1/4 p^2 x b8 + 1/4 p^2 x^2 b8 - 16 b a Sin@2 p xD A2 A1 / 3,
y1,9@xD ® 16 b a Sin@2 p xD A1 A2 / 3 + b9/32 - 1/32 Cos@4 p xD b9,
y1,10@xD ® b10/8 - 1/8 Cos@2 p xD b10 + 1/3 b a A3^2 - 1/3 b a Cos@2 p xD A3^2 + 2/3 b p x a Sin@2 p xD A3^2,
y1,11@xD ® 1/3 b a A3^2 - 1/3 b a Cos@2 p xD A3^2 + 2/3 b p x a Sin@2 p xD A3^2 + b11/8 - 1/8 Cos@2 p xD b11,
y1,12@xD ® b12/32 - 1/32 Cos@4 p xD b12 + b a A4^2 - b a Cos@4 p xD A4^2,
y1,13@xD ® b a A4^2 - b a Cos@4 p xD A4^2 + b13/32 - 1/32 Cos@4 p xD b13>
```

Then, according to **yForm**, we pick up the spatial function forms for **uSymbolv** as

```
bait1 =
```

```
8a_ . x^n_ . :> x^n . ; FreeQ@a, xD, a_ f_@b_ xD :> f@b xD . ; FreeQ@a, xD, a_ x f_@b_ xD :> x f@b xD<;
```

```
funcList = Table@
```

```
Prepend@Cases@yForm@@i, 2DD, #D & . ž bait1 . Flatten . Union, 1D, 8i, Length@yFormD<D
```

```
881, x, x^2, Cos@2 p xD<, 81, Cos@2 p xD, Cos@4 p xD<, 81, Cos@2 p xD, Cos@4 p xD<,
81, Cos@4 p xD<, 81, Cos@4 p xD<, 81, Cos@4 p xD, Sin@2 p xD<, 81, x, x^2, Sin@2 p xD<,
81, x, x^2, Sin@2 p xD<, 81, Cos@4 p xD, Sin@2 p xD<, 81, Cos@2 p xD, x Sin@2 p xD<,
81, Cos@2 p xD, x Sin@2 p xD<, 81, Cos@4 p xD<, 81, Cos@4 p xD<<
```

We assume the solution for $y_{1,i}$ as a linear combination of `funcList[[i]]`, substitute it into `yForm[[i]]`, equate both sides, solve for the undetermined coefficients, and obtain the solution for $y_{1,i}$ as

```
psi1Sol = Table@
  form = funcList@@kDD;
  cList = Table@c_i, 8i, Length@formD<D;
  sol = cList.form; algExpr =
sol - yForm@@k, 2DD •. b_k_ -> Integrate@Expand@y^t@xD D@sol, xD •. yRuleD, 8x, 0, 1<D;
  algEq = Append@Coefficient@algExpr, Rest@formDD,
  algExpr •. Thread@Rest@formD -> 0DD == 0 •• Thread;
  y1,k -> Function@x, sol •. Solve@algEq, cListD@@1DD •• Expand •• Evaluated,
  8k, Length@yFormD<D
```

Solve::svars : Equations may not give solutions for all "solve" variables.

Solve::svars : Equations may not give solutions for all "solve" variables.

$$\begin{aligned}
& y_{1,1} \otimes \text{FunctionBx}, -\frac{32}{3} b a A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D + \frac{48 b p^2 x a A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3 + p^2} - \\
& \frac{48 b p^2 x^2 a A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3 + p^2} + \frac{32}{3} b a \cos 2 p x D A_1 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D - \\
& \frac{48 b p^2 x a A_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 + p^2} + \frac{48 b p^2 x^2 a A_2 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 + p^2} - \\
& \frac{12 b p^2 x a A_3 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D}{3 + p^2} + \frac{12 b p^2 x^2 a A_3 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D}{3 + p^2} - \\
& \frac{48 b p^2 x a A_4 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D}{3 + p^2} + \frac{48 b p^2 x^2 a A_4 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D}{3 + p^2} F, y_{1,2} \otimes \text{FunctionAx}, \\
& \frac{4}{9} b a \dot{A}_1 @ T_1, T_2 D^2 - \frac{16}{9} b a \cos 2 p x D \dot{A}_1 @ T_1, T_2 D^2 + \frac{4}{3} b a \cos 4 p x D \dot{A}_1 @ T_1, T_2 D^2 E, y_{1,3} \otimes \\
& \text{FunctionAx}, \frac{4}{9} b a A_1 @ T_1, T_2 D^2 - \frac{16}{9} b a \cos 2 p x D A_1 @ T_1, T_2 D^2 + \frac{4}{3} b a \cos 4 p x D A_1 @ T_1, T_2 D^2 E, \\
& y_{1,4} \otimes \text{FunctionAx}, \frac{4}{3} b a \dot{A}_2 @ T_1, T_2 D^2 - \frac{4}{3} b a \cos 4 p x D \dot{A}_2 @ T_1, T_2 D^2 E, \\
& y_{1,5} \otimes \text{FunctionAx}, \frac{4}{3} b a A_2 @ T_1, T_2 D^2 - \frac{4}{3} b a \cos 4 p x D A_2 @ T_1, T_2 D^2 E, \\
& y_{1,6} \otimes \text{FunctionBx}, \frac{16 b a \sin 2 p x D \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot 3} F, \\
& y_{1,7} \otimes \text{FunctionBx}, -\frac{16 b a \sin 2 p x D \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot 3} F, \\
& y_{1,8} \otimes \text{FunctionBx}, -\frac{16 b a \sin 2 p x D \dot{A}_2 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D}{3 \cdot 3} F, \\
& y_{1,9} \otimes \text{FunctionBx}, \frac{16 b a \sin 2 p x D \dot{A}_1 @ T_1, T_2 D \dot{A}_2 @ T_1, T_2 D}{3 \cdot 3} F, \\
& y_{1,10} \otimes \text{FunctionAx}, -c_2 + \cos 2 p x D c_2 + \frac{2}{3} b p x a \sin 2 p x D \dot{A}_3 @ T_1, T_2 D^2 E, \\
& y_{1,11} \otimes \text{FunctionAx}, -c_2 + \cos 2 p x D c_2 + \frac{2}{3} b p x a \sin 2 p x D A_3 @ T_1, T_2 D^2 E, \\
& y_{1,12} \otimes \text{FunctionAx}, \frac{4}{3} b a \dot{A}_4 @ T_1, T_2 D^2 - \frac{4}{3} b a \cos 4 p x D \dot{A}_4 @ T_1, T_2 D^2 E, \\
& y_{1,13} \otimes \text{FunctionAx}, \frac{4}{3} b a A_4 @ T_1, T_2 D^2 - \frac{4}{3} b a \cos 4 p x D A_4 @ T_1, T_2 D^2 E >
\end{aligned}$$

We note that all of the spatial dependent functions satisfy the boundary conditions.

The **sol2Formv1** consists of two parts: a part is related to the secular terms, which consists of terms proportional to $E^{\pm I w_1 T_0}$ and $E^{\pm I w_2 T_0}$ and the other part consists of terms that are not related to the secular terms. The positions of the former are

```

pos@1D =
HPosition@# sol2Formv1 . expRule1@1D . Exp@_ + _ T0D -> 0, a_ .; a != 0, 1D . Flatten .
RestL & . 9 E^{-I w1 T0}, E^{I w1 T0} =
8811<, 810<<

```



```
pos@2D =
HPosition@# sol2Formv1 •. expRule1@2D •. Exp@_ . + _ T0D -> 0, a_ •; a != 0, 1D •• Flatten ••
RestL & •ž 9E-I w2 T0, EI w2 T0 =

8811<, 810<<
```

Since **pos[1]** and **pos[2]** are the same, we simply define

```
8pos1, pos1cc< = pos@1D •• Flatten

811, 10<
```

Next, we augment $y_{1,pos1}$ and $h_{1,pos1}$ by a linear combination of all possible homogeneous solutions. The unknown coefficients can be determined by the orthogonality conditions. ($y_{1,pos1cc}$ and $h_{1,pos1cc}$ are simply the complex conjugates of $y_{1,pos1}$ and $h_{1,pos1}$, respectively.) First, we express their solution forms using **psi1Sol** and **eta1Sol** as

```
STFormv =
8y1,pos1@xD + Sum@c1,j Fj@xD, 8j, 2<D, h1,pos1@xD + Sum@c1,j I wj Fj@xD, 8j, 2<D< •. eta1Sol •.
psi1Sol •. ci_ -> 0 •. modeshapes •. params •• Expand

: $ \frac{2}{3} c_{1,1} - \frac{2}{3} \cos@2 p xD c_{1,1} + \frac{1}{2} \sin@2 p xD c_{1,2} + \frac{2}{3} b p x a \sin@2 p xD A_3@T_1, T_2D^2,

2 I \frac{2}{3} p c_{1,1} - 2 I \frac{2}{3} p \cos@2 p xD c_{1,1} + 2 I \frac{1}{2} p \sin@2 p xD c_{1,2} +

\frac{2}{3} I b p a A_3@T_1, T_2D^2 - \frac{2}{3} I b p a \cos@2 p xD A_3@T_1, T_2D^2 + \frac{4}{3} I b p^2 x a \sin@2 p xD A_3@T_1, T_2D^2 >
```

Demanding that **STFormv** be orthogonal to the **adjointC[[i]]**, we obtain the unknown coefficients

```
c1Rule =
Solve@int@adjointC@@#DD.STFormv, 8x, 0, 1<D == 0 •. modeshapes •. params •. intRule1 ••.
intRule2 •. int -> Integrate, c1,#D & •ž 81, 2< •• Flatten

: c1,1 @ 0, c1,2 @ - \frac{b p a A_3@T_1, T_2D^2}{3 \cdot \frac{1}{2}} >
```

Then, we adjust **psi1Sol** with

```
psi1Sol@@pos1DD = y1,pos1 -> Function@x, STFormv@@1DD •. c1Rule •• Evaluated

y1,11 @ FunctionAx, - \frac{1}{3} b p a \sin@2 p xD A_3@T_1, T_2D^2 + \frac{2}{3} b p x a \sin@2 p xD A_3@T_1, T_2D^2 E

psi1Sol@@pos1ccDD =
y1,pos1cc -> Function@x, STFormv@@1DD •. c1Rule •. conjugateRule •• Evaluated

y1,10 @ FunctionAx, - \frac{1}{3} b p a \sin@2 p xD \dot{A}_3@T_1, T_2D^2 + \frac{2}{3} b p x a \sin@2 p xD \dot{A}_3@T_1, T_2D^2 E
```

Ÿ Horizontal Modes

The **sol2Formh** consists of two parts: a part is related to the secular terms, which consists of terms proportional to $E^{\pm I w_3 T_0}$ and $E^{\pm I w_4 T_0}$ and the other part consists of terms that are not related to the secular terms. The positions of the former are

```
pos@3D =
HPosition@# sol2Formh •. expRule1@3D •. Exp@_. + _ T0D -> 0, a_ •; a != 0, 1D •• Flatten ••
RestL & •ž 9E-I w3 T0, EI w3 T0=

882<, 83<<

pos@4D =
HPosition@# sol2Formh •. expRule1@4D •. Exp@_. + _ T0D -> 0, a_ •; a != 0, 1D •• Flatten ••
RestL & •ž 9E-I w4 T0, EI w4 T0=

88<, 8<<
```

Hence, we assume that the solution for $u_{2,2}$ and $v_{2,2}$ consists of two parts corresponding to the parts in **sol2Formh**. The part, which is not related to the secular terms, is

```
sol2Formha = Delete@sol2Formh, pos@3DD

8E-I T0 w1 - I T0 w3 Sin@p xD, EI T0 w1 + I T0 w3 Sin@p xD, E-I T0 w1 - I T0 w4 Sin@2 p xD,
EI T0 w1 - I T0 w4 Sin@2 p xD, E-I T0 w1 + I T0 w4 Sin@2 p xD, EI T0 w1 + I T0 w4 Sin@2 p xD<
```

We define the corresponding undetermined coefficients as

```
uSymbolha = Table@y2,i, 8i, Length@sol2FormhaD<D;
vSymbolha = uSymbolha •. y -> h;
```

Hence, the general solution in terms of the **sol2Formha** can be written as

```
sol2ha = 8u2,2 -> Function@8x, T0, T1, T2<, uSymbolha.sol2Formha •• Evaluated,
v2,2 -> Function@8x, T0, T1, T2<, vSymbolha.sol2Formha •• Evaluated<;
```

The part of **sol2Formh**, which is related to the secular terms, is

```
sol2Formhb = sol2Formh@@pos@3D@@1DDDD

8EI T0 w1 - I T0 w3 Sin@p xD<
```

and define the undetermined coefficients as

```
uSymbolhb = Table@ys2,i, 8i, Length@sol2FormhbD<D;
```

where the undetermined coefficients for $v_{2,2}$ are related to the **uSymbolhb** due to the orthogonality condition. Therefore, the general solution in terms of the **sol2Formhb** can be written as

```
sol2hb = 8u2,2 -> Function@8x, T0, T1, T2<, uSymbolhb.sol2Formhb •• Evaluated,
v2,2 -> Function@8x, T0, T1, T2<, -I w3 uSymbolhb.sol2Formhb •• Evaluated<;
```

The total solution is a combination of **sol2ha**, **sol2hb**, and the complexconjugate of **sol2hb**.

Substituting **sol2ha** into **order2Eqm[[{3,4}]]** and equating the coefficients of **sol2Formha** on both sides, we have

```
algEqa =
Flatten@Coefficient@Subtract žž # •. sol2ha, sol2FormhaD & •ž order2Eqm@@83, 4<DDD == 0 ••
Thread;
```

Solving for the **uSymbolha** and the **vSymbolha** yields

```
symbolaRule = Solve@algEqa, uSymbolha~Join~vSymbolhaD@@1DD •. params

: y2,1 ®  $\frac{2 b a \dot{A}_1 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D}{3}$ , y2,2 ®  $\frac{2 b a A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D}{3}$ ,
y2,3 ®  $\frac{16 b a \dot{A}_1 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D}{3}$ , y2,4 ®  $\frac{16 b a A_1 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D}{3}$ ,
y2,5 ®  $\frac{16 b a A_4 @ T_1, T_2 D \dot{A}_1 @ T_1, T_2 D}{3}$ , y2,6 ®  $\frac{16 b a A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3}$ ,
h2,1 ®  $-2 I \cdot \frac{11}{3} b p a \dot{A}_1 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D$ , h2,2 ®  $2 I \cdot \frac{11}{3} b p a A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D$ ,
h2,3 ®  $\frac{64 I b p a A_1 @ T_1, T_2 D \dot{A}_4 @ T_1, T_2 D}{3}$ , h2,4 ® 0, h2,5 ® 0, h2,6 ®  $\frac{64 I b p a A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} >$ 
```

Substituting **sol2hb** into either of **order2Eqm[[{3,4}]]**, equating the coefficients of **sol2Formhb** on both sides, and solving for the **uSymbolhb**, we have

```
symbolbRule =
Solve@Coefficient@Subtract žž order2Eqm@@3DD •. sol2hb, sol2FormhbD == 0 •• Thread,
uSymbolhbD@@1DD •. params

: y2,1 ®  $\frac{4 b a A_1 @ T_1, T_2 D \dot{A}_3 @ T_1, T_2 D}{3} >$ 
```

Substituting **symbolaRule** and **symbolbRule** into **sol2ha** and **sol2hb** yields

```
sol2haForm = 8u2,2@x, T0, T1, T2D, v2,2@x, T0, T1, T2D< •. sol2ha •. symbolaRule;
sol2hbForm = 8u2,2@x, T0, T1, T2D, v2,2@x, T0, T1, T2D< •. sol2hb •. symbolbRule;
```

```

sol2h = 8u2,2 ® Function@8x, T0, T1, T2<,
  sol2haForm@@1DD + sol2hbForm@@1DD + Hsol2hbForm@@1DD •. conjugateRuleL •. Expand •.
  EvaluatedD, v2,2 ® Function@8x, T0, T1, T2<, sol2haForm@@2DD +
  sol2hbForm@@2DD + Hsol2hbForm@@2DD •. conjugateRuleL •. Expand •. EvaluatedD<

:u2,2 ® FunctionB8x, T0, T1, T2<, 
$$\frac{2 b E^{i T_0 w_1 + i T_0 w_3} a \sin(p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D}{3} +$$


$$\frac{16 b E^{i T_0 w_1 + i T_0 w_4} a \sin(2 p x D A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} -$$


$$\frac{4 b E^{-i T_0 w_1 + i T_0 w_3} a \sin(p x D A_2 @ T_1, T_2 D A_1 @ T_1, T_2 D}{3} -$$


$$\frac{16 b E^{-i T_0 w_1 + i T_0 w_4} a \sin(2 p x D A_1 @ T_1, T_2 D A_1 @ T_1, T_2 D}{3} -$$


$$\frac{4 b E^{i T_0 w_1 - i T_0 w_3} a \sin(p x D A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D}{3} +$$


$$\frac{2 b E^{-i T_0 w_1 - i T_0 w_3} a \sin(p x D A_1 @ T_1, T_2 D A_2 @ T_1, T_2 D}{3} -$$


$$\frac{16 b E^{i T_0 w_1 - i T_0 w_4} a \sin(2 p x D A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} +$$


$$\frac{16 b E^{-i T_0 w_1 - i T_0 w_4} a \sin(2 p x D A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} F,$$

v2,2 ® FunctionB8x, T0, T1, T2<, 
$$2 i \frac{1}{3} b E^{i T_0 w_1 + i T_0 w_3} p a \sin(p x D A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D +$$


$$\frac{64 i b E^{i T_0 w_1 + i T_0 w_4} p a \sin(2 p x D A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} -$$


$$\frac{4 i b E^{-i T_0 w_1 + i T_0 w_3} a \sin(p x D w_3 A_3 @ T_1, T_2 D A_1 @ T_1, T_2 D}{3} +$$


$$\frac{4 i b E^{i T_0 w_1 - i T_0 w_3} a \sin(p x D w_3 A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D}{3} -$$


$$2 i \frac{1}{3} b E^{-i T_0 w_1 - i T_0 w_3} p a \sin(p x D A_1 @ T_1, T_2 D A_3 @ T_1, T_2 D -$$


$$\frac{64 i b E^{-i T_0 w_1 - i T_0 w_4} p a \sin(2 p x D A_1 @ T_1, T_2 D A_4 @ T_1, T_2 D}{3} F>$$


```

9.2.3 Solvability Conditions

Substituting **sol1**, **sol2v**, and **sol2h** into **eqEps[[3]]** yields

```

order3Eq =
  llinearSys •. u-k,1 -> uk,3M == llinearSys •. u-k,1 -> uk,3M - H#@@1DD & •ž eqEps@@3DDL •.
    sol1 •. sol2v •. eta1Sol •. psi1Sol •. sol2h •. intRule1 •. intRule2 •.
    modeshapes •. yRule •. int -> Integrate •• ExpandM •• Thread; •• Timing

843.242 Second, Null<

```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I_{w_1} T_0}$, from the equations governing the vertical modes, we have

$$ST2v = \text{CoefficientA}[\text{@@}2DD] \& \cdot \text{ž } \text{order3Eq}[\text{@@}81, 2<DD] \cdot \text{expRule1}[\text{@\#D}], E^{I_{w_1} T_0} E \& \cdot \text{ž } 81, 2<;$$

Collecting the terms that may lead to secular terms from the equations governing the horizontal modes, we have

$$ST2h = \text{CoefficientA}[\text{@@}2DD] \& \cdot \text{ž } \text{order3Eq}[\text{@@}83, 4<DD] \cdot \text{expRule1}[\text{@\#D}], E^{I_{w_1} T_0} E \& \cdot \text{ž } 83, 4<;$$

Demanding that $ST2v[i]$ be orthogonal to the $\text{adjointC}[i]$, we obtain the solvability conditions

SCond2v =

SolveAHint@adjointC@@#DD.ST2v@@#DD, 8x, 0, 1<D.. intRule1... intRule2.. modeshapes..

params.. int -> IntegrateL == 0,

A_#^{H0,1L}@T₁, T₂DE@@1DD & •ž 81, 2<.. Flatten.. ExpandAll;

SCond2v.. displayRule

$$\begin{aligned}
 :D_2 A_1 \otimes & - \frac{I E^{I T_1 s_1 + I t_1} \int_0^1 H P_1 @ x D - \cos @ 2 p x D P_1 @ x D \hat{a} x}{4 \cdot \frac{1}{6} p} - A_1 m_1 + \frac{2}{3} I p^3 a A_1^2 \dot{A}_1 - \\
 & \frac{224}{3} I b^2 p a^2 A_1^2 \dot{A}_1 + \frac{256 I b^2 p a^2 A_1^2 \dot{A}_1}{3 + p^2} + \frac{64 I b^2 p^3 a^2 A_1^2 \dot{A}_1}{9 + 3 p^2} + \frac{2}{3} I E^{2 I T_1 s_2} p^3 a A_2^2 \dot{A}_1 - \\
 & \frac{352}{9} I b^2 E^{2 I T_1 s_2} p a^2 A_2^2 \dot{A}_1 + \frac{2}{3} I E^{2 I T_1 s_4} p^3 a A_4^2 \dot{A}_1 - \frac{352}{9} I b^2 E^{2 I T_1 s_4} p a^2 A_4^2 \dot{A}_1 + \\
 & \frac{4}{3} I p^3 a A_1 A_2 \dot{A}_2 - \frac{256}{9} I b^2 p a^2 A_1 A_2 \dot{A}_2 - \frac{256 I b^2 p a^2 A_1 A_2 \dot{A}_2}{3 + p^2} - \frac{64 I b^2 p^3 a^2 A_1 A_2 \dot{A}_2}{9 + 3 p^2} + \\
 & \frac{1}{3} I p^3 a A_1 A_3 \dot{A}_3 - \frac{4}{3} I b^2 p a^2 A_1 A_3 \dot{A}_3 - \frac{64 I b^2 p a^2 A_1 A_3 \dot{A}_3}{3 + p^2} - \frac{16 I b^2 p^3 a^2 A_1 A_3 \dot{A}_3}{9 + 3 p^2} + \\
 & \frac{4}{3} I p^3 a A_1 A_4 \dot{A}_4 - \frac{256}{9} I b^2 p a^2 A_1 A_4 \dot{A}_4 - \frac{256 I b^2 p a^2 A_1 A_4 \dot{A}_4}{3 + p^2} - \frac{64 I b^2 p^3 a^2 A_1 A_4 \dot{A}_4}{9 + 3 p^2}, \\
 D_2 A_2 \otimes & - \frac{I E^{I T_1 s_1 - I T_1 s_2 + I t_1} \int_0^1 \sin @ 2 p x D P_1 @ x D \hat{a} x}{4 \cdot \frac{1}{2} p} - A_2 m_1 + \frac{4}{3} I p^3 a A_1 A_2 \dot{A}_1 - \\
 & \frac{1024}{9} I b^2 p a^2 A_1 A_2 \dot{A}_1 + \frac{64 I b^2 p^3 a^2 A_1 A_2 \dot{A}_1}{3 + p^2} + \frac{2}{3} I E^{-2 I T_1 s_2} p^3 a A_1^2 \dot{A}_2 - \\
 & \frac{352}{9} I b^2 E^{-2 I T_1 s_2} p a^2 A_1^2 \dot{A}_2 + 6 I p^3 a A_2^2 \dot{A}_2 + \frac{32}{3} I b^2 p a^2 A_2^2 \dot{A}_2 - \frac{64 I b^2 p^3 a^2 A_2^2 \dot{A}_2}{3 + p^2} + \\
 & 2 I E^{-2 I T_1 s_2 + 2 I T_1 s_4} p^3 a A_4^2 \dot{A}_2 + \frac{32}{3} I b^2 E^{-2 I T_1 s_2 + 2 I T_1 s_4} p a^2 A_4^2 \dot{A}_2 + \\
 & I p^3 a A_2 A_3 \dot{A}_3 - \frac{16 I b^2 p^3 a^2 A_2 A_3 \dot{A}_3}{3 + p^2} + 4 I p^3 a A_2 A_4 \dot{A}_4 - \frac{64 I b^2 p^3 a^2 A_2 A_4 \dot{A}_4}{3 + p^2} >
 \end{aligned}$$

Demanding that ST2h[[i]] be orthogonal to the adjointC[[i+2]], we obtain the solvability conditions

SCond2h =

```
SolveAHint@adjointC@@#+ 2DD.ST2h@@#DD, 8x, 0, 1<D •. intRule1 •. intRule2 •. modeshapes •.
  params •. int -> IntegrateL == 0,
  AH0,1L#+2@T1, T2DE@@1DD & •ž 81, 2< •. Flatten •. ExpandAll;
```

SCond2h •. displayRule

$$\begin{aligned}
 & :D_2 A_3 \otimes - A_3 m_2 + \frac{2}{3} I p^3 a A_1 A_3 \dot{A}_1 - \frac{136}{3} I b^2 p a^2 A_1 A_3 \dot{A}_1 + \\
 & \frac{32 I b^2 p^3 a^2 A_1 A_3 \dot{A}_1}{3 + p^2} + 2 I p^3 a A_2 A_3 \dot{A}_2 - \frac{32 I b^2 p^3 a^2 A_2 A_3 \dot{A}_2}{3 + p^2} + \frac{3}{4} I p^3 a A_3^2 \dot{A}_3 - \\
 & \frac{4}{3} I b^2 p a^2 A_3^2 \dot{A}_3 - \frac{8 I b^2 p^3 a^2 A_3^2 \dot{A}_3}{3 + p^2} + 2 I p^3 a A_3 A_4 \dot{A}_4 - \frac{32 I b^2 p^3 a^2 A_3 A_4 \dot{A}_4}{3 + p^2}, \\
 & D_2 A_4 \otimes - \frac{I E^{I T_1 s_1 - I T_1 s_4 + I t_2} \int_0^1 \sin @ 2 p x D P_2 @ x D \hat{a} x}{4 \cdot \frac{1}{2} p} - A_4 m_2 + \frac{4}{3} I p^3 a A_1 A_4 \dot{A}_1 - \\
 & \frac{1024}{9} I b^2 p a^2 A_1 A_4 \dot{A}_1 + \frac{64 I b^2 p^3 a^2 A_1 A_4 \dot{A}_1}{3 + p^2} + 4 I p^3 a A_2 A_4 \dot{A}_2 - \\
 & \frac{64 I b^2 p^3 a^2 A_2 A_4 \dot{A}_2}{3 + p^2} + I p^3 a A_3 A_4 \dot{A}_3 - \frac{16 I b^2 p^3 a^2 A_3 A_4 \dot{A}_3}{3 + p^2} + \\
 & \frac{2}{3} I E^{-2 I T_1 s_4} p^3 a A_1^2 \dot{A}_4 - \frac{352}{9} I b^2 E^{-2 I T_1 s_4} p a^2 A_1^2 \dot{A}_4 + 2 I E^{2 I T_1 s_2 - 2 I T_1 s_4} p^3 a A_2^2 \dot{A}_4 + \\
 & \frac{32}{3} I b^2 E^{2 I T_1 s_2 - 2 I T_1 s_4} p a^2 A_2^2 \dot{A}_4 + 6 I p^3 a A_4^2 \dot{A}_4 + \frac{32}{3} I b^2 p a^2 A_4^2 \dot{A}_4 - \frac{64 I b^2 p^3 a^2 A_4^2 \dot{A}_4}{3 + p^2} >
 \end{aligned}$$

We can rewrite **SCond2v** and **SCond2h** in compact form by collecting the coefficients of the same form as follows:

```
collectForm = TableA9Ai@T1, T2D EI w1 T0, Ai@T1, T2D E-I w1 T0, 8i, 4<E •. Flatten;

cubicTerms = Nest@Outer@Times, collectForm, #D &, collectForm, 2D •. Flatten •. Union;

cubicST@i_D := E-I w1 T0 cubicTerms •. expRule1@iD •. Exp@_ T0 + _.D -> 0 •. Union •. Rest
```

```
moduEq = MapIndexed@Hterm = cubicST@#2@@1DDD; coef = Coefficient@#1@@2DD, termD;
#1@@1DD == H#1@@2DD *. Thread@term -> 0DL + coef.termL &, Join@Scond2v, Scond2hDD;
moduEq *. displayRule
```

$$\begin{aligned}
D_2 A_1 &= - \frac{I E^{I T_1 s_1 + I t_1} \dot{U}_0^1 H P_1 @ x D - \text{Cos}@2 p x D P_1 @ x D L \hat{a} x}{4 \cdot \frac{1}{2} p} - \\
& A_1 m_1 + \left\{ \frac{2}{k^3} I p^3 a - \frac{224}{3} I b^2 p a^2 + \frac{256 I b^2 p a^2}{3 + p^2} + \frac{64 I b^2 p^3 a^2 y}{9 + 3 p^2} \right\} A_1^2 \dot{A}_1 + \\
& E^{2 I T_1 s_2} \left\{ \frac{2}{k^3} I p^3 a - \frac{352}{9} I b^2 p a^2 \right\} A_2^2 \dot{A}_1 + E^{2 I T_1 s_4} \left\{ \frac{2}{k^3} I p^3 a - \frac{352}{9} I b^2 p a^2 \right\} A_4^2 \dot{A}_1 + \\
& \left\{ \frac{4}{k^3} I p^3 a - \frac{256}{9} I b^2 p a^2 - \frac{256 I b^2 p a^2}{3 + p^2} - \frac{64 I b^2 p^3 a^2 y}{9 + 3 p^2} \right\} A_1 A_2 \dot{A}_2 + \\
& \left\{ \frac{1}{k^3} I p^3 a - \frac{4}{3} I b^2 p a^2 - \frac{64 I b^2 p a^2}{3 + p^2} - \frac{16 I b^2 p^3 a^2 y}{9 + 3 p^2} \right\} A_1 A_3 \dot{A}_3 + \\
& \left\{ \frac{4}{k^3} I p^3 a - \frac{256}{9} I b^2 p a^2 - \frac{256 I b^2 p a^2}{3 + p^2} - \frac{64 I b^2 p^3 a^2 y}{9 + 3 p^2} \right\} A_1 A_4 \dot{A}_4, \\
D_2 A_2 &= - \frac{I E^{I T_1 s_1 - I T_1 s_2 + I t_1} \dot{U}_0^1 \text{Sin}@2 p x D P_1 @ x D \hat{a} x}{4 \cdot \frac{1}{2} p} - A_2 m_1 + \\
& \left\{ \frac{4}{k^3} I p^3 a - \frac{1024}{9} I b^2 p a^2 + \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_1 A_2 \dot{A}_1 + E^{-2 I T_1 s_2} \left\{ \frac{2}{k^3} I p^3 a - \frac{352}{9} I b^2 p a^2 \right\} A_1^2 \dot{A}_2 + \\
& \left\{ 6 I p^3 a + \frac{32}{3} I b^2 p a^2 - \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_2^2 \dot{A}_2 + E^{-2 I T_1 s_2 + 2 I T_1 s_4} \left\{ 2 I p^3 a + \frac{32}{3} I b^2 p a^2 \right\} A_4^2 \dot{A}_2 + \\
& \left\{ I p^3 a - \frac{16 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_2 A_3 \dot{A}_3 + \left\{ 4 I p^3 a - \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_2 A_4 \dot{A}_4, \\
D_2 A_3 &= - A_3 m_2 + \left\{ \frac{2}{k^3} I p^3 a - \frac{136}{3} I b^2 p a^2 + \frac{32 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_1 A_3 \dot{A}_1 + \left\{ 2 I p^3 a - \frac{32 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_2 A_3 \dot{A}_2 + \\
& \left\{ \frac{3}{k^4} I p^3 a - \frac{4}{3} I b^2 p a^2 - \frac{8 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_3^2 \dot{A}_3 + \left\{ 2 I p^3 a - \frac{32 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_3 A_4 \dot{A}_4, \\
D_2 A_4 &= - \frac{I E^{I T_1 s_1 - I T_1 s_4 + I t_2} \dot{U}_0^1 \text{Sin}@2 p x D P_2 @ x D \hat{a} x}{4 \cdot \frac{1}{2} p} - A_4 m_2 + \\
& \left\{ \frac{4}{k^3} I p^3 a - \frac{1024}{9} I b^2 p a^2 + \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_1 A_4 \dot{A}_1 + \left\{ 4 I p^3 a - \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_2 A_4 \dot{A}_2 + \\
& \left\{ I p^3 a - \frac{16 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_3 A_4 \dot{A}_3 + E^{-2 I T_1 s_4} \left\{ \frac{2}{k^3} I p^3 a - \frac{352}{9} I b^2 p a^2 \right\} A_1^2 \dot{A}_4 + \\
& E^{2 I T_1 s_2 - 2 I T_1 s_4} \left\{ 2 I p^3 a + \frac{32}{3} I b^2 p a^2 \right\} A_2^2 \dot{A}_4 + \left\{ 6 I p^3 a + \frac{32}{3} I b^2 p a^2 - \frac{64 I b^2 p^3 a^2 y}{3 + p^2} \right\} A_4^2 \dot{A}_4 >
\end{aligned}$$

These solvability conditions satisfy the symmetry properties because

```
formList = TableAAj@T1, T2D TableAAi@T1, T2D A1@T1, T2D, 8i, 4<E, 8j, 4<E;
formList *. displayRule
```

$$\begin{aligned}
& 88 A_1^2 \dot{A}_1, A_1 A_2 \dot{A}_2, A_1 A_3 \dot{A}_3, A_1 A_4 \dot{A}_4, 8 A_1 A_2 \dot{A}_1, A_2^2 \dot{A}_2, A_2 A_3 \dot{A}_3, A_2 A_4 \dot{A}_4, \\
& 8 A_1 A_3 \dot{A}_1, A_2 A_3 \dot{A}_2, A_3^2 \dot{A}_3, A_3 A_4 \dot{A}_4, 8 A_1 A_4 \dot{A}_1, A_2 A_4 \dot{A}_2, A_3 A_4 \dot{A}_3, A_4^2 \dot{A}_4 <<
\end{aligned}$$


```

notationList = Table@si,j, {8i, 4<, 8j, 4<D
88s1,1, s1,2, s1,3, s1,4<, 8s2,1, s2,2, s2,3, s2,4<, 8s3,1, s3,2, s3,3, s3,4<, 8s4,1, s4,2, s4,3, s4,4<<

coefRule = MapThread@Rule, {notationList,
  Coefficient@H2 w# IL moduEq@@@#, 2DD, formList@@@DDD & .ž Range@4D<, 2D •• Flatten;

testList = Table@si,j - sj,i, {8i, 4<, 8j, i, 4<D •• Flatten •• Union •• Rest

8s1,2 - s2,1, s1,3 - s3,1, s2,3 - s3,2, s1,4 - s4,1, s2,4 - s4,2, s3,4 - s4,3<

testList •. coefRule •. params •• Factor

80, 0, 0, 0, 0, 0, 0<

```

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